DEMONSTRATIO MATHEMATICA
Vol. XXVII No 2 1994

Grażyna Anioł

ON THE RATE OF CONVERGENCE OF SOME DISCRETE OPERATORS

We consider a certain class of discrete approximation operators L_n which include e.g. the Bernstein polynomials, the Baskakov operators, the Meyer–König and Zeller operators or the Favard operators. For bounded or some locally bounded functions f on an interval I there is estimated the rate of convergence of $L_n[f](x)$ at these points x at which the one-sided limits $f(x\pm 0)$ exist. In the main theorems the Chanturiya's modulus of variation is used.

1. Preliminaries

Let I be a finite or infinite interval and let $M(I)$ [resp. $C(I)$] be the class of all complex-valued functions bounded [continuous] on I. In the case when I is not compact interval, denote by $M_{\text{loc}}(I)$ the class of all functions defined on I and bounded on every compact subinterval of I. Introduce, formally, for functions f belonging to these classes the discrete operators L_n given by

(1) $$L_n[f](x) := \sum_{j\in J_n} f(\xi_{j,n})p_{j,n}(x) \quad (x\in I, n\in N),$$

where $N := \{1,2,\ldots\}$, $J_n \subseteq Z := \{0,\pm1,\pm2,\ldots\}$, $\xi_{j,n}\in I$, $p_{j,n}\in C(I)$.

Suppose that

(2) $$\sum_{j\in J_n} |p_{j,n}(x)| \le \varphi_1(x) \quad \text{for all } x\in I \text{ and } n\in N.$$

where φ_1 is a positive function (with finite values) on I. In this case, operators (1) are well defined in the whole class $M(I)$. Assume further that, for every $x\in I$,

(3) $$\varrho_n(x) := \sum_{j\in J_n} p_{j,n}(x) - 1 \to 0 \quad \text{as } n\to\infty$$

Research supported by KBN grant 2 1079 91 01

and

$$(4) \qquad |\mu_{2,n}|(x) := \sum_{j \in J_n} (\xi_{j,n} - x)^2 |p_{j,n}(x)| < \infty \quad \text{if } n \in N .$$

Then (see e.g. [12], pp. 28–29), at every point x of continuity of $f \in M(I)$ at which $|\mu_{2,n}|(x) \to 0$ as $n \to \infty$,

$$\lim_{n \to \infty} L_n[f](x) = f(x) .$$

In case $f \in C(I) \cap M(I)$ this convergence is uniform on every interval $Y \subseteq I$ such that $\varphi_1 \in M(Y)$ and

$$\sup_{x \in Y} |\varrho_n(x)| \to 0, \qquad \sup_{x \in Y} |\mu_{2,n}|(x) \to 0 \quad \text{as } n \to \infty .$$

The same is also true for unbounded functions $f \in C(I)$, satisfying a suitable growth condition. The rate of this uniform convergence is evaluated in [8]. Also, in [8] there are applied the recent results of W. Kratz and U. Stadtmüller [6] concerning the moduli of continuity of $L_n[f]$, and there are investigated the degrees of approximation of $f \in C(I)$ by operators (1) in the Hölder type norms.

In this paper we present some inequalities for the rate of pointwise convergence of $L_n[f](x)$ for functions $f \in M(I)$ (or $f \in M_{\text{loc}}(I)$) at these points $x \in I$ at which the one-sided limits $f(x \pm 0)$ exist. For the sake of brevity we use the notation

$$s(x) := \{f(x+0) + f(x-0)\}/2, \qquad r(x) := \{f(x+0) - f(x-0)\}/2 ,$$

Our main estimates concerning the difference $\{L_n[f](x) - s(x)\}$ are expressed in terms of the modulus of variation of the function

$$g_x(u) := \begin{cases} f(u) - f(x+0) & \text{if } u > x , \\ 0 & \text{if } u = x, \\ f(u) - f(x-0) & \text{if } u < x \end{cases} \qquad (u \in I) .$$

Given any positive integer k, the modulus of variation $v_k(g;Y)$ of a bounded function g on a finite or infinite interval Y is defined as the upper bound of all numbers

$$\sum_{j=1}^{k} |g(\beta_j) - g(\alpha_j)|$$

over all systems Π_k of k non-overlapping intervals (α_j, β_j) contained in Y. If $k = 0$ we take $v_0(g;Y) = 0$. Clearly, $v_k(g;Y)$ is a non-decreasing function of k. Some basic properties of this modulus can be found e.g. in [1], [9].

In our considerations, the integral part of a real number u is denoted by $[u]$.

2. Main results

Let $f \in M(I)$ (or $f \in M_{\mathrm{loc}}(I)$) and let (1) be meaningful. Consider a point $x \in \operatorname{Int} I$ at which both limits $f(x \pm 0)$ exist.

