DEMONSTRATIO MATHEMATICA
Vol. XXVII No 2 1994

Eugeniusz Janiec

ON STARLIKE AND CONVEX MAPS OF A BANACH SPACE INTO THE COMPLEX PLANE

1. Introduction

Let E be a complex Banach space, E^* the space dual to E, and let $B = \{x \in E : \|x\| < 1\}$ and $\widehat{B} = \{x \in E : \|x\| = 1\}$. For any $A \in E^*$, put

$$\varkappa(A) = \{x \in E : A(x) \neq 0\}, \quad \gamma(A) = E - \varkappa(A).$$

If $A \in E^*$ and $A \neq 0$, then $\varkappa(A)$ is dense in E and $\varkappa(A) \cap \widehat{B}$ is dense in $\widehat{B}$ (for then $\gamma(A)$ is a hyperplane).

Set $K = \{z \in \mathbb{C} : |z| < 1\}$. The symbol S^* (resp. S^C) will denote the well-known class of starlike (resp. convex) functions $f : K \to \mathbb{C}$ of the form $f(z) = z + a_2 z^2 + \ldots$, $z \in K$.

Let $D \subset E$, $D \neq \emptyset$, be any bounded and open set such that $zD \subset D$ for $z \in \mathbb{C}$, $|z| \leq 1$. Clearly, $0 \in D$. Denote by $\mathcal{H}(D)$ the family of all functions $f : D \to \mathbb{C}$, $f(0) = 0$, which are holomorphic in D, i.e. have the Fréchet derivative $f'(x)$ at each point $x \in D$. It is known that each function $f \in \mathcal{H}(D)$ is in some neighbouhood U of the point 0 a sum of the series $\sum_{m=1}^{\infty} P_{m,f}(x)$ uniformly convergent on U, where $P_{m,f} : E \to \mathbb{C}$ are continuous and homogeneous polynomials of degree m. For any $f \in \mathcal{H}(D)$ and $a \in \overline{D} \cap \varkappa(P_{1,f})$, put

$$f_a(z) = \frac{f(za)}{P_{1,f}(a)}, \quad z \in K.$$

Clearly, $f_a(0) = 0$ and $f_a'(0) = 1$. Moreover, it is easy to check that

$$f_a^{(n)}(z) = \frac{f^{(n)}(za)(a, \ldots, a)}{P_{1,f}(a)}, \quad n \in \mathbb{N},\ z \in K. \tag{1}$$

Let $A \in E^*$, $A \neq 0$. Denote by $S_A^C(D)$ (resp. $S_A^*(D)$) the family of all

$f \in \mathcal{H}(D)$ whose expansion in a Taylor series with centre at 0 has the form

$$f(x) = A(x) + \sum_{n=2}^{\infty} P_{n,f}(x),$$

such that, for any $a \in \varkappa(A) \cap \mathrm{Fr}(D)$, f is univalent on $D_a := \{za : z \in K\}$ and maps D_a onto a convex (resp. starlike with respect to the point 0) domain.

It will turn out (see Corollary 1) that all $f \in S_A^*(D) \cup S_A^C(D)$ vanish on D_a for $a \in \gamma(A) \cap \mathrm{Fr}(D)$.

In the case when $n \in \mathbb{N}$, $n \geq 2$, $E = \mathbb{C}^n$, $A(z_1, \dots, z_n) \equiv z_1 + \dots + z_n$ and $D \subset \mathbb{C}^n$ is any bounded complete Reinhardt domain containing the origin, the above-defined families $S_A^*(D)$ and $S_A^C(D)$ are, by Theorems 1, 2, 3, identical with the classes $S^*(D)$ and $S^C(D)$, respectively, considered in [2]. The majority of the result of the present paper imply the corresponding results of [2]. This also concerns the case when $D = B$, for both the sets $D_1 = \{(z_1, \dots, z_n) : |z_1| < 1, \dots, |z_n| < 1\}$ and $D_2 = \{(z_1, \dots, z_n) : |z_1| + \dots + |z_n| < 1\}$ (considered in [2]) may be treated as the unit balls in $\mathbb{C}^n$.

In papers [1, 4] the authors considered the families M and N of all functions $f : B \to \mathbb{C}$, $f(0) = 0$, holomorphic in B and such that, for any $y \in E$, $\|y\| = 1$, the function $g(z) = zf(zy)$, $z \in K$, is univalent, and starlike or convex, respectively.

