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Ivan Chajda

A REPRESENTATION OF THE ALGEBRA OF QUASIORDERED LOGIC BY BINARY FUNCTIONS

Dedicated to Professor Tadeusz Traczyk

Some algebras are representable by suitable sets of functions. The well-known examples are the Cayley Theorem for groups, the Holland Theorem for 1-groups etc. In [2], the authors developed a representation of the Boolean algebra by a certain set G of binary functions $f : A^2 \to A$, the so called guards. These functions satisfy the following conditions:

idempotency:	$f(x,x) = x$
diagonality:	$f(f(x,y), f(u,v)) = f(x,v)$
commutativity:	$f(g(x,y), g(u,v)) = g(f(x,u), f(y,v))$

for each $f, g \in G$.

The origin of guards is explained in [1]. The guard is properly a switching function under (the condition) P:

$$f(x,y) := \text{if } P \text{ then } x \text{ else } y$$

in the computer science terminology, see [6] for more detailes. Hence, the guard is a Boolean condition which must hold before the procedure it guards can begin. It is easy to verify that guards are binary functions satisfying idempotency, diagonality and commutativity.

For some reasons, there was developed another logic based on the so called algebra of quasiordered logic, see [3], [4]. In such a logic, we can make differences between empirical true values (given values) and those obtained by some logical reasoning (calculated values). For some detailes, see [4]. The motivation of this paper is to give a representation of such algebras by binary functions satisfying conditions which generalize guards.

DEFINITION 1. A *generalized guard* algebra G on a set A is a set of binary functions $f : A^2 \to A$ (the so called *generalized guards*) satisfying the following four properties (for all $f, g, h \in G$):
uniformity (U): $f(x,x) = g(x,x)$
conditional diagonality (CD):

$$\langle x,y\rangle \neq \langle u,v\rangle \Rightarrow f(f(x,y), f(u,v)) = f(x,v)$$

conditional commutativity (CC):

$$\langle x,y\rangle \neq \langle u,v\rangle \text{ and } \langle x,u\rangle \neq \langle y,v\rangle \text{ imply}$$
$$f(g(x,y), g(u,v)) = g(f(x,u), f(y,v))$$

insertion property (IP):

$$\text{if } f \neq pr_2 \text{ then } f(x,y) = f(x, g(y,y))$$
$$\text{if } f \neq pr_1 \text{ then } f(x,y) = f(h(x,x), y)$$
$$\text{if } pr_1 \neq f \neq pr_2 \text{ then } f(x,y) = f(h(x,x), g(y,y)),$$

where pr_i is the i-th binary projection, i.e. $pr_i(x_1, x_2) = x_i$.

It is almost evident that idempotency, diagonality and computativity imply (U), (CD), (CC) and (IP) but not vice versa in a general case.

The rather long name "an algebra of quasiordered logic" adapted from [4] will be shorten to a "q-algebra":

DEFINITION 2. By a q-algebra is meant an algebra $(A; \vee, \wedge, ', 0, 1)$ of the similarity type (2,2,1,0,0) satisfying the following conditions:
associativity: $a \vee (b \vee c) = (a \vee b) \vee c \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$
commutativity: $a \vee b = b \vee a \quad a \wedge b = b \wedge a$
weak absorption: $a \vee (a \wedge b) = a \vee a \quad a \wedge (a \vee b) = a \wedge a$
weak idempotence: $a \vee (b \vee b) = a \vee b \quad a \wedge (b \wedge b) = a \wedge b$
equalization: $a \vee a = a \wedge a$
distributivity: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
0-1 laws: $a \wedge 0 = a$ and $a \vee 1 = 1$
complementation: $a \wedge a' = 0 \quad a \vee a' = 1$.

An element $b \in A$ is an idempotent whenever $b \vee b = b$ (or, equivalently, $b \wedge b = b$). In the whole paper, we will suppose $0 \neq 1$.

It is clear tht the set of all idempotents of a q-algebra A forms a subalgebra of A which is a Boolean algebra (where $\vee$ is join, $\wedge$ is meet, $'$ is complementation, 0 is zero and 1 is unit).

