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ON POWERS OF SETS IN LINEAR GROUPS

Dedicated to Professor Tadeusz Traczyk

Let $GL_1(n,K)$ denote the subbgroup of matrices $A \in GL(n,K)$ such that $\det(A) = \pm 1$, $K_2 = \{g \in G : o(g) = 2\}$, where G is a group. In the paper [1] it has been proved that in the linear group $GL_1(3,K)$, $SL(3,K)$ and $PSL(3,K)$ the sets K_2^4 cover these groups. In the proof the computer results from the paper [5] were used.

In this paper we will give another proof without the use of the mentioned computer results. It can be observed that the statement $GL_1(3,K) = K_2^4$ improves the result obtained in the paper [3] for $n = 3$.

The following Lemma will be useful in the sequel.

LEMMA 1. (see [2]) *Let G be a group. An element $g \in K_2^m (m \geq 2)$ if and only if there is an element $x \in K_2^{m-1}$, $x \neq g^{-1}$ such that $(gx)^2 = 1$.*

THEOREM 2. *In the group $GL_1(3,K)$, $K_2^4 = GL_1(3,K)$.*

Proof. If $A \in K_2$, then $\det(A) = \pm 1$. Thus $K_2^4 \subseteq GL_1(n,K)$. We will prove the inverse inclusion for $n = 3$. Each matrix $A \in GL_1(3,K)$ is similar to the one of the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -b_2 & -b_1 & 0 \\ 0 & 0 & \pm b_2 \end{bmatrix}, \quad A_3 = aE,$$

where $a_3^2 = 1$, $a^3 = \pm 1$.

We will consider three possible cases: (i) $\mathrm{char}(K) \neq 2$, (ii) $\mathrm{char}(K) = 2$, $|K| \neq 2$, (iii) $|K| = 2$.

(i). The matrix A_1 fulfils conditions

$$(1) \qquad T_1 A_1 \neq E, \quad (T_1 A_1)^2 = E, \quad \text{where } T_1 = S_1 R_1,$$

$$S_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in K_2, \quad R_1 = \begin{bmatrix} 0 & 0 & a_3^{-1} \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The matrix R_1 fulfils conditions

$$\text{(2)} \qquad M_1R_1 \neq E, \quad (M_1R_1)^2 = E, \quad \text{where } M_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in K_2.$$

By (2) and Lemma 1 it follows that $R_1 \in K_2^2$. Hence $T_1 = S_1R_1 \in K_2^2$, so $A_1 \in K_2^4$, by (1) and Lemma 1.

For the matrix A_2 we have

$$\text{(3)} \qquad T_2A_2 \neq E, \quad (T_2A_2)^2 = E, \quad \text{where } T_2 = \begin{bmatrix} 0 & b_2^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -b_2 \end{bmatrix}.$$

If $b_2 = 1$, then $T_2 \in K_2$ and $A_2 \in K_2^2 \subseteq K_2^4$, by Lemma 1. If $b_2 \neq 1$, then

$$T_2 = S_2R_2, \quad S_2 = \begin{bmatrix} b_2^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_2 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in K_2.$$

The matrix S_2 fulfils conditions

$$\text{(4)} \qquad M_2S_2 \neq E, \quad (M_2S_2)^2 = E, \quad \text{where } M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in K_2.$$

Hence $S_2 \in K_2^2$, by Lemma 1. Therefore $T_2 = S_2R_2 \in K_2^3$ and $A_2 \in K_2^4$, by (3) and Lemma 1.

The matrix A_3, $a^3 = 1$, fulfils conditions

$$\text{(5)} \qquad T_3A_3 \neq E, (T_3A_3)^2 = E, \text{where } T_3 = S_3R_3,$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & a^{-1} \\ 0 & -1 & 0 \\ a^{-2} & 0 & 0 \end{bmatrix} \in K_2.$$

The matrix S_3 fulfils conditions

$$\text{(6)} \qquad M_3S_3 \neq E, (M_3S_3)^2 = E, \text{where } M_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in K_2.$$

Thus $S_3 \in K_2^2$, by (6) and Lemma 1. Therefore $T_3 = S_3R_3 \in K_2^3$, so $A_3 \in K_2^4$, by (5) and Lemma 1.

