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STRONGNESS IN LATTICES

Dedicated to Professor Tadeusz Traczyk

Faigle [3] introduced the notion of a strong join-irreducible element in lattices of finite length. In this note we extend the concept of strongness to arbitrary lattices being not necessarily of finite length. Here we give a generalization of some results of papers [3] and [5].

1. Preliminaries

Let L be a lattice. If L contains a least or a greatest element, these elements will be denoted by 0 or 1, respectively. An element $u \in L - \{0\}$ is called join-irreducible iff, for all $a, b \in L$ $u = a \vee b$ implies $u = a$ or $u = b$. Denote by $J(L)$ the set of all join-irreducible elements of L. We say that a is covered by b and write $a \prec b$ if $a < b$ and if $a \leq c < b$ implies $c = a$ for all c. For two elements $a, b \in L (a \leq b)$ we define $[a, b] = \{c \in L : a \leq c \leq b\}$.

A lattice L is called strongly dually atomic if for any $a, b \in L$ with $a < b$ there is $p \in [a, b]$ such that $p \prec b$. (This notion is the dual of the concept "strongly atomic" as used in [2].) In a strongly dually atomic lattice L the unique lower cover of a join-irreducible element u is denoted by u'.

A lattice L is said to be consistent (see [4]) iff $a \in L$ and $u \in J(L)$ imply that $a \vee u \in J([a, a \vee u])$. We remark that in [1], p. 249, one finds a condition equivalent to dual consistence (see also [2], p. 53).

A complete lattice L is lower continuous, if

$$a \vee \bigwedge C = \bigwedge (a \vee c : c \in C)$$

for all $a \in L$ and for all chains C in L.

Now we introduce the concept of a strong lattice. For lattices of finite length the definition of strongness is given by Stern (see [8] or [9]) by the property

(S) $u \in J(L), a \in L$ and $u \leq a \vee u'$ imply $u \leq a$.

We extend the notion of strongness from lattices of finite length to arbitrary lattices. Namely, we say that a lattice L is strong if the following condition is satisfied:

(S') $u \in J(L), a, b \in L$ and $b < u \leq a \vee b$ imply $u \leq a$.

It is easy to see that in strongly dually atomic lattices (in particular: in lattices of finite length) properties (S), (S') and

(DO) $u \in J(L), a \in L$ and $u \nleq a$ imply $a \vee u' < a \vee u$

(given in [7], p. 68) are equivalent.

We remark that any atomistic lattice (in particular: each geometric lattice) is strong. (Indeed, each join-irreducible element of an atomistic lattice is an atom.)

Now we observe that any modular lattice is strong. Let L be a modular lattice and let $u \in J(L)$, $a, b \in L$ with $b < u \leq a \vee b$. By the modular law, $u = (a \wedge u) \vee b$. Since u is join-irreducible this implies $u = a \wedge u$, that is $u \leq a$ which means that L is strong.

Also, it is not difficult to give examples of lattices which are strong but neither modular nor atomistic (see Section 23 of [10]).

As a preparation for the next result we need the following

LEMMA 1. *Let L be a strong lattice and $c, d \in L$ with $c \prec d$. If $u \in J(L)$ and $b \in L$ are elements such that $b < u \leq d$ but $u \nleq c$, then $b \leq c$.*

Proof. Suppose that $b \nleq c$. We have $c \leq b \vee c \leq d$ and $c \prec d$. Then $u \leq d = b \vee c$ and strongness implies $u \leq c$, a contradiction. Therefore, $b \leq c$.

