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MONOTONICITY THEOREMS FOR THE I-PROXIMAL LOCAL SYSTEM

In paper [1] it was proved that if a local system S fulfils the so-called condition (SD), then the monotonicity of a function f is implied by some conditions of (S)-limits and (S)-derivatives of f. In our paper we show that the local system which consists of complements of sets I-sparse at a point fulfils the condition (SD). We also formulate some monotonicity theorems which follow from this fact.

Throughout the paper, I denotes the ideal of meager sets and B the family of sets having the Baire property on the real line R. For any $A \subset R$ and $x \in R$, we put $A - x = \{a - x; a \in A\}$ and $x \cdot A = \{xa; a \in A\}$. By an n-subinterval of an interval $[a, b]$ we mean each interval $[a + \frac{i}{n}(b - a), a + \frac{i+1}{n}(b - a)]$ for $i = 0, 1, \ldots, n - 1$. By a B-measurable kernel of a set $A \subset R$ we mean a set $\widetilde{A} \subset A$ having the Baire property, such that $E \in I$ whenever $E \subset A \setminus \widetilde{A}$ and $E \in B$. It is known that every subset of the real line has a B-measurable kernel.

Let E be a set having the Baire property. We write $\underline{d}_I^+(E, x) = 0$ if there is a sequence $\{t_n\}$ of real numbers tending to infinity such that $\chi_{t_n(E-x)\cap[0,1]} \to 0$ I-a.e. (i.e. the set of points for which the convergence does not hold is of the first category). We say that a set $A \subset R$ is I-sparse on the right at a point x, if there exists a set $E \supset A$ having the Baire property, such that $\underline{d}_I^+(E \cup F, x) = 0$ for any $F \in B$ with $\underline{d}_I^+(F, x) = 0$. A set I-sparse on the left at x is defined similarly. We say that E is I-sparse at x if it is I-sparse on the right and on the left at this point. Conditions equivalent to those definitions can be found in [2].

THEOREM 1. *Let* $A \subset [0, 1)$ *and* $B = [0, 1) \setminus A$. *If*

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(i) $0 \in A$,
(ii) *B is I sparse on the right at each point of A,*
(iii) *A is I-sparse on the left at each point of B,*
then $B = \emptyset$.

Proof. First, we show that A has the Baire property. Let $\widetilde{A}$ be a B-measurable kernel of A and G_A an open set such that $\widetilde{A} \triangle G_A \in I$. For any $E \subset R$, put

$$\phi^+(E) = \{x; R \setminus E \text{ is } I\text{-sparse on the right at } x\}.$$

Then (ii) implies $\widetilde{A} \subset A \subset \phi^+(A) = \phi^+(\widetilde{A}) = \phi^+(G_A) \subset \overline{G}_A$. Therefore, as $\overline{G}_A \setminus \widetilde{A} \in I$, $A \in B$. Let $A_0 = A \cap G_A$ and $B_0 = B \cap G_B$ where G_B is an open set such that $B \triangle G_B \in I$. Then for each $x \in A_0$, there is a positive number ε such that $(x - \varepsilon, x + \varepsilon) \setminus A \in I$, and a similar property holds also for B_0.

Now, suppose to the countrary that $B \neq \emptyset$. Then, evidently, $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$. We shall define inductively two sequences $\{a_n\}$ and $\{b_n\}$. Since $0 \in A$, there are points $a_1 \in A_0$ and $b_1 \in B_0$ with $a_1 < b_1$. Assume we have defined points $a_n \in A_0$, $b_n \in B_0$ such that $a_n < b_n$. Let $c_n = \inf\{x; [x, b_n] \setminus B \in I\}$. Then as a_{n+1} we take any point from $A_0 \cap (a_n, c_n) \cap (c_n - \frac{b_n - c_n}{n}, c_n)$. It is easy to see that, for each $t \in [a_{n+1}, b_n)$,

(1) $I \cap B_0$ is of the second category for any n-subinterval I of the interval $[t, b_n]$.

In the same way one can define b_{n+1} as a point of $B_0 \cap (a_{n+1}, b_n) \cap (a_{n+1}, a_{n+1} + \frac{1}{n})$ such that for each $t \in (a_{n+1}, b_{n+1}]$,

(2) $I \cap A_0$ is of the second category for any n-subinterval I of the interval $[a_{n+1}, t]$.