2. The estimation of $|P_{n,f}(a)|$ and $\|P_{n,f}\|$ in the families $S_A^*(D)$ and $S_A^C(D)$

LEMMA 1. *If $f \in S_A^*(D)$ (resp. $f \in S_A^C(D)$), $n \geq 2$ and $a \in \mathrm{Fr}(D)$, then*

$$|P_{n,f}(a)| \leq n|A(a)| \quad (\textit{resp. } |P_{n,f}(a)| \leq |A(a)|). \tag{2}$$

In the case when $D = B$, estimate (2) is sharp and the equality holds for the function

$$f(x) = \frac{A(x)}{(1 - H(x))^2}, \quad x \in B$$
$$\left(\textit{resp. } f(x) = \frac{A(x)}{1 - H(x)}, \quad x \in B\right),$$

where $H \in E^*$, $H(a) = 1$, $\|H\| = 1$.

Proof. Suppose that $f \in S_A^*(D)$ and $n \geq 2$ (in the case when $f \in S_A^C(D)$, the proof runs similarly). If $a \in \mathrm{Fr}(D) \cap \varkappa(A)$, then $f_a \in S^*$; hence we get (2). So, let us assume that $a \in \mathrm{Fr}(D) \cap \gamma(A)$.

Clearly, $a = \lim_{k \to \infty} a_k$ where $a_k \in \varkappa(A)$ for $k \in \mathbb{N}$. Since the domain D is bounded, therefore, for any $k \in \mathbb{N}$, there exists $r_k \in \mathbb{R}_+$ such that

$a_k/r_k \in \mathrm{Fr}(D)$. Clearly, $(r_k)_{k\in\mathbb{N}}$ is bounded (for 0 is an interior point of D). Since $a_k/r_k \in \mathrm{Fr}(D) \cap \varkappa(A)$ for $k \in \mathbb{N}$, we have, by the first part of the proof,

$$\left|P_{n,f}\left(\frac{a_k}{r_k}\right)\right| \le n\left|A\left(\frac{a_k}{r_k}\right)\right|, \quad k \in \mathbb{N}.$$

Hence

$$|P_{n,f}(a_k)| \le n r_k^{n-1}|A(a_k)|, \quad k \in \mathbb{N}.$$

By taking the limit with $k \to \infty$, we get $P_{n,f}(a) = 0$, which ends the proof of (2).

COROLLARY 1. *All $f \in S_A^*(D)$ and all $g \in S_A^C(D)$ vanish on $\gamma(A) \cap D$.*

COROLLARY 2. *If $f \in S_A^*(B)$ and $n \ge 2$, then*

$$\|P_{n,f}\| \le n\|A\|.$$

If $g \in S_A^*(B)$ and $n \ge 2$, then

$$\|P_{n,g}\| \le \|A\|.$$

The above bounds are sharp, being attained by

$$f(x) = \frac{A(x)}{(1-H(x))^2}, \quad g(x) = \frac{A(x)}{1-H(x)}, \quad x \in B.$$

The estimates from Corollary 2 can be generalized to the case of any domain D (considered in this paper). For the purpose, put $M_D = \sup\{M \in \mathbb{R}_+ : B_M \subset D\}$ where $B_M = \{x \in E : \|x\| < M\}$. We have the following

COROLLARY 3. *If $f \in S_A^*(D)$, then*

$$\|P_{n,f}\| \le M_D^{1-n} n\|A\| \quad \textit{for } n \ge 2.$$

If $g \in S_A^C(D)$, then

$$\|P_{n,g}\| \le M_D^{1-n}\|A\| \quad \text{for } n \ge 2.$$

Proof (when $g \in S_A^C(D)$). Let $n \in \mathbb{N}$, $n \ge 2$, and let $a \in \widehat{B}$. There exists $r_a \in \mathbb{R}_+$ such that $r_a a \in \mathrm{Fr}(D)$. Clearly, $|r_a| \ge M_D$. Hence, by Lemma 1, $|P_{n,g}(r_a a)| \le |A(r_a a)|$, and, in consequence,

$$\|P_{n,g}(a)\| \le r_a^{1-n}|A(a)| \le M_D^{1-n}|A(a)|,$$

which, by the arbitrariness of $a \in \widehat{B}$, ends the proof.