If $a^3 = -1$. then A_3 fulfils conditions

$$\text{(7)} \qquad T_4A_3 \neq E, (T_4A_3)^2 = E, \text{where } T_4 = S_4R_4,$$

$$S_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0 & 0 & -a^{-1} \\ 0 & -1 & 0 \\ a^{-2} & 0 & 0 \end{bmatrix} \in K_2 .$$

The matrix S_4 fulfils conditions

$$(8) \qquad M_3S_4 \neq E, (M_3S_4)^2 = E .$$

From (8) and Lemma 1 it follows that $S_4 \in K_2^2$. Hence $T_4 = S_4R_4 \in K_2^3$. Therefore $A_3 \in K_2^4$, by (7) and Lemma 1.

(ii). If $a_1 \neq 0$ or $a_2 \neq 0$, then A_1 fulfils conditions (1), so $A_1 \in K_2^4$. If $a_1 = a_2 = 0$, then

$$T_5A_1 \neq E, (T_5A_1)^2 = E, \text{where } T_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in K_2 .$$

Thus $A_1 \in K_2^2 \subseteq K_2^4$, by Lemma 1.

If $b_1 \neq 0$, then the conditions (3) are fulfilled and the proof for A_2 is the same as in the case (i). If $b_1 = 0$ and $b_2^3 \neq 1$, then in $SL(3,K), A_2$ is similar to the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & b_2 & b_2^{-1} \end{bmatrix}$$

which is a type of A_1, so $A_2 \in K_2^4$. If $b_1 = 0, b_2^3 = 1$ and $b_2 \neq 1$, then $b_2^2 \neq 1$. From Theorem 1 (see [1]) we know that $GL_1(3,K) = SL(3,K) \subseteq C_VC_V\underline{Z}$, where C_V denotes a class of conjugate elements of the matrix $V = diag(1, b_2, b_2^2)$. From the equality $b_2E = VT_6VT_6^{-1}$, where

$$T_6 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that $Z \subseteq C_VC_V$. Hence $GL_1(3,K) \subseteq C_VC_V$. The matrix V fulfils conditions $VT_7 \neq E, (VT_7)^2 = E$, where

$$T_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in K_2 .$$

Thus $GL_1(3,K) \subseteq K_2^4$ and particularly $A_2 \in K_2^4$.

If $b_1 = 0, b_2 = 1$, then the matrix A_2 fulfils conditions

$$(9) \qquad T_8A_2 \neq E, (T_8A_2)^2 = E,$$

where

$$T_8 = \begin{bmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix T_8 belongs to K_2 because the equality $x^2 + y^2 = 1$ has a solution different from $(1,0),(0,1)$, in the field $GF(2^s), s \neq 1$. Hence $A_2 \in K_2^2 \subseteq K_2^4$, by (9) and Lemma 1.

The proof of the relation $A_3 \in K_2^4$ is the same as in the case (i).

(iii). In the paper [3] it has been proved that $K_2^4 = PSL(2,q)$, $q \geq 5$. But $GL_1(3,K) = SL(3,K) \simeq PSL(2,7)$, so this proves our case.

We have proved that matrices $A_1 \in GL_1(3,K)$ belong to the set K_2^4. The set K_2^4 is a normal set so $GL_1(3,K) \subseteq K_2^4$.

Observe that if $A_i \in SL(3,K)$, then $T_i, R_i, S_i, M_i \in SL(3,K)$. Therefore we have the following corollary.

COROLLARY 1.1. *$SL(3,K) = K_2^4$ and $PSL(3,K) = K_2^4$.*

References

[1] J. Ambrosiewicz, *Powers of sets in linear groups*, Demonstratio Math. 23 (1990), 395–403.

[2] J. Ambrosiewicz, *On the square of sets of linear groups*, Rend. Sem. Mat. Univ. Padova 75 (1986), 253–256.

[3] J. Ambrosiewicz, *Powers of sets of elements of order two in linear groups*, Demonstratio Math. 27 (1994), 109–114.

[4] E. Ambrosiewicz, *Powers of sets of involutions in linear groups*, Demonstratio Math. 24 (1991), 311–314.

[5] A. Dold, B. Eckmann, Z. Arad, M. Herzog, *Products of conjugacy classes in groups*, Lecture Notes in Mathematics 1112, Springer-Verlag, Berlin Heidelberg, 1985, 53–196.

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