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SEMI-DIRECT PRODUCTS AND BOL LOOP

Dedicated to Professor Tadeusz Traczyk

1. Introduction

If K is a loop[1] satisfying some loop-theoretic property $\mathcal{P}$ (usually described by means of identities), it is often of interest to extend K to loops G which also satisfy $\mathcal{P}$. In such situations, the following question is relevant: Given loops H and K with K satisfying a property $\mathcal{P}$, can one use the Cartesian product set $H \times K$ as the carrier set for various "semi-direct" product loops $G = H \times K$ which also satisfy property $\mathcal{P}$ and which have a subloop $\overline{K}$ isomorphic to K serving as the kernel of a (loop) homomorphism of G onto H?

In §2 we discuss how and precisely under what circumstances (external) semi-direct products of loops can be formed, and show exactly when a loop is an (internal) semi-direct product of a pair of subloops. In this section, we include some examples, and show that some constructions in the literature — seemingly *ad hoc* — are in reality semi-direct products.

In §3 we select a specific loop-theoretic property $\mathcal{P}$, namely, that of satisfying the (right) Bol identity: $(xy\cdot z)y = x(yz\cdot y)$. The class of Bol loops has the advantage of being general enough to include Bruck loops, Moufang

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[1] We assume that the reader is familiar with basic facts concerning loops (e.g., subloops, normal subloops, inner maps, nuclei, autotopisms, loop isotopes, etc.), all of which can be found in R.H. Bruck's treatise [2], as well as in H.O. Pflugfelder's recent textbook [6].

loops, extra loops, and groups as special members and specific enough to accommodate definitive results. In Theorem 3.7, we present necessary and sufficient conditions for a semi-direct product of loops to be a Bol loop, and at the same time call attention to the role played by right nuclear automorphisms in such investigations. We discuss the availability of such automorphisms for Bol loops.

In §4 we illustrate the applicability of our results by constructing some new Bol loops.

2. Semi-direct products[2]

There is ample group-theoretic motivation for

CONSTRUCTION C. Let H and K be loops, let $\mathrm{Sym}\,K$ be the symmetric group on the set K, and let $\theta : H \to \mathrm{Sym}\,K$ be such that $\theta(e_H)$ is the identity element of $\mathrm{Sym}\,K$ and such that $(e_K)\theta(h) = e_K$ for all $h \in K$. Now for all $(h_1, k_1), (h_2, k_2) \in H \times K$ define

$$(2.1) \qquad (h_1, k_1) \cdot (h_2, k_2) = (h_1 h_2, k_1\theta(h_2) \cdot k_2).$$

It is easy to see that $H \times K$ is a loop with respect to this binary operation, the identity element being (e_H, e_K).

DEFINITION 2.1. Any loop constructed from loops H and K in accordance with Construction C is called the *external semi-direct product of H and K with respect to θ* and is denoted by $H \times_\theta K$.

The reader should recognize that Construction C and the terminology and notation of Definition 2.1 are taken from group theory (see, for instance, the chapter on extensions in J.J. Rotman [13]). In particular, if K is a group, if H is the automorphism group of K, and if $\theta : H \to \mathrm{Sym}\,K$ is the insertion map, then the loop $H \times_\theta K$ of Definition 2.1 is a group. This particular group is the classical holomorph of K and shows that any group K can be embedded in and then identified with a normal subgroup $\overline{K}$ of a group $H \times_\theta K$ in such a way that every automorphism of $\overline{K}$ can be viewed as the restriction of an inner automorphism of $H \times_\theta K$ to $\overline{K}$. (For a loop-theoretic analogue of a holomorph, see Example 2.7 and the two references cited

[2] In this and the remaining sections of this paper, we write loop operations multiplicatively (using juxtaposition or dots) and do not make any notational distinction between a loop and its carrier set. We often employ a mixture of juxtaposition, dots, parentheses, brackets, etc. to indicate in a readable, unambiguous way how associations are to be made. For instance, we prefer to avoid awkward expressions like "$x \cdot (((x \cdot y) \cdot z) \cdot y)$" by writing, instead, "$x[(xy \cdot z)y]$". We use e_G to denote the identity element of a loop G, and employ a right-sided functional notation, writing $x\alpha$ to indicate the result of applying a map α to an element x. In this regard, whenever x and y are members of a loop G, we write $xy = xR(y) = yL(x)$, so $R(y)$ and $L(x)$ become members of $\mathrm{Sym}\,G$.

there). But now in the paragraphs below we adopt procedures which permit us to extend the notion of semi-direct product to loop theory.

Recall that the associator (x, y, z) of members x, y, z of a loop G is that element in G such that $xy \cdot z = (x \cdot yz)(x, y, z)$. Whenever A, B and C are subloops of G it is customary to use (A, B, C) to denote the subloop of G generated by the associators (a, b, c) for all $a \in A$, $b \in B$ and $c \in C$.

DEFINITION 2.2. A loop G is an *internal semi-direct product of subloops H and K* means that K is normal in G, that $G = HK$, and that $H \cap K = (K, H, K) = (H, H, K) = (H, K, G) = \{e_G\}$.

THEOREM 2.3. *If a loop G is an internal semi-direct product of subloops H and K, then G is isomorphic to the external semi-direct product $H \times_\theta K$ where for each $h \in H$ the map $\theta(h)$ is the restriction of the inner map $R(h)L(h)^{-1}$ of G to the normal subloop K.*

Proof. Since $G = HK$, each $g \in G$ can be expressed as $g = hk$ for some $h \in H$ and some $k \in K$. As for uniqueness, let $h_1, h_2 \in H$ and $k_1, k_2 \in K$ with $h_1k_1 = h_2k_2$. Since $(H, H, K) = \{e_G\}$, it is easy to see that $h_1L(h_2)^{-1}R(k_1) = h_1R(k_1)L(h_2)^{-1}$, and so

$$\begin{aligned} h_1L(h_2)^{-1} &= h_1L(h_2)^{-1}R(k_1)R(k_1)^{-1} = h_1R(k_1)L(h_2)^{-1}R(k_1)^{-1} \\ &= (h_1k_2)L(h_2)^{-1}R(k_1)^{-1} = (h_2k_2)L(h_2)^{-1}R(k_1)^{-1} = k_2R(k_1)^{-1}. \end{aligned}$$

But now with $h_1L(h_2)^{-1} = k_2R(k_1)^{-1}$ and $H \cap K = \{e_G\}$, we conclude that $h_1 = h_2$ and $k_1 = k_2$.