It is clear that, for all $u \in I$,

$$f(u) = g_x(u) + s(x) + r(x)\operatorname{sgn}_x(u) + \{f(x) - s(x)\}\delta_x(u),$$

where

$$\operatorname{sgn}_x(u) := \begin{cases} 1 & \text{if } u > x, \\ 0 & \text{if } u = x, \\ -1 & \text{if } u < x, \end{cases} \qquad \delta_x(u) := \begin{cases} 1 & \text{if } u = x, \\ 0 & \text{if } u \neq x. \end{cases}$$

Therefore,

$$L_n[f](x) - s(x) = L_n[g_x](x) + \Delta_n(f;x) + s(x)\varrho_n(x), \tag{5}$$

with

$$\Delta_n(f;x) := r(x)L_n[\operatorname{sgn}_x](x) + \{f(x) - s(x)\}L_n[\delta_x](x). \tag{6}$$

In order to evaluate the term $L_n[g_x](x)$ it is convenient to write

$$\begin{aligned} L_n[g_x](x) = & \sum_{|\xi_{j,n}-x|\le a} g_x(\xi_{j,n})p_{j,n}(x) + \\ & + \vartheta_{x,a} \sum_{|\xi_{j,n}-x|>a} g_x(\xi_{j,n})p_{j,n}(x) \quad (a > 0), \end{aligned} \tag{7}$$

where $\vartheta_{x,a} = 0$ if neither of the points $x \pm a$ belongs to $\operatorname{Int} I$, and $\vartheta_{x,a} = 1$ otherwise.

LEMMA. *Suppose that f is bounded on an interval $[x-a, x+a] \cap I$ and that conditions* (2), (4) *are satisfied. Choose a positive null sequence (d_n) such that $d_n \le \frac{1}{2}$ and write $m := [1/d_n]$. Then, for every $n \in N$,*

$$\begin{aligned} \Big| \sum_{|\xi_{j,n}-x|\le a} g_x(\xi_{j,n})p_{j,n}(x)\Big| \le & \\ & \le P_n(a,x)\Big\{\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; I_j^a) + \frac{1}{m^2} v_m(g_x; I^a)\Big\}, \end{aligned} \tag{8}$$

where $I^a = I^a(x) := [x-a, x+a] \cap I$, $I_j^a = I_j^a(x) := [x - jad_n, x + jad_n] \cap I$, $j = 1, 2, \ldots, m-1$ and $P_n(a,x) := 2\{\varphi_1(x) + 8|\mu_{2,n}|(x)/(ad_n)^2\}$.

Proof. Write $g_x = g$. Introduce the points

$$t_j := x + jad_n, \quad j = 1, 2, \ldots, m, \quad t_{m+1} := x + a,$$

and the intervals $T_j := [x, t_j] \cap I$, $j = 1, 2, \ldots, m+1$. Denote by l the integer not greater than m, such that $t_l \in I$, $t_{l+1} \notin I$, and put $l = m$ in the case when all points t_j, $j = 1, \dots, m+1$, belong to I. Then

$$(9) \quad \sum_{x \le \xi_{j,n} \le x+a} g(\xi_{j,n}) p_{j,n}(x) = \sum_{x \le \xi_{j,n} \le t_1} g(\xi_{j,n}) p_{j,n}(x) +$$

$$+ \sum_{k=1}^{l} g(t_k) \sum_{t_k < \xi_{j,n} \le t_{k+1}} p_{j,n}(x) + \sum_{k=1}^{l} \sum_{t_k < \xi_{j,n} \le t_{k+1}} \{g(\xi_{j,n}) - g(t_k)\} p_{j,n}(x) =$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \quad \text{say}.$$

Obviously, by (2),

$$\Big|\sum\nolimits_1\Big| \le \sum_{x \le \xi_{j,n} \le t_1} |g(\xi_{j,n}) - g(x)||p_{j,n}(x)| \le \varphi_1(x) v_1(g; T_1).$$

Applying the inequality

$$(10) \quad \Big| \sum_{|\xi_{j,n} - x| \ge t} p_{j,n}(x) \Big| \le \frac{1}{t^2} |\mu_{2,n}|(x) \quad (t > 0),$$

and arguing similarly to the proof of Theorem 1 in [9] we conclude that

$$\Big|\sum\nolimits_2\Big| \le \frac{|\mu_{2,n}|(x)}{a^2 d_n^2} \Big\{ 2 \sum_{j=1}^{l-1} \frac{1}{j^3} v_j(g; T_j) + \frac{1}{l^2} v_l(g; T_{l+1}) \Big\}$$

and

$$\Big|\sum\nolimits_3\Big| \le \frac{|\mu_{2,n}|(x)}{a^2 d_n^2} \Big\{ 6 \sum_{j=2}^{l} \frac{1}{j^3} v_j(g; T_j) + \frac{1}{l^2} v_l(g; T_{l+1}) \Big\}.$$

Thus, we get the estimate of the left-hand side of equality (9).

By symmetry, we obtain the analogous estimate for the sum of $g(\xi_{j,n}) \cdot p_{j,n}(x)$ when $x - a \le \xi_{j,n} \le x$, and our assertion follows.