3. Necessary and sufficient conditions for functions to belong to the families $S_A^*(D)$ and $S_A^C(D)$

THEOREM 1. *If $f \in S_A^*(D)$, then, for any $x \in \varkappa(A) \cap D$, $f(x) \ne 0$ and*

$$\mathrm{re}\frac{f'(x)(x)}{f(x)} > 0. \tag{3}$$

If $f \in S_A^C(D)$, then, for any $x \in \varkappa(A) \cap D$, $f'(x)(x) \neq 0$ and

$$\operatorname{re}\left(1 + \frac{f''(x)(x,x)}{f'(x)(x)}\right) > 0 \tag{4}$$

Proof. Assume first that $f \in S_A^*(D)$ and $x \in \varkappa(A) \cap D$. Since $f_x \in S^*$ and f_x is holomorphic in $\overline{K}$,

$$\operatorname{re}\frac{zf_x'(z)}{f_x(z)} > 0 \quad \text{for } |z| \leq 1\,.$$

Hence $f(x) = f_x(1)A(x) \neq 0$ and from the equality

$$\frac{f'(x)(x)}{f(x)} = \frac{f_x'(1)}{f_x(1)}$$

we get (3).

Assume now that $f \in S_A^C(D)$ and $x \in \varkappa(A) \cap D$. Since $f_x \in S^C$ and f_x is holomorphic in $\overline{K}$,

$$\operatorname{re}\left(1 + \frac{zf_x''(z)}{f_x'(z)}\right) > 0 \quad \text{for } |z| \leq 1\,. \tag{5}$$

Hence $f'(x)(x) = f_x'(1)A(x) \neq 0$. By (1), it is easily seen that

$$\frac{f_x''(1)}{f_x'(1)} = \frac{f''(x)(x,x)}{f'(x)(x)}\,,$$

which, by (5), gives (4).

THEOREM 2. *If $f \in \mathcal{H}(D)$, $f'(0) = A$ and, for any $x \in D$ such that $f(x) \neq 0$,*

$$\operatorname{re}\frac{f'(x)(x)}{f(x)} > 0\,,$$

then $f \in S_A^*(D)$.

Proof. Let $a \in \varkappa(A) \cap \operatorname{Fr}(D)$. For any $z \in K$ such that $f_a(z) \neq 0$, we have

$$\operatorname{re}\frac{zf_a'(z)}{f_a(z)} = \operatorname{re}\frac{f'(za)(za)}{f(za)} > 0\,. \tag{6}$$

So, it suffices to prove that $f_a(z) \neq 0$ for all $0 \neq z \in K$. Suppose to the contrary that $f_a(z_0) = 0$ for a certain $0 \neq z_0 \in K$. Since the function $zf_a'(z)/f(z)$, $z \in K$, has a pole in z_0, there exists $z \in K$ such that $f_a(z) \neq 0$ and $\operatorname{re}(zf_a'(z)/f(z)) < 0$, which contradicts (6).

Similarly we get

THEOREM 3. *If $f \in \mathcal{H}(D)$, $f'(0) = A$ and, for any $x \in D$ such that $f'(x)(x) \neq 0$,*

$$\operatorname{re}\left(1 + \frac{f''(x)(x,x)}{f'(x)(x)}\right) > 0\,,$$

then $f \in S_A^C(D)$.

If $f \in \mathcal{H}(D)$, then the function $h(x) := f'(x)(x)$, $x \in D$, belongs to $\mathcal{H}(D)$, and

$$h'(x)(p) = f'(x)(p) + f''(x)(p, x)$$

for $x \in D$, $p \in E$; in particular,

$$h'(x)(x) = f'(x)(x) + f''(x)(x, x)$$

for $x \in D$. From this, Theorems 1, 2, 3 and Corollary 1 we easily obtain the following

THEOREM 4. *If $f \in \mathcal{H}(D)$ and $f'(0) = A$, then $f \in S_A^C(D)$ if and only if the function $h(x) := f'(x)(x)$, $x \in B$, belongs to $S_A^*(D)$.*