For each $h \in H$, let $\theta(h)$ be the restriction of the inner map $R(h)L(h)^{-1}$ to K. Since K is a normal subloop of G, it is clear that $\theta(h) \in \operatorname{Sym} K$.

Finally, let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Taking advantage of $(H, K, G) = (K, H, K) = (H, H, K) = \{e_G\}$, we see that

$$\begin{aligned} (h_1k_1)(h_2k_2) &= h_1(k_1 \cdot h_2k_2) = h_1(k_1R(h_2)L(h_2)^{-1}L(h_2) \cdot k_2) \\ &= h_1[(h_2 \cdot k_1\theta(h_2))k_2] \\ &= h_1[h_2(k_1\theta(h_2) \cdot k_2)] \\ &= (h_1h_2)(k_1\theta(h_2) \cdot k_2). \end{aligned}$$

Thus, $hk \mapsto (h, k)$ is an isomorphism of G onto $H \times_\theta K$, and our proof is complete. ∎

THEOREM 2.4. *If a loop G is an external semi-direct product of loops G and K, that is, $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$, then there are subloops $\overline{H}$ and $\overline{K}$ of G isomorphic to H and K respectively so that G is an internal semi-direct product of $\overline{H}$ and $\overline{K}$.*

Proof. Let $\overline{H} = \{(h, e_K)|$ all $h \in H\}$ and $\overline{K} = \{(e_H, k)|$ all $k \in K\}$. Since $(e_K)\theta(h) = e_K$ for all $h \in H$, it is clear from (2.1) that $\overline{H}$ is a subloop of G isomorphic to H. Since $k\theta(e_H) = k$ for all $k \in K$, it is equally evident from (2.1) that $\overline{K}$ is a subloop of G isomorphic to K. Since $k\theta(e_H) = k$ for all $k \in K$, it is equally evident from (2.1) that $\overline{K}$ is a subloop of G isomorphic to K. Also from (2.1) it is easy to see that $G = \overline{H}\overline{K}$ and that $\overline{H} \cap \overline{K} = (\overline{K}, \overline{H}, \overline{K}) = (\overline{H}, \overline{H}, \overline{K}) = (\overline{H}, \overline{K}, G) = \{(e_H, e_K)\} = \{e_G\}$. Since $(h, k) \to (h, e_K)$ is a homomorphism of G onto $\overline{H}$ with kernel $\overline{K}$, the subloop $\overline{K}$ is normal in G. This completes our proof. ■

The two preceding theorems reveal how intimately external and internal semi-direct products are related to each other, and also how easy it is to shift one's emphasis from one to the other, as illustrated in the following.

Remark 2.5 (i) Let G, H, and K be loops such that $G = H \times_\theta K$. Then $G = H \times K$, the ordinary direct product, id and only if $\theta(h)$ is the identity element od Sym K for all $h \in H$.

(ii) Now let G be an internal semi-direct product of subloops H and K. Then using (i) and the preceding theorems one sees that G is the direct product of H and K if and only if G (as well as K) is a normal subloop of G.

In addition to the familiar examples (direct products of loops, semi-direct products of groups, etc.), we wish to call attention to some loop-theoretic constructions in the literature which can be viewed as semi-direct products.

EXAMPLE 2.6. Let R be an alternative ring, let G, H, and K be given by

$$G = \left\{ \begin{pmatrix} 1 & a & x & z \\ 0 & 1 & b & y \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \text{all } a, b, x, y, z \in R \right\},$$

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & y \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \text{all } b, y \in R \right\},$$

$$K = \left\{ \begin{pmatrix} 1 & a & x & y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \text{all } a, x, y \in R \right\}.$$

Then under matrix multiplication G is a loop, and H and K are subloops of G. The conditions of Definition 2.2 hold, so G is an internal semi-direct product of H and K. This loop happens to be a Bol loop, and whenever R is not associative, it is not Moufang. Furthermore, if R is an alternative division ring with Char $R \neq 2$, it has been shown (see D.A. Robinson [8, 10]) that every loop isotopic to G is in fact isomorphic to G.

EXAMPLE 2.7. Let H be a subgroup of the automorphism group of a loop K. Then H is a subgroup of $\operatorname{Sym} K$. Let $\theta : H \to \operatorname{Sym} K$ be given as the "insertion" map: $\theta(h) = h$ for all $h \in H$. It is easy to see that θ satisfies the requirements of Construction C, so we can form the external semi-direct product $H \times_\theta K$ of H and K with respect to θ. This loop is recognized as the H-holomorph of K. Its origins in group theory are familiar. In its loop-theoretic context see, for instance, R.H. Bruck [1] and D.A. Robinson [9].

EXAMPLE 2.8. We turn now to a class of loops constructed by Karl Robinson [12]. For each positive integer k let C_k be the finite cyclic group of order k. For any integer $n \geq 2$ and any odd integer m with $m > 1$ let $H = C_2 \times C_2 \times \ldots \times C_2$ be the elementary Abelian 2-group of order 2^n and let $K = C_m$. Enumerate the members of H as $x_1, x_2, x_3, \ldots, x_{2^n}$ with $x_1 = e_H$ and $x_2 x_3 = x_4$. For each $s = 4, 5, \ldots, 2^n$ define $\theta_s : H \to \operatorname{Sym} K$ by

$$\theta_s(x_i) = \begin{cases} \tau & \text{if } 1 < i \leq s \\ \text{identity element of} \operatorname{Sym} K & \text{otherwise} \end{cases}$$

where $k\tau = k^{-1}$ for all $k \in K$. Then each $\theta_s(x_i)$ is indeed an automorphism of K, and for each $s = 4, 5, \ldots, 2^n$, we can form the semi-direct product $H \times_{\theta_s} K$. Karl Robinson [12] denotes such a loop by $B(2^n, s, m)$, shows that each such loop is a Bol loop which is not Moufang, and proves that for each such m and n these $2^n - 3$ Bol loops of order $2^n m$ are pair-wise non-isomorphic. This construction can be generalized, as we shall see in §4.