Now, we are ready to give the functional representation. Let $A = (A; \vee, \wedge, ', 0, 1)$ be a q-algebra and $X = A \times A$. Denote by $G(A)$ the set of all binary functions over X defined as follows: for each $b \in A$ we put

$$f_b(x,y) = \begin{cases} b & \text{if } \langle x,y\rangle = \langle 1,0\rangle \\ (x \wedge b) \vee (y \wedge b') & \text{if } \langle x,y\rangle \neq \langle 1,0\rangle. \end{cases}$$

LEMMA 1. *Let A be a q-algebra, $b \in A$ and $f_b \in G(A)$. Then $f_b(x,x) = x$ if and only if x is an idempotent of A.*

Proof. As it was shown in [3], [4], we have

$$x \vee 0 = x \vee x$$

for each $x \in A$. By the definition of f_b,

$$f_b(x,x) = (x \wedge b) \vee (x \wedge b') = x \vee (b \wedge b') = x \vee 0 = x \vee x$$

whence the assertion is evident. ∎

LEMMA 2. *Let A be a q-algebra. For each $x, y \in A$, the elements $x \vee y$, $x \wedge y$, x', 0 and 1 are idempotents.*

For the easy proof, see e.g. [3], [4].

THEOREM 1. *Let A be a q-algebra. The set $G(A)$ is a generalized guard algebra.*

Proof. We have to verify the conditions U, CD, CC, IP of Definition 1.

ad (U): Let $b, c \in A$. Since $x = y$, we have $\langle x,y\rangle \neq \langle 1,0\rangle$,

thus $f_b(x,x) = (x \wedge b) \vee (x \wedge b') = x \vee (b \wedge b') = x \vee 0 = x \vee x = x \vee (c \wedge c') = (x \wedge c) \vee (x \wedge c') = f_c(x,x)$.

ad (CD):

(i) Suppose $\langle 1,0\rangle \neq \langle x,y\rangle \neq \langle u,v\rangle \neq \langle 1,0\rangle$. Then, by Lemma 2, we have

$$\begin{aligned} f_b(f_b(x,y), f_b(u,v)) &= (((x \wedge b) \vee (y \wedge b') \wedge b) \vee (((u \wedge b) \vee (v \wedge b')) \wedge b') \\ &= (x \wedge b) \vee (v \wedge b') = f_b(x,v). \end{aligned}$$

(ii) Suppose $\langle x,y\rangle = \langle 1,0\rangle \neq \langle u,v\rangle$. Then

$$\begin{aligned} f_b(f_b(1,0), f_b(u,v)) = f_b(b, f_b(u,v)) &= (b \wedge b) \vee ((u \wedge b) \vee (v \wedge b') \wedge b') \\ &= (1 \wedge b) \vee (v \wedge b') = f_b(1,v). \end{aligned}$$

(iii) The case $\langle x,y\rangle \neq \langle 1,0\rangle = \langle u,v\rangle$ is analogous to that of (ii).

ad (CC): Let $\langle x,y\rangle \neq \langle u,v\rangle$ and $\langle x,u\rangle \neq \langle y,v\rangle$ and $b, c \in A$.

(i) Suppose that all of four pairs are different from $\langle 1,0\rangle$.

Then

$$\begin{aligned} f_b(f_c(x,y), f_c(u,v)) &= ((x \wedge c) \vee (y \wedge c')) \wedge b) \vee (((u \wedge c) \vee (v \wedge c')) \wedge b') \\ &= (((x \wedge b) \vee (u \wedge b') \wedge c) \vee (((y \wedge b) \vee (v \wedge b')) \wedge c') \\ &= f_c(f_b(x,u), f_b(y,v)). \end{aligned}$$

(ii) If $\langle x,y\rangle = \langle 1,0\rangle$ and all remaining pairs are different from $\langle 1,0\rangle$ then

$$f_b(f_c(1,0), f_c(u,v)) = f_b(c, f_c(u,v)) = (c\wedge b)\vee(((u\wedge c)\vee(v\wedge c')\wedge b') \\ = f_c(f_b(1,u), f_b(0,v)).$$

All other possible cases can be count in a similar way.

ad (IP): Let $b,c\in A$. If $f_b \neq pr_2$ and $\langle x,y\rangle \neq \langle 1,0\rangle$, we have

$$f_b(x, f_c(y,y)) = (x\wedge b)\vee(((y\wedge c)\vee(y\wedge c'))\wedge b') = (x\wedge b)\vee(y\wedge b') \\ = f_b(x,y);$$

if $\langle x,y\rangle = \langle 1,0\rangle$ then

$$f_b(1, f_c(0,0) = f_b(1, 0\vee 0) = f_b(1,0).$$

All other cases of (IP) are similar. ∎

In accordance with Theorem 1, the generalized guard algebra $G(A)$ for a given q-algebra A will be called the *induced generalized guard algebra.*

For the proof of Theorem 2, the following technical lemma is usefull:

LEMMA 3. *Let G be a generalized guard algebra and $f,g,h\in G$ are different from pr_1, pr_2 and $h(x,x)=x$. Then*

$$g(x, f(h(x,y),y)) = f(g(x,h(x,y),g(x,y)).$$

Proof. If $x\neq y$ then by (IP), we conclude

$$g(x, f(h(x,y),y)) = g(f(x,x), f(h(x,y),y)) \text{ and, by } (CC), \\ = f(g(x,h(x,y)), g(x,y)).$$

If $x=y$, then by (IP) and (U) we obtain

$$g(x, f(h(x,x),x)) = g(f(x,x), f(h(x,x),x) = g(f(x,x), f(x,x)) \\ = g(x,x) = f(x,x) = f(g(x,x), g(x,x) \\ = f(g(x,h(x,x)), g(x,x)).$$

THEOREM 2. *Let G be a generalized guard algebra. Define the operations $\vee$, $\wedge$, $'$, o, j as follows*

$$(f\vee g)(x,y) = f(x, g(x,y))$$
$$(f\wedge g)(x,y) = f(g(x,y),y)$$
$$f'(x,y) = f(y,x)$$
$$o(x,y) = f(y,y),\ j(x,y) = f(x,x).$$

Then $A(G) = (G;\vee,\wedge,',o,j)$ is a q-algebra, the so called induced q-algebra.

P r o o f. For associativity, we can check easily

$$((f \wedge g) \wedge h)(x,y) = f(g(h(x,y),y),y) = (f \wedge (g \wedge h))(x,y)$$

and also dually, for the operation $\vee$.

Prove commutativity of $\wedge$. If $x = y$, we have by (U) $(f \wedge g)(x,x) = (g \wedge f)(x,x)$. If $x \neq y$ and $f \neq pr_1 \neq g$ then

$$(f \wedge g)(x,y) = f(g(x,y),y) = f(g(x,y),g(y,y)) = g(f(x,y),f(y,y))$$
$$= g(f(x,y),y) = (g \wedge f)(x,y).$$

If $f = pr_1$ then

$$(f \wedge g)(x,y) = pr_1(g(x,y),y) = g(x,y) = g(pr_1(x,y),y) = (g \wedge f)(x,y).$$

Analogously, we can testify it in the remaining cases and dually also for $\vee$.

Prove weak absorption:

If $f \neq pr_1$ and $x \neq y$ then

$$(f \wedge (f \wedge g))(x,y) = f(x,f(g(x,y),y) = f(f(x,x),f(g(x,y),y)) = f(x,y)$$
$$= f(f(x,x),f(x,y)) = f(x,f(x,y)) = (f \vee f)(x,y).$$

If $f = pr_1$ then

$$(f \vee (f \wedge g))(x,y) = f(x,f(g(x,y),y) = x = f(x,f(x,y) = (f \vee f)(x,y).$$

If $x = y$ (and $f \neq pr_1$) we obtain

$$(f \vee (f \wedge g))(x,x) = f(x,f(g(x,x),x)) = f(x,f(x,x)) = (f \vee f)(x,x).$$

Dually it can be done $f \wedge (f \wedge g) = f \wedge f$.

Weak idempotency:

Suppose $f \neq pr_1$ and $g(x,y) \neq x$. Then

$$(f \vee (f \vee g))(x,y) = f(x,f(x,g(x,y))) = f(f(x,x),f(x,g(x,y)))$$
$$= f(x,g(x,y)) = (f \vee g)(x,y).$$

If $f = pr_1$, the proof is evident. If $f \neq pr_1$ and $g(x,y) = x$, we have

$$(f \vee (f \vee g))(x,y) = f(x,f(x,g(x,y))) = f(x,f(x,x)) = f(x,x)$$
$$= f(x,g(x,x)) = (f \vee g)(x,y).$$

Dually we can prove $f \wedge (f \wedge g) = f \wedge g$.

Equalization:

If $pr_1 \neq f \neq pr_2$ and $x \neq y$, we obtain

$$(f \wedge f)(x,y) = f(f(x,y),y) = f(f(x,y),f(y,y)) = f(x,y)$$
$$= f(f(x,x),f(x,y)) = f(x,f(x,y)) = (f \vee f)(x,y).$$

If $f = pr_1$ then

$$(f \wedge f)(x,y) = f(f(x,y),y) = x = f(x,f(x,y)) = (f \vee f)(x,y).$$

The case $f = pr_2$ is similar and the case $x = y$ follows directly by (U).