4. The estimation of $|f(x)|$ and $|f^{(n)}(x)(x, \ldots, x)|$ in the families $S_A^*(D)$ and $S_A^C(D)$

THEOREM 5. *If $f \in S_A^*(B)$ and $w \in B$, then*

$$\frac{|A(w)|}{(1 + \|w\|)^2} \le |f(w)| \le \frac{|A(w)|}{(1 - \|w\|)^2}. \tag{7}$$

If $g \in S_A^C(B)$ and $w \in B$, then

$$\frac{|A(w)|}{1 + \|w\|} \le |g(w)| \le \frac{|A(w)|}{1 - \|w\|}. \tag{8}$$

The above bounds are sharp, being attained by

$$f_1(x) = \frac{A(x)}{[1 + H(x)]^2}, \quad f_2(x) = \frac{A(x)}{[1 - H(x)]^2}, \quad x \in B, \tag{9}$$

$$g_1(x) = \frac{A(x)}{1 + H(x)}, \quad g_2(x) = \frac{A(x)}{1 - H(x)}, \quad x \in B, \tag{10}$$

respectively, where $H \in E^*$, $H(w) = \|w\|$, $\|H\| = 1$.

Proof (when $f \in S_A^*(B)$). Inequalities (7) are obvious for $w \in \gamma(A) \cap B$. So, let us assume that $0 \ne w \in \varkappa(A) \cap B$ and put $a = w/\|w\|$. Since $f_a \in S^*$,

$$\frac{|z|}{(1 + |z|)^2} \le |f_a(z)| \le \frac{|z|}{(1 - |z|)^2}$$

for $z \in K$. Putting $z = \|w\|$, we get (7).

THEOREM 6. *Let $n \in \mathbb{N}$ and $0 \ne w \in B$.*

(a) *If $f \in S_A^*(B)$, then*

$$\frac{|A(w)|(n - \|w\|)}{(1 + \|w\|)^{n+2}} \le \frac{|f^{(n)}(w)(w, \ldots, w)|}{n!\|w\|^{n-1}} \le \frac{|A(w)|(n + \|w\|)}{(1 - \|w\|)^{n+2}}. \tag{11}$$

(b) *If $g \in S_A^C(B)$, then*

$$(12) \qquad \frac{|A(w)|}{(1+\|w\|)^{n+1}} \le \frac{|g^{(n)}(w)(w,\ldots,w)|}{n!\|w\|^{n-1}} \le \frac{|A(w)|}{(1-\|w\|)^{n+1}}.$$

(c) *Bounds* (11) *and* (12) *are sharp, being attained by functions* (9) *and* (10), *respectively.*

Proof (of part (b)). Inequalities (12) are obvious for $w \in \gamma(A) \cap B$. So, let us assume that $w \in \varkappa(A) \cap B$ and put $a = w/\|w\|$. Since $g_a \in S^C$,

$$\frac{n!}{(1+|z|)^{n+1}} \le |g_a^{(n)}(z)| \le \frac{n!}{(1-|z|)^{n+1}}$$

for $z \in K$. But, by (1),

$$g_a^{(n)}(z) = \frac{g^{(n)}(za)(za,\ldots,za)}{A(za)z^{n-1}}$$

for $0 \neq z \in K$. Thus, putting $z = \|w\|$, we get (12).

Parts (a) and (b) of the above theorem and inequalities (7), (8) can easily be generalized to the case when $f \in S_A^*(D)$ while $g \in S_A^C(D)$. The difference between estimates (7), (8), (11), (12) and the new ones lies in replacing $\|w\|$ by $\inf\{\lambda > 0 : w \in \lambda D\}$ only.

References

[1] I. I. Bawrin, *Ob odnom klasse odnolistnych otobraženij w Banachowych prostranstwach*, Soobšč. Akad. Nauk Gruzin. SSR, Nr 2 (1980).

[2] K. Dobrowolska, I. Dziubiński, *On starlike and convex functions of many variables*, Demonstratio Math. 11 (1978), 545–556.

[3] I. Dziubiński, R. Sitarski, *On classes of holomorph functions of many variables starlike and convex on some hypersurfaces*, Demonstratio Math. 13 (1980), 619–632.

[4] P. Liczberski, *On Bawrin's families of holomorphic maps of Banach spaces into the complex plane*, Folia Sci. Univ. Tech. Resov. 48 (1988), Mat. Fiz., z. 7, 77–85.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ
S. Banacha 22
90-238 ŁÓDŹ, POLAND

Received July 14, 1992.