3. Bol Loops

Let G be a loop. Recall that an ordered triple $A = \langle \alpha, \beta, \gamma \rangle$ where $\alpha, \beta, \gamma \in \operatorname{Sym} G$ is an autotopism of G provided that $x\alpha \cdot y\beta = (xy)\gamma$ for all $x, y \in G$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be in $\operatorname{Sym} G$. If $A = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $B = \langle \alpha_2, \beta_2, \gamma_2 \rangle$, we define AB by $AB = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \rangle$ and note that if any two of A, B and AB are autotopisms of G so is the third. Recall from §1 that G is a Bol loop provided that $(xy \cdot z)y = (x(yz \cdot y)$ for all $x, y, z \in G$. Basic facts concerning Bol loops can be found in [7], where it is shown that

LEMMA 3.1. *A loop G is a Bol loop if and only if*

$$A(x) = \langle R(x)^{-1}, L(x)R(x), R(x) \rangle$$

is an autotopism of G for all $x \in G$.

Now we approach the problem of determining precisely when semi-direct products of loops produce Bol loops.

LEMMA 3.2. *Let G, H, and K be loops such that $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$. Then G is a Bol loop if and only if*

(i) *H is a Bol loop, and*
(ii) *the equation*

(3.1) $[(k_1\theta(h_2)\cdot k_2)\theta(h_3)\cdot k_3]\theta(h_2)\cdot k_2 = k_1\theta(h_2h_3\cdot h_2)[(k_2\theta(h_3)\cdot k_3)\theta(h_2)\cdot k_2]$

holds for all $h_2, h_3 \in H$, *all* $k_1, k_2, k_3 \in K$.

Proof. Let $x = (h_1, k_1)$, $y = (h_2, k_2)$, and $z = (h_3, k_3)$ be in G. Now use (2.1) to exhibit $(xy \cdot z)y$ and $x(yz \cdot y)$. Comparison of the first and second "components" gives (i) and (ii) respectively. ■

LEMMA 3.3. *Let* G, H *and* K *be loops such that* $G = H \times_\theta K$ *for some* $\theta : H \to \operatorname{Sym} K$. *If* $\theta(x)$ *is an automorphism of* K *for all* $x \in H$ *and if* $\theta(xy \cdot x) = \theta(x)\theta(y)\theta(x)$ *for all* $x, y \in H$, *then* (*3.1*) *holding for all* $h_2, h_3 \in H$, *all* $k_1, k_2, k_3 \in K$ *is equivalent to*

$$B(k, x, y) = \langle R(k)^{-1}, L(k)R(k\theta(x)\theta(y)), R(k\theta(x)\theta(y))\rangle$$

being an autotopism of K *for all* $x, y \in H$, *all* $k \in K$.

Proof. Since G is a loop, we have $e_K\theta(h) = e_K$ and $k\theta(e_H) = k$ for all $h \in H$, all $k \in K$. From the hypothesis we have

$$\theta(x) = \theta(xx^\rho \cdot x) = \theta(x)\theta(x^\rho)\theta(x)$$

for all $x \in H$ where x^ρ is the right inverse of x. So

$$\theta(x)^{-1} = \theta(x^\rho) \tag{3.2}$$

for all $x \in H$. Now note that (3.1) holds for all $h_2, h_3 \in H$, all $k_1, k_2, k_3 \in K$ if and only if

$$\begin{aligned}(3.3)\quad [(k_1\theta(h_2)\theta(h_3)\theta(h_2) \cdot k_2\theta(h_3)\theta(h_2)) \cdot k_3\theta(h_2)]k_2 \\ = k_1\theta(h_2)\theta(h_3)\theta(h_2 \cdot [(k_2\theta(h_3)\theta(h_2) \cdot k_3\theta(h_2)) \cdot k_2]\end{aligned}$$

for all $h_2, h_3 \in H$, all $k_1, k_2, k_3 \in K$. But (3.3) holds all $h_2, h_3 \in H$, all $k_1, k_2, k_3 \in K$ if and only if

$$(uv \cdot w)v\theta(r)^{-1}\theta(s)^{-1} = u(vw \cdot v\theta(r)^{-1}\theta(s)^{-1}) \tag{3.4}$$

for all $r, s \in H$, all $u, v, w \in K$. Using (3.2), replacing r and s by x^λ and y^λ respectively (λ here indicates left inverse) and writing k for v, we see that (3.4) holds for all $r, s \in H$, all $u, v, w \in K$ if and only if

$$(uk \cdot w) \cdot k\theta(x)\theta(y) = u[kw \cdot k\theta(x)\theta(y)] \tag{3.5}$$

for all $x, y \in H$, all $u, k, w \in K$. Then merely replacing u by $uR(k)^{-1}$, we see that (3.5) holds for all $x, y \in H$, all $u, k, w \in K$ if and only if

$$uR(k)^{-1} \cdot [wL(k)R(k\theta(x)\theta(y))] = (uw)R(k\theta(x)\theta(y)) \tag{3.6}$$

for all $x, y \in H$, all $u, w, k \in K$. But this is equivalent to $B(k, x, y)$ being an autotopism of K for all $x, y \in H$, all $k \in K$, and our proof is complete. ■

LEMMA 3.4. *Let G, H, and K be loops such that $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$. Then G is a Bol loop if and only if*

(i) *H is a Bol loop,*

(ii) *$\theta(x)$ is an automorphism of K for all $x \in H$,*

(iii) *$\theta(xy \cdot x) = \theta(x)\theta(y)\theta(x)$ for all $x, y \in H$,*

(iv) *$B(k,x,y) = \langle R(k)^{-1}, L(k)R(k\theta(x)\theta(y)), R(k\theta(x)\theta(y))\rangle$ is an autotopism of K for all $x, y \in H$, all $k \in K$.*

Proof. In view of Lemmas 3.2 and 3.3 we need only show that (ii) and (iii) hold whenever G is a Bol loop. So let us assume that G is a Bol loop. By Lemma 3.2 we know that (3.1) holds for all $h_2, h_3 \in H$, all $k_1, k_2, k_3 \in K$. Setting $h_3 = e_H$ and $k_2 = e_K$ in (3.1), we obtain

$$(k_1\theta(h_2) \cdot k_3)\theta(h_2) = k_1\theta(h_2^2) \cdot (k_3\theta(h_2)) \tag{3.7}$$

for all $h_2 \in H$, all $k_1, k_3 \in K$. Setting $k_3 = e_K$ in (3.7), we get $k_1\theta(h_2)^2 = k_1\theta(h_2^2)$ for all $h_2 \in H$, all $k_1 \in K$. Using this in (3.7) and replacing $k_1\theta(h_2)$ by k, we get

$$(kk_3)\theta(h_2) = k\theta(h_2) \cdot k_3\theta(h_2)$$

for all $h_2 \in H$, all $k, k_3 \in K$. So (ii) holds.