Since $a_n < a_{n+1} < b_{n+1} < b_n$ and $b_{n+1} - a_{n+1} < \frac{1}{n}$, there is a number $z \in (0,1)$ for which $z = \lim a_n = \lim b_n$. But, since $(0,1) \subset A \cup B$, z must belong to A or to B. Without loss of generality we may suppose that $z \in A$. From [3, Lemma 4] it follows that there is a positive integer k_0 such that, for any $h \in (0,1)$, there exists a k_0-subinterval I of $[z, z + h]$ for which $I \setminus A$ is of the first category. Put $n_0 = 2k_0 + 1$. Let I_1 be a k_0-subinterval of $[z, b_{n_0}]$ with $I_1 \setminus A \in I$ and let I_2 be an n_0-subinterval of $[z, b_{n_0}]$ included in I_1. As $a_{n_0+1} < z$, condition (1) implies that $I_2 \cap B_0$ is of the second category. On the other hand, $I_2 \cap B_0 \subset I_1 \cap B = I_1 \setminus A \in I$. This contradiction ends the proof.

Theorem 1 can be expressed as follows:

"The local system

(*) $S(x) = \{A \subset R; x \in A \text{ and } R \setminus A \text{ is } I\text{-sparse at } x\}$

has the condition (SD)" (see [1]).

In order to formulate the monotonicity theorems announced before, we give a lot of definitions. By an I-proximal upper (lower) limit of a real function f at a point x we mean

$$I\text{-pr-}\limsup_{y\to x} f(y) = \inf\{t; f^{-1}[t,\infty) \text{ is } I\text{-sparse at } x\},$$

$$I\text{-pr-}\liminf_{y\to x} f(y) = \sup\{t; f^{-1}(-\infty,t] \text{ is } I\text{-sparse at } x\},$$

If, in the preceding definitions, one replaces the phrase "I-sparse" by "I-sparse on the right" ("I-sparse on the left"), then one obtains the definitions of right-hand (left-hand) I-proximal extreme limits. They are denoted $I\text{-pr-}\limsup_{y\to x^+} f(y)$, $I\text{-pr-}\liminf_{y\to x^+} f(y)$, $I\text{-pr-}\limsup_{y\to x^-} f(y)$ and $I\text{-pr-}\liminf_{y\to x^-} f(y)$, respectively.

By an I-proximal lower derivative of a function f we mean the I-proximal lower limit of its differente quotient, i.e.

$$I\text{-pr-}\underline{D}f(x) = I\text{-pr-}\liminf_{y\to x} \frac{f(y)-f(x)}{y-x}.$$

In an analogous way we define the I-proximal Dini derivates $I\text{-pr-}\overline{D}^+f(x)$, $I\text{-pr-}\underline{D}^+f(x)$, $I\text{-pr-}\overline{D}^-f(x)$, $I\text{-pr-}\underline{D}^-f(x)$.

In Thomson's book [4] and in [3] one can find definitions of limits and derivatives of a function with respect to a local system S (the so-called (S)-limits and (S)-derivates). It is evident that, for the local system S defined by $(*)$, we have

$$I\text{-pr-}\liminf_{y\to x} f(y) = (S) - \liminf_{y\to x} f(y),$$

$$I\text{-pr-}\liminf_{y\to x^+} f(y) = (S^+) - \liminf_{y\to x} f(y),$$

$$I\text{-pr-}\underline{D}f(x) = (S) - \underline{D}f(x), \qquad I\text{-pr-}\underline{D}^+f(x) = (S^+) - \underline{D}f(x)$$

and similar equations hold also for the remaining limits and derivates (where $S^+(x) = \{A; (-\infty,x)\cup A \in S(x)\}$ and $S^-(x) = \{A; (x,\infty)\cup A \in S(x)\}$). Thus Theorems 1 and 2 and Corollary 3 from [1] imply the following theorems.

THEOREM 2. *If a real function f satisfies the following properties:*

(a) $I\text{-pr-}\limsup_{y\to x^-} f(y) \le f(x)$ *for every x,*
(b) $I\text{-pr-}\underline{D}^+f(x) \ge 0$ *almost everywhere,*
(c) $I\text{-pr-}\underline{D}^+f(x) > -\infty$ *everywhere except possibly at points of the denumerable set whose every point satisfies the inequality* $f(x) \le I\text{-pr-}\liminf_{y\to x^+} f(y)$,

then f is nondecreasing

THEOREM 3. *If a real function f satisfies the following properties:*

(a) $I\text{-pr-}\limsup_{y\to x^-} f(y) \le f(x) \le I\text{-pr-}\liminf_{y\to x^+} f(y)$ *for every* x,

(b) $f(E)$ *has void interior, where*

$$E = \{x; I\text{-pr-}\underline{D}^+ f(x) \le 0 \quad \textit{and} \quad I\text{-pr-}\underline{D}^- f(x) \le 0\},$$

then f *is nondecreasing.*

THEOREM 4. *If* f *is a real function such that* $I\text{-pr-}\underline{D} f(x) \ge 0$ *for every* x, *then* f *is nondecreasing.*

References

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