2. Results

The first major result is

THEOREM 1. *A semimodular lower continuous strongly dually atomic lattice is consistent iff it is strong.*

Proof. Let L be a semimodular lower continuous strongly dually atomic lattice. Assume first that L is consistent but not strong. Let a join-irreducible element $u \in J(L)$ be such that $u \leq a \vee u'$ and $u \nleq a$ for some $a \in L$. Thus the set $T := \{x \in L : u \leq x \vee u'$ and $u \nleq x\}$ is not empty. Let C be a chain in T. The lower continuity follows

$$u' \vee \bigwedge C = \bigwedge (c \vee u' : c \in C) \geq u .$$

Clearly, $u \not\leq \bigwedge C$. Therefore $\bigwedge C \in T$, and T contains a minimal element b, by Zorn's lemma. Since L is strongly dually atomic we may choose $p \in L$ with $p \prec b$. Observe that

$$(1) \qquad p \vee u' < p \vee u\,.$$

Indeed, if $p \vee u' = p \vee u$, then $p \in T$, contradicting the minimality of b.

Now we observe that $b \leq p \vee u'$ is not possible, since $b \leq p \vee u'$ would imply, $b \vee u' \leq p \vee u' < p \vee u \leq b \vee u'$, a contradiction. Since $p \prec b$ and $b \not\leq p \vee u'$ we get

$$b \wedge (p \vee u') = p \prec b\,.$$

Hence, by semimodularity we conclude that

$$p \vee u' \prec b \vee p \vee u' = b \vee u' = b \vee u\,.$$

Thus we have

$$p \vee u' < p \vee u \leq b \vee u \text{ and } p \vee u' \prec b \vee u\,.$$

Consequently,

$$(2) \qquad p \vee u = b \vee u\,,$$

and therefore $p \vee u = (p \vee u') \vee b$. Consistence implies that $p \vee u$ is a join-irreducible element of the sublattice $[p, 1]$. This together with (1) and (2) yields $b \vee u = p \vee u = b$, which contradicts the fact that $u \not\leq b$. It follows that L must be strong.

Conversely assume that L is strong but not consistent. This means that there exist $a \in L$ and $u \in J(L)$ with $a \vee u \notin J([a, 1])$. Thus there are two distinct elements $c_1, c_2 \in [a, a \vee u]$ which are covered by $a \vee u$. Since $u \not\leq c_1, c_2$, by Lemma 1 we get $u' \leq c_i$ for $i = 1, 2$. Thus we have $u' \leq u \wedge (c_1 \wedge c_2) \leq u$ and obviously $u \not\leq c_1 \wedge c_2$. Hence we obtain $u \wedge (c_1 \wedge c_2) = u' \prec u$. By semimodularity we conclude that

$$c_1 \wedge c_2 \prec (c_1 \wedge c_2) \vee u = a \vee u\,.$$

This is a contradiction since by our construction we have $c_1 \wedge c_2 < c_1 \prec a \vee u$.

Remark 1. The preceding theorem generalizes Theorem 27.1 of [10] (see also [3], p. 33 and [5], p. 125), since any lattice of finite length is lower continuous and strongly dually atomic.

As a preparation for the next result we recall the following concepts:

If a is an element of a complete lattice L, then a representation $a = \bigvee T$ with $T \subseteq J(L)$ is called a join-decomnposition of a. A join-decomposition $a = \bigvee T$ is irredundant if $\bigvee(T - \{t\}) \neq a$ for all $t \in T$.

We say that a complete lattice L has the Kuroš-Ore replacement property (KORP) for join-decompositions (see [10], p. 30) if each element of L has at least one irredundant join-decomposition, and whenever $a = \bigvee T = \bigvee R$ are two irredundant join-decompositions of an element $a \in L$, for each $t_0 \in T$ there exists an $r_0 \in R$ such that $a = \bigvee(T - \{t_0\}) \vee r_0$ is also an irredundant join decomposition.

G. Richter ([6], Theorem 11) has shown that a strongly dually atomic lower continuous lattice has the KORP for join-decompositions iff it is consistent. From this and Theorem 1 we obtain

THEOREM 2. *Let L be a semimodular, lower continuous, strongly dually atomic lattice. Then the following conditions are equivalent:*

(i) *L has the KORP for join-decompositions,*

(ii) *L is consistent,*

(iii) *L is strong.*

Remark 2. From this theorem we obtain Corollary 27.2 of [10] (see also [5], Theorem 4).

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