Finally, letting $k_2 = k_3 = e_K$ in (3.1), we get

$$k_1\theta(h_2)\theta(h_3)\theta(h_2) = k_1\theta(h_2h_3 \cdot h_2)$$

for all $h_2, h_3 \in H$, all $k_1 \in K$. So (iii) holds, and our proof of Lemma 3.4 is complete.

LEMMA 3.5. *Let K be a Bol loop. If $a, b \in K$ and if $ab \in N_\rho(K)$, the right nucleus of K, then $k \cdot ab = ka \cdot b$ for all $k \in K$.*

Proof. Since K is a (right) Bol loop (and more generally for any loop satisfying the right inverse property) we have $N_\rho(K) = N_\mu(K)$, the middle nucleus of K. So we have

$$k \cdot ab = [(k \cdot ab)b^{-1}]b = [k(ab \cdot b^{-1})]b = ka \cdot b$$

for all $k \in K$. ∎

DEFINITION 3 .6. An automorphism θ of a Bol loop K is *right nuclear* means that $y^{-1} \cdot y^\theta \in N_\rho(K)$ for all $y \in K$, where $N_\rho(K)$ again is the right nucleus of K.

THEOREM 3.7. *Let G, H and K be loops such that $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$. Then G is a Bol loop if and only if*

(i) *H and K are Bol loops,*

(ii) *$\theta(x)$ is a right nuclear automorphism of K for each $x \in H$,*

(iii) *$\theta(xy \cdot x) = \theta(x)\theta(y)\theta(x)$ for all $x, y \in H$.*

Proof. First, let G be a Bol loop. Then H and K are Bol loops, since they are isomorphic to subloops of G. So, by Lemma 3.4, we see that (i) and (iii) hold, that $\theta(x)$ is an automorphism of K for all $x \in H$, and that $B(k,x,y)$ of Lemma 3.3 is an autotopism of K for all $x, y \in H$, all $k \in K$. Then, in particular, $BB(k, e_H, e_H)$ and $B(k, x, e_H)$ are autotopisms of K for all $x \in H$, all $k \in K$. But then note that

$$B(k, e_H, e_H)\langle I, R((k)^{-1}R(k\theta(x)), R(k)^{-1}R(k\theta(x))\rangle = B(k, x, e_H)$$

for all $x \in H$, all $k \in K$, where I is the identity map on K. So

$$\langle I, R(k)^{-1}R(k\theta(x)), R(k)^{-1}R(k\theta(x))\rangle \tag{3.8}$$

must be an autotopism of K for all $x \in H$, all $k \in K$. So we have

$$a \cdot bR(k)^{-1}R(k\theta(x)) = (ab)R(k)^{-1}R(k\theta(x))$$

for all $x \in H$, all $a, b, k \in K$. Setting $b = e_K$, we get

$$R(k^{-1} \cdot k\theta(x)) = R(k)^{-1}R(k\theta(x))$$

for all $x \in H$, all $k \in K$. So autotopism (3.8) above becomes

$$\langle I, R(k^{-1} \cdot k\theta(x)), R(k^{-1} \cdot k\theta(x))\rangle$$

for all $x \in H$, all $k \in K$. So $k^{-1} \cdot k\theta(x) \in N_\rho(K)$ for all $x \in H$, all $k \in K$. So $\theta(x)$ is indeed right nuclear for each $x \in H$, and (ii) holds.

Conversely, assume now that (i), (ii), and (iii) hold. Then we see that $k^{-1} \cdot k\theta(x) \in N_\rho(k)$ and $(k\theta(x))^{-1} \cdot k\theta(x)\theta(y) \in N_\rho(K)$ for all $x, y \in H$, all $k \in K$. Since K is a Bol loop, we can use Lemma 3.5 to get

$$\begin{aligned}(k^{-1} \cdot k\theta(x))&[(k\theta(x))^{-1} \cdot k\theta(x)\theta(y)] \\ &= [(k^{-1} \cdot k\theta(x)) \cdot (k\theta(x))^{-1}] \cdot (k\theta(x)\theta(y)) = k^{-1} \cdot k\theta(x)\theta(y)\end{aligned}$$

for all $x \in H$, all $k \in K$. But since $k^{-1} \cdot k\theta(x)\theta(y)$ has just been shown to be a product of two elements in $N_\rho(K)$, it is in $N_\rho(K)$ too. Thus

$$\langle I, R(k^{-1} \cdot k\theta(x)\theta(y)), R(k^{-1} \cdot k\theta(x)\theta(y))\rangle \tag{3.9}$$

is an autotopism of K for all $x, y \in H$, all $k \in K$. Since K is a Bol loop, we recall that $N_\rho(K) = N_\mu(K)$. So we see that $k^{-1} \cdot k\theta(x)\theta(y)$ is also in $N_\mu(K)$ for all $x, y \in H$, all $k \in K$. Hence, we have

$$[a(k^{-1} \cdot k\theta(x)\theta(y)] \cdot (k\theta(x)\theta(y))^{-1} = a[(k^{-1} \cdot k\theta(x)\theta(y)) \cdot (k\theta(x)\theta(y))^{-1}]$$

for all $x, y \in H$, all $a, k \in K$. But K satisfies the right inverse property, so we see that

$$[a(k^{-1} \cdot k\theta(x)\theta(y))] \cdot (k\theta(x)\theta(y)^{-1} = ak^{-1}$$

for all $x, y \in H$, all $a, k \in K$. Using the right inverse property once again, we get

$$a(k^{-1} \cdot k\theta(x)\theta(y)) = (ak^{-1}) \cdot (k\theta(x)\theta(y))$$

for all $x, y \in H$, all $a, k \in K$. So autotopism (3.9) can be rewritten as

$$(3.10) \qquad \langle I, R(k^{-1})R(k\theta(x)\theta(y)), R(k^{-1})R(k\theta(x)\theta_y))\rangle$$

for all $x, y \in H$, all $k \in K$. But recall that $R(k^{-1}) = R(k)^{-1}$ and that by Lemma 3.1

$$(3.11) \qquad \langle R(k)^{-1}, L(k)R(k), R(k)\rangle$$

is an autotopism of K for all $k \in K$. So multiplying autotopisms (3.11) and (3.10), we see that $B(k, x, y)$ is an autotopism of K for all $x, y \in H$, all $k \in K$. So, by Lemma 3.4, G is a Bol loop and our proof is complete. ∎

The corollaries which follow are immediate consequences of Theorem 3.7, since Moufang loops — those loops which satisfy the identity $(xy \cdot z)y = x(y \cdot zy)$ — are necessarily Bol loops. In fact, Moufang loops are seen to be precisely those Bol loops that are di-associative.