Distributivity:

If $f = pr_1$ or $f = pr_2$ (or, similarly, for g or h), the proof is straightforward. Let f, g, h differ from pr_1, pr_2.

Then $(f \wedge (g \vee h))(x,y) = f(g(x, h(x,y)), y)$ and, by Lemma 3,

$$((f \wedge g) \vee (f \wedge h))(x,y) = f(g(x, f(h(x,y),y)), f(h(x,y),y)) = \\ = f(f(g(x,h(x,y),g(x,y)), f(h(x,y),y)) = f(g(x,h(x,y)),y)$$

proving distributivity.

0-1 laws: If $f = pr_1$ then $(f \wedge o)(x,y) = pr_1(o(x,y),y) = o(x,y)$. If $f \neq pr_1$, we have

$$(f \wedge o)(x,y) = f(o(x,y),y) = f(f(y,y),y) = f(y,y) = o(x,y).$$

Analogously we can prove $(f \vee j)(x,y) = j(x,y)$.

It remains to prove complementation. Immediately, we have $(f' \wedge f)(x,y) = f'(f(x,y),y) = f(y, f(x,y))$.

If $x = y$ then $f(y,y) = o(x,y)$ and we obtain $f' \wedge g = o$.

If $x \neq y$ and $f = pr_2$, then

$$f(y, f(x,y)) = f(y,y) = o(x,y).$$

If $x \neq y$ and $f \neq pr_2$, we have by (IP) and (CD)

$$f(y, f(x,y)) = f(f(y,y), f(x,y)) = f(y,y) = o(x,y).$$

The proof of $f' \vee f = j$ is dual. ∎

COROLLARY . *Every q-algebra* $A = (A; \vee, \wedge, ', 0, 1)$ *is isomorphic with* $A(G(A)) = (G(A); \vee; \wedge, ', f_0(x,y), f_1(x,y))$, *where* $G(A)$ *is indiced generalized guard algebra and the isomorphism is given as follows*:

$$b \mapsto f_b(x,y) = \begin{cases} b & \text{for } \langle x,y\rangle = \langle 1,0\rangle \\ (x \wedge b) \vee (y \wedge b') & \text{for } \langle x,y\rangle \neq \langle 1,0\rangle. \end{cases}$$

The operations $\vee, \wedge, '$ *in* $A(G(A))$ *are defined by formulas*:

$$(f_b \vee f_c)(x,y) = f_b(x, f_c(x,y))$$
$$(f_b \wedge f_c)(x,y) = f_b(f_c(x,y),y)$$
$$f_b'(x,y) = f_b(y,x).$$

Proof. $f_o(x,y) = (x \wedge 0) \vee (y \wedge 1) = y \wedge 1 = y \wedge (b \vee b') = (y \wedge b) \vee (y \wedge b') = f_b(y,y) = o(x,y)$, dually $f_1(x,y) = j(x,y)$ (the cases $\langle x,y\rangle = \langle 1,0\rangle$ are trivial), thus $A(G(A))$ is a q-algebra. It is clear that the mapping $b \mapsto f_b(x,y)$ is a bijection. Moreover,

$$f_{b\vee c}(x,y) = (x \wedge (b \vee c)) \vee (y \wedge (b \vee c)') = f_b(x, f_c(x,y)) = (f_b \vee f_c)(x,y).$$

Analogously, we can show $f_{b\wedge c} = f_b \wedge f_c$ and $f_{b'} = f'_b$ thus $b \mapsto f_b(x,y)$ is an isomorphism. ■

References

[1] S. L. Bloom, Z. Ésik: *Cayley iff Stone*, EATCS Bulletin, Feb. 1991, 159–161.
[2] S. L. Bloom, Z. Ésik, E. Manes: *A Cayley Theorem for Boolean algebras*, Amer. Math. Monthly, 97(1990), 831–833.
[3] I. Chajda: *Lattices of quasiordered sets*, Acta UPO (Olomouc), 31(1992), 6–12.
[4] I. Chajda: *Subdirectly irreducible algebras of quasiordered logic*, Acta U P (Olomouc), 32(1993), to appear.
[5] I. Chajda, M. Kotrle: *Subdirectly irreducible and congruence distributive q-lattices*, Czech. Math. J., to appear 1993.
[6] E. G. Manes, M. A. Arbib: *Algebraic Approaches to Program Semantics*, Springer Verlag (Berlin) 1986.

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PRIR. FAK. UP OLOMOUC
Tomkova 38
779 00 OLOMOUC, CZECH REPUBLIC

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