COROLLARY 3.7.1. *Let H be a subgroup of the automorphism group of a loop K. Then the H holomorph of K (see Example 2.7) is a Bol loop if and only if K is a Bol loop and each $h \in H$ is right nuclear.*

COROLLARY 3.7.2. *Let G, H and K be loops with $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$. Then G is a Moufang loop if and only if*

(i) *H and K are Moufang loops,*

(ii) *$\theta(x)$ is a right nuclear automorphism of K for each $x \in H$,*

(iii) *$\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in H$.*

COROLLARY 3.7.3. *Let G, H and K be loops with $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$. Then G is a group if and only if*

(i) *H and K are groups,*

(ii) *$\theta(x)$ is an automorphism of K for each $x \in H$,*

(iii) *$\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in H$.*

COROLLARY 3.7.4. (Karl Robinson [12]). *Let G, H and K be loops with $G = H \times_\theta K$ for some $\theta : H \to \operatorname{Sym} K$ and assume furthermore that H and K are groups. Then G is a Bol loop which is not Moufang if and only if*

(i) *$\theta(x)$ is an automorphism of K for each $x \in H$,*

(ii) *$\theta(xyx) = \theta(x)\theta(y)\theta(x)$ for all $x, y \in H$,*

(iii) *$\theta(uv) \neq \theta(u)\theta(v)$ for some $u, v \in H$.*

(It was shown in Example 2.7 how H, K and θ can be selected to satisfy the conditions of the preceding corollary. The loop $H \times_\theta K$ which results is also mentioned in [4, p. 40]. Incidentally, this type of construction was recently employed by Karl Robinson and one of the authors (see [11]) to show that the nucleus of a Bol loop need not be normal.)

In order to use semi-direct products of Bol loops to produce new Bol loops — especially ones which are not Moufang — close attention must be paid to conditions (ii) and (iii) of Theorem 3.7. We address our concerns with a few remarks and results.

Remark 3.8. Let H and K be groups with H cyclic. Suppose that a semi-direct product $H \times_\theta K$ with respect to some $\theta : H \to \operatorname{Sym} K$ is a Bol loop. Consulting Theorem 3.7, we see that $\theta(xyx) = \theta(x)\theta(y)\theta(x)$ for all $x, y \in H$. This condition, together with the requirement from §2 that $\theta(e_H) = e_{\operatorname{Sym} K}$, allows one to prove that $\theta(x^n) = \theta(x)^n$ for all $x \in H$, all integers n. But now let a be any generator for the cyclic group H. For $x, y \in H$, we have $x = a^m$ and $y = a^n$ for some integers m and n, so $\theta(xy) = \theta(a^m a^n) = \theta(a^{m+n}) = \theta(a)^{m+n} = \theta(a)^m\theta(a)^n = \theta(a^m)\theta(a^n) = \theta(x)\theta(y)$. From Corollary 3.7.3, we see now that $H \times_\theta K$ must also be a group. In other words, to use a semi-direct product of groups H and K to produce a Bol loop which is not a group (and for that matter not Moufang) we must make certain that the group H is not cyclic. (Note that in Example 2.8 the group H is not cyclic, but rather an elementary Abelian 2-group).

Remark 3.9. To utilize Theorem 3.7 we must have at our disposal right nuclear automorphisms of Bol loops. If G is an extra loop, in that $(xy \cdot z)x = x(y \cdot zx)$ for all $x, y, z \in G$, then G is a Bol loop (in fact, it is Moufang) and all of its inner maps $R(x)R(y)R(xy)^{-1}$, all $x, y \in G$, are right nuclear automorphisms of G (see D.A. Robinson [9]).

LEMMA 3.10. *Let G be a Bol loop. If $a, b \in G$ with $ab \in N_\rho(G)$, then $a^{-1} \cdot ab = b$ and $b^{-1}a^{-1} = (ab)^{-1}$.*

Proof. Since G satisfies the right inverse property, for each $u \in G$, we recall that u^{-1} is that element in G so that $u^{-1}u = uu^{-1} = e_G$. From Lemma 3.5 we have $x \cdot ab = xa \cdot b$ for all $x \in G$. Setting $x = a^{-1}$, we get $a^{-1} \cdot ab = b$; setting $x = (ab)^{-1}$ and using the right inverse property twice, we get $v^{-1}a^{-1} = (ab)^{-1}$. ∎

THEOREM 3.11. *Let G be a Bol loop. If α, β and γ are automorphisms of G with α and β right nuclear, then $\alpha\beta, \alpha^{-1}$ and $\gamma^{-1}\alpha\gamma$ are right nuclear automorphisms of G.*

Proof. Clearly $\alpha\beta, \alpha^{-1}$ and $\gamma^{-1}\alpha\gamma$ are automorphisms of G. We need only establish that they are right nuclear.

Since α and β are right nuclear and since $N_\rho(G)$ is group, it follows that $y^{-1} \cdot y\alpha \in N_\rho(G)$, $(y\alpha)^{-1} \cdot (y\alpha)\beta \in N_\rho(G)$ and also $(y^{-1} \cdot y\alpha) \cdot [(y\alpha)^{-1} \cdot (y\alpha)\beta] \in N_\rho(G)$ for all $y \in G$. Now using Lemma 3.10, we get $y^{-1} \cdot y\alpha\beta = y^{-1} \cdot (y\alpha)\beta = y^{-1}[(y\alpha) \cdot ((y\alpha)^{-1} \cdot (y\alpha)\beta)] = (y^{-1} \cdot y\alpha) \cdot ((y\alpha)^{-1} \cdot (y\alpha)\beta)$. So we have $y^{-1} \cdot y\alpha\beta \in N_\rho(G)$ for all $y \in G$. Thus $\alpha\beta$ is right nuclear.

Since α is right nuclear and $(y\alpha^{-1})^{-1} \cdot y = (y\alpha^{-1})^{-1} \cdot (y\alpha^{-1})\alpha$, it follows that $(y\alpha^{-1})^{-1} \cdot y \in N_\rho(G)$ for all $y \in G$. But then too we have $((y\alpha^{-1})^{-1} \cdot y)^{-1} \in N_\rho(G)$. By Lemma 3.10 we have $((y\alpha^{-1})^{-1} \cdot y)^{-1} = y^{-1} \cdot y\alpha^{-1}$. So $y^{-1} \cdot y\alpha^{-1} \in N_\rho(G)$ for all $y \in G$. Thus α^{-1} is right nuclear.

It is easy to show that $u\gamma \in N_\rho(G)$ whenever $u \in N_\rho(G)$. So with α right nuclear we note that $(y^{-1}\gamma^{-1} \cdot y\gamma^{-1}\alpha)\gamma \in N_\rho(G)$ for all $y \in G$. But we have $(y^{-1}\gamma^{-1} \cdot y\gamma^{-1}\alpha)\gamma = y^{-1} \cdot y\gamma^{-1}\alpha\gamma$, so $y^{-1} \cdot y\gamma^{-1}\alpha\gamma \in N_\rho(G)$ for all $y \in G$, and $\gamma^{-1}\alpha\gamma$ is right nuclear. This completes our proof. ∎

Since the identity automorphism is right nuclear, Theorem 3.11 has some immediate consequences.

COROLLARY 3.11.1. *Let G be a Bol loop. The set of all right nuclear automorphisms of G is a normal subgroup of the automorphism group of G.*

COROLLARY 3.11.2. *Let G be a Bol loop. If S is a set of right nuclear automorphisms of G, then the subgroup $\langle S\rangle$ of the automorphism group of G generated by S is a group of right nuclear automorphisms of G.*

COROLLARY 3.11.3. *Let G be a Bol loop. Then G has a non-trivial group of right nuclear automorphisms if and only if G has at least two right nuclear automorphisms (i.e., at least one other than the identity map).*

We now show that many Bol loops (in addition to the extra loops mentioned in Remark 3.9) do have non-trivial right nuclear automorphisms. This becomes crucial in §4, where some applications are discussed.

Recall that an element $\alpha \in \operatorname{Sym} G$ for any loop G is a *pseudo-automorphism* of G provided that $\langle \alpha, \alpha R(a), \alpha R(a)\rangle$ is an autotopism for some $a \in G$. Such an element a is called a *companion* of the pseudo-automorphism α. For all $x, y \in G$, let $R(x,y)$ be the inner map of G given by $R(x,y) = R(x)R(y)R(xy)^{-1}$ and let x^{-1} be such that $xx^{-1} = e_G$. Now let $[x,y]$ be defined by $[x,y] = (x^{-1}y^{-1})(xy)$ for all $x, y \in G$. Note that for any right inverse property loop G we have $xx^{-1} = x^{-1}x = e_G$ and $x^{-1}y^{-1} = [x,y](xy)^{-1}$ for all $x, y \in G$. So for a right inverse property loop the elements $[x,y]$ indicate to what extent G departs from satisfying the automorphic inverse property.

THEOREM 3.12. *If G is a Bol loop, then $R(x,y)$ is a pseudo-automorphism of G with companion $[x,y]$.*

Proof. Since G is a Bol loop, we know that $A(x)$ given by $A(x) = \langle R(x)^{-1}, L(x)R(x), R(x)\rangle$ is an autotopism of G for each $x \in G$ (see Lemma 3.1). So corresponding to each pair of elements x and y of G there is a map $\beta \in \operatorname{Sym} G$ so that $B(x,y) = A(x)^{-1}A(y)^{-1}A(xy) = \langle R(x)R(y)R(xy)^{-1}, \beta, R(x)^{-1}R(y)^{-1}R(xy)\rangle$ is an autotopism of G. So

$$aR(x)R(y)R(xy)^{-1} \cdot b\beta = (ab)R(x)^{-1}R(y)^{-1}R(xy) \tag{3.12}$$

for all $a, b \in G$. Setting $a = e_G$ in (3.12), we see that $\beta = R(x)^{-1}R(y)^{-1}R(xy)$; setting $b = e_G$ in (3.12), we get $R(x,y)R(e_G\beta) = R(x)^{-1}R(y)^{-1}R(xy)$. Thus

$$B(x,y) = \langle R(x,y), R(x,y)R(e_G\beta), R(x,y)R(e_G\beta)\rangle$$

and $e_G\beta = (x^{-1}y^{-1})(xy) = [x,y]$. Our proof is complete. ∎

COROLLARY 3.12.1. *Let G be a Bol loop. Then $R(x,y)$ is an automorphism of G if and only if $[x,y] \in N_\rho(G)$.*

Recall that a Bol loop G is a Bruck loop provided that $x \to x^2$ is a permutation of G and that $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$. So for all such loops $[x,y] = e_G$ for all $x, y \in G$. Thus, with all $[x,y] \in N_\rho(G)$, we have the following

COROLLARY 3.12.2 (G. Glauberman [5]). *If G is a Bruck loop, then $R(x,y)$ is an automorphism of G for all $x, y \in G$.*

COROLLARY 3.12.3. *Bol loops which are not groups have non-trivial pseudo-automorphisms.*

COROLLARY 3.12.4. *Bruck loops which are not groups have non-trivial automorphisms.*

One can verify that the Bol loop G given in Example 2.6 has the interesting feature that $[x,y] \in N_\rho(G)$ for all $x, y \in G$. So its inner maps $R(x,y)$ are automorphisms of G for all $x, y \in G$.

The smallest Bol loops which are not groups have order 8. There are six of them (see R.P. Burn [3]), and they all have non-trivial right nuclear automorphisms.

4. Some applications

It is our intention here to exploit Theorem 3.7 to construct some new Bol loops.

A. Let us return to the semi-direct product discussed in Example 2.8. It is easy to see that each of the θ_s, given there, for $s = 4, 5, \ldots, 2^n$ (recall that n is an integer with $n > 1$), satisfies the conditions of Theorem 3.7. Thus, the loops $B(2^n, s, m)$ (recall that m is an odd integer with $m > 1$) for $s = 4, 5, \ldots, 2^n$ are indeed Bol loops of order $2^n m$ and, since $\theta_s(x_2x_3) \neq \theta_s(x_2)\theta_s(x_3)$, none are Moufang. As Karl Robinson [12] points out, there are precisely $2^n + (m-1)(s-1)$ elements x in $B(2^n, s, m)$ such that $x = x^{-1}$, and so the $2^n - 3$ loops obtained are pair-wise non-isomorphic.

In the very same context as that of Example 2.8 (recall that the members of $H = C_2 \times C_2 \times \ldots \times C_2$ are given as $x_1, x_2, \ldots, x_{2^n}$ with $x_1 = e_H$ and $x_2x_3 = x_4$) we now select θ_2 and, whenever $n > 2, \theta_3$ as follows

$$\theta_2(x_2) = \tau, \quad \theta_2(x_i) = I \qquad \text{for } i \neq 2;$$

$$\theta_3(x_2) = \theta_3(x_5) = \tau, \quad \theta_3(x_i) = I \qquad \text{for } i \neq 2, 5$$

where, as before, I is the identity map on K and $k\tau = k^{-1}$ for all $k \in K$.

It is easy to see that θ_2 and θ_3 both satisfy the conditions of Theorem 3.7. Thus, $H \times_{\theta_2} K$ and, provided $n > 2$, $H \times_{\theta_3} K$ are both Bol loops of order $2^n m$, denoted here by $B(2^n, 2, m)$ and $B(2^n, 3, m)$ respectively. Neither of the Bol loops is Moufang, because with $j = 2, 3$, we have $\theta_j(x_2 x_3) = \theta_j(x_4) = I$ whereas $\theta_j(x_2)\theta_j(x_3) = \tau$. Note that $B(2^n, 2, m)$ has exactly $2^n + (m - 1)$ elements x such that $x = x^{-1}$. So $B(2^n, 2, m)$ is not isomorphic to any of the Bol loops of order $2^n m$ of Example 2.8. Furthermore, if $n > 2$, the loop $B(2^n, 3, m)$ has exactly $2^n + (s - 1)(m - 1)$ elements x such that $x = x^{-1}$. Thus, $B(2^n, 3, m)$ is not isomorphic to $B(2^n, 2, m)$, nor to any of the loops of order $2^n m$ of Example 2.8.

In conclusion, with these "new" Bol loops, we assert that *for any positive integers m, n with m odd, $m > 1$, and $n > 2$ there are at least $2^n - 1$ pair-wise non-isomorphic Bol loops of order $2^n m$ which are not Moufang and that for any odd integer $m > 1$ there are at least 2 non-isomorphic Bol loops of order $4m$ which are not Moufang.*

B.1. Of the six Bol loops of order 8 which are not groups (they are not Moufang either) there is just one of exponent 2 (see R.P. Burn [3]). It can be realized as follows (see D.A. Robinson [7, p. 346]): Let $R = \{0, 1\}$ be the ring of integers modulo 2, let $B = R \times R \times R$, and define multiplication for B by

$$(i, j, k) \cdot (p, q, r) = (i + p, j + q, k + r + jp(q + 1))$$

for all $(i, j, k), (p, q, r) \in B$. Then $e = e_B = (0, 0, 0)$ and $a = (0, 0, 1)$ are the only elements of B which are in all three nuclei of B and commute element-wise with B. The elements comprising B can be listed as

$$e = (0,0,0),\ a = (0,0,1),\ x_1 = (0,1,1),\ x_2 = (1,0,1),\ x_3 = (1,1,0),$$
$$ax_1 = (0,1,0),\ ax_2 = (1,0,0),\ ax_3 = (1,1,1)$$

and the multiplication table for B is as follows:

	e	a	x_1	x_2	x_3	ax_1	ax_2	ax_3
e	e	a	x_1	x_2	x_3	ax_1	ax_2	ax_3
a	a	e	ax_1	ax_2	ax_3	x_1	x_2	x_3
x_1	x_1	ax_1	e	ax_3	x_2	a	x_3	ax_2
x_2	x_2	ax_2	x_3	e	x_1	ax_3	a	ax_1
x_3	x_3	ax_3	x_2	ax_1	e	ax_2	x_1	a
ax_1	ax_1	x_1	a	x_3	ax_2	e	ax_3	x_2
ax_2	ax_2	x_2	ax_3	a	ax_1	x_3	e	x_1
ax_3	ax_3	x_3	ax_2	x_1	a	x_2	ax_1	e

Now define

$$(i,j,k)\alpha = (i,j,i+j+k)$$

for all $(i,j,k) \in B$. Then α is a right nuclear automorphism of B of order 2, fixing each of e, a, x_3, and ax_3, interchanging x_1 and ax_1, and interchanging x_2 and ax_2.

[Remark .Let $(i,j,k)\beta = (i,j,j+k)$ for all $(i,j,k) \in B$. Then β is also a right nuclear automorphism of B. Actually, the automorphism group, Aut B, of B is the elementary Abelian 2-group of order 8. Although for each $\theta \in$ Aut B and each $y \in B, y^{-1} \cdot y\theta$ is in at least one of the three nuclei of B, only those automorphisms in the subgroup generated by α and β are right nuclear.]

B.2. Let B and α be the same as in Part B.1 above. For each positive even integer m let $H = C_m$ be the cyclic group of order m, and let c be a generator of H. Now let $K = B$, and define $\theta(c^i) = \alpha^i$ for all integers i. Then with m even we see that θ is well defined. It follows that $\theta : H \to \operatorname{Aut} K$ and that $\theta(x)$ is a right nuclear automorphism of K for each $x \in H$. For all integers i and j note that $\theta(c^i c^j \cdot c^i) = \theta(c^{2i+j}) = \alpha^{2i+j} = \alpha^i \alpha^j \alpha^i = \theta(c^i)\theta(c^j)\theta(c^i)$. Since the conditions of Theorem 3.7 hold, it follows that $H \times_\theta K$ is a Bol loop of order $8m$ with a subgroup isomorphic to the Bol loop B. Since B is not Moufang, neither is $H \times_\theta K$. It is clear then that *for each integer $r > 3$ and each positive integer n there is a Bol loop of order $2^r n$ with a subloop isomorphic to the non-Moufang Bol loop B of exponent 2 and order 8 described in B.1.*

The loop B can be generated by two elements x and y such that $y \neq x \cdot xy$ (for instance, take $x = x_1$ and $y = x_2$ in Part B.1 above). An examination of the elements of order 2 in the Bol loops of Example 2.8 (see also Part A of the present section) shows that they cannot accomodate elements x and y with $y \neq x \cdot xy$ and so they have no subloop isomorphic to B. This guarantees that the loops just constructed have not already occurred in Example 2.8.

B.3. Now let H be the Bol loop B of Part B.1, and let K be any Bol loop which has a right nuclear automorphicm γ of order 2. For each integer $i = 1,2,3,5,6,7$ ($i = 4$ is intentionally omitted for reasons explained later) we select $\theta_i : H \to \operatorname{Aut} K$ as follows:

$$\theta_1(x) = \begin{cases} \gamma & \text{if } x = a \\ I & \text{otherwise} \end{cases}$$

$$\theta_2(x) = \begin{cases} \gamma & \text{if } x = x_1 \text{ or } ax_1 \\ I & \text{otherwise} \end{cases}$$

$$\theta_3(x) = \begin{cases} \gamma & \text{if } x = a, x_1, \text{ or } ax_1 \\ I & \text{otherwise} \end{cases}$$

$$\theta_5(x) = \begin{cases} \gamma & \text{if } x = a, x_1, ax_1, x_2, \text{ or } ax_2 \\ I & \text{otherwise} \end{cases}$$

$$\theta_6(x) = \begin{cases} \gamma & \text{if } x = x_1, ax_1, x_2, ax_2, x_3, \text{ or } ax_3 \\ I & \text{otherwise} \end{cases}$$

$$\theta_7(x) = \begin{cases} \gamma & \text{if } x \neq e_H \\ I & \text{otherwise} \end{cases}$$

where I is the identity automorphism on K.

As noted B.1 the element a is the only member of B other than e_B which is in all three nuclei of B and which commutes element-wise with B, that is, $\{e_B, a\}$ is the center of B. A glance at the multiplication table of B in B.1 reveals $uv = vu$ or $uv = a \cdot vu$ whenever u and v are members of B. Thus, $\{e_B, a\}$ is also the commutator subloop of B. Recall too that B has exponent 2, is a (right) Bol loop, and is right alternative. Now let θ be any one of the θ_i for $i = 1, 2, 3, 5, 6, 7$ defined above, and note that $\theta(ay) = \theta(y)$ for all $y \in B$ with $y \neq e_B$ and $y \neq a$. With this information it is easy to examine $\theta(xy \cdot x)$ for all $x, y \in B$. If $xy = yx$, then we have $\theta(xy \cdot x) = \theta(yx \cdot x) = \theta(yx^2) = \theta(y) = \theta(x)^2\theta(y) = \theta(x)\theta(y)\theta(x)$; if $xy \neq yx$, we have $\theta(xy \cdot x) = \theta(yx \cdot a)x) = \theta(y(xa \cdot x)) = \theta(a \cdot yx^2) = \theta(ay) = \theta(y) = \theta(x)^2\theta(y) = \theta(x)\theta(y)\theta(x)$. Thus, $\theta(xy \cdot x) = \theta(x)\theta(y)\theta(x)$ for all $x, y \in B$. Since the conditions of Theorem 3.7 are satisfied, the six loops $H \times_{\theta_i} K$ are Bol loops. None are Moufang, since each contains a subloop isomorphic to B.

Note that each θ_i above maps i members of B to γ, and the remaining $8 - i$ to I. Since it does not seem possible to select a map θ_4 which maps exactly 4 members of B to γ while satisfying the conditions of Theorem 3.7, a map θ_4 is conspicuously missing above.

As a special case, one can take $K = C_m$, the cyclic group of order m, where m is odd with $m > 1$, and then select γ by $k\gamma = k^{-1}$ for all $k \in K$. For each $i = 1, 2, 3, 5, 6, 7$, the resulting Bol loop $B \times_{\theta_i} K$ of order $8m$ has precisely $8 + i(m-1)$ elements x such that $x^{-1} = x$. Hence, the six loops are pair-wise non-isomorphic. Since each of these loops has a subloop isomorphic to B and those of Example 2.8 do not (recall an argument sketched in B.2 above), these Bol loops cannot be isomorphic to any loops in Example 2.8. Furthermore, comparing the orders of these six Bol loops with those of B.2, we see that these are not isomorphic to any of those constructed in B.2.

Replacing B above by $B \times G$ where $G = C_2 \times C_2 \times \ldots \times C_2$ is the elementary Abelian 2-group of order 2^{n-3} for any integer $n > 3$, but keeping $K = C_m$ and γ as above, we can define maps $\alpha_j : G \to \{I, \gamma\}$ in various ways to guarantee that the loops $(B \times G) \times_{\theta_{ij}} C_m$, with $\theta_{ij}(b, g) = \theta_i(b)\alpha_j(g)$ for all $(b, g) \in B \times G$, are Bol loops of order $2^n m$ which are not Moufang. But now questions concerning which of the loops are non-isomorphic cannot be answered by merely determining the number of solutions to $x^{-1} = x$, and seem difficult to resolve.

As we have seen in this section, the Bol loop B of Part B.1 is well suited to this semi-direct product technique for constructing new Bol loops. It may be profitable to examine the remaining five Bol loops of order 8 which are not groups to see whether or not they too can be exploited in a similar manner.

5. Epilogue

There are, of course, many interesting constructions of quasigroups and loops throughout the literature which are not semi-direct products. One of the best surveys of such constructions is due to O. Chein (see Chapter II of O. Chein, H.O. Pflugfelder, J.D.H. Smith [4]).

References

[1] R. H. Bruck, *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.

[2] R. H. Bruck, *A survey of binary systems*, Ergeb. Math. Grenzgeb., vol. 20, Springer Verlag, 1958.

[3] R. P. Burn, *Finite Bol loops*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 377–385.

[4] O. Chein, H. O. Pflugfelder and J. D. H. Smith (editors), *Quasigroups and loops: theory and applications*, Heldermann Verlag, Berlin 1990.

[5] G. Glauberman, *On loops of odd order*, J. Algebra **1** (1964), 374–396.

[6] H. O. Pflugfelder, *Quasigroups and loops: Introduction*, Heldermann Verlag, Berlin, 1990.

[7] D. A. Robinson, *Bol loops*, Trans. Amer. Math. Soc. **123** (1966), 341–354.

[8] D. A. Robinson, *A Bol loop isomorphic to all loop isotopes*, Proc. Amer. Math. Soc. **19** (1968), 671–672.

[9] D. A. Robinson, *Holomorphy theory of extra loops*, Publ. Math. Debrecen **18** (1971), 59–64.

[10] D. A. Robinson, *Some Moufang G-loops in a matrix setting*, Comm. Algebra **17** (1989), 843–853.

[11] D. A. Robinson and Karl Robinson, *A class of Bol loops whose nuclei are not normal*, submitted.

[12] Karl Robinson, *A note on Bol loops of order* $2^n k$, Aequationes Math. **22** (1981), 302–306.

[13] J. J. Rotman, *An introduction to the theory of groups*, third ed., Allyn and Bacon, Inc., New York, 1984.

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