If the function f is bounded on I and if at least one of the points $x \pm a$ belongs to $\operatorname{Int} I$ then inequality (10) yields

$$(11) \quad \Big| \sum_{|\xi_{j,n} - x| > a} g_x(\xi_{j,n}) p_{j,n}(x) \Big| \le \frac{|\mu_{2,n}|(x)}{a^2} v_1(g_x; I).$$

Denoting by $\omega(g_x; \cdot)_I$ the modulus of continuity of g_x on I and using its well-known properties one can get the estimate

$$(12) \quad \Big| \sum_{|\xi_{j,n} - x| > a} g_x(\xi_{j,n}) p_{j,n}(x) \Big| \le \omega(g_x; d_n)_I \Big\{ \varphi_1(x) + \frac{1}{a d_n} |\mu_{2,n}|(x) \Big\}.$$

Suppose now that there exists a null sequence (d_n), $0 < d_n \le 1/2$ and a positive function φ_2 on I, such that

$$|\mu_{2,n}|(x) \le \varphi_2(x)d_n^2 \quad \text{for all } x \in I,\ n \in N. \tag{13}$$

Taking into account identities (5), (7), our Lemma and inequalities (11), (12) we can state the first main result as follows.

THEOREM 1. *Let conditions* (2), (13) *be satisfied. Suppose that* $f \in M(I)$ *and that at a fixed point* $x \in \operatorname{Int} I$ *the one-sided limits* $f(x \pm 0)$ *exist. Then, for all positive integers* n,

$$|L_n[f](x) - s(x)| \le P_1(x)\left\{\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; I_j^1) + \frac{1}{m^2} v_m(g_x; I^1)\right\} + \\ + \vartheta_{x,1}\varphi_2(x)d_n^2 v_1(g_x; I) + |\Delta_n(f;x)| + |s(x)||\varrho_n(x)|,$$

where m, I^1, I_j^1 *are as in the Lemma (with* $a = 1$*),* $P_1(x) := 2\{\varphi_1(x) + 8\varphi_2(x)\}$, $\Delta_n(f;x)$ *and* $\varrho_n(x)$ *are defined by* (6) *and* (3), *respectively. For continuous* f *the term* $\varphi_2(x)d_n^2 v_1(g_x; I)$ *can be replaced by* $\{\varphi_1(x) + \varphi_2(x)d_n\} \cdot \cdot\omega(g_x; d_n)_I$.

Now a result for unbounded f will be given.

THEOREM 2. *Let* $I = [0, \infty)$ *or* $I = (-\infty, \infty)$ *and let conditions* (2), (13) *be fulfilled. Suppose that a function* f *of class* $M_{\mathrm{loc}}(I)$ *satisfies the growth condition*

$$|f(x)| \le \psi(x) \quad (x \in I) \tag{14}$$

with a positive function $\psi \in C(I)$ *such that for all* $n \ge n_0 \in N$

$$\sum_{j \in J_n} \psi^2(\xi_{j,n})|p_{j,n}(x)| \le \varphi_3(x) \quad (x \in I, 0 < \varphi_3(x) < \infty). \tag{15}$$

If at a point $x \in \operatorname{Int} I$ *the limits* $f(x \pm 0)$ *exist and if* A *is an arbitrary positive number for which* $|x| \le A$ *then, for every integer* $n \ge n_0$, *we have*

$$|L_n[f](x) - s(x)| \le P_A(x)\left\{\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; I_j^A) + \frac{1}{m^2} v_m(g_x; I^A)\right\} + \\ + \Lambda(x)d_n + |\Delta_n(f;x)| + |s(x)||\varrho_n(x)|,$$

where m, I^A, I_j^A *are as in the Lemma (with* $a = A$*),* $P_A(x) := 2\{\varphi_1(x) + 8\varphi_2(x)/A^2\}$, $\Lambda(x) := \frac{1}{A}(\varphi_3(x)\varphi_2(x))^{1/2} + \frac{1}{2A^2}\psi(x)\varphi_2(x)$, $\Delta_n(f;x)$ *and* $\varrho_n(x)$ *are defined by* (6) *and* (3), *respectively.*

Proof. Take $A > 0$ and write $L_n[g_x](x)$ in the form (7) with $a = A$. In view of (14),

$$\Big| \sum_{|\xi_{j,n} - x| > A} g_x(\xi_{j,n})p_{j,n}(x)\Big| \le \sum_{|\xi_{j,n} - x| \ge A} \{\psi(\xi_{j,n}) + \psi(x)\}|p_{j,n}(x)| \le$$

$$\leq \frac{1}{A} \sum_{|\xi_{j,n}-x|\geq A} \psi(\xi_{j,n})|\xi_{j,n} - x||p_{j,n}(x)| + \psi(x) \sum_{|\xi_{j,n}-x|\geq A} |p_{j,n}(x)| \leq$$

$$\leq \frac{1}{A}(\varphi_3(x)\varphi_2(x))^{1/2}d_n + \frac{1}{A^2}\psi(x)\varphi_2(x)d_n^2 \leq \Lambda(x)d_n\,,$$

by the Cauchy–Schwarz inequality and (13), (15). The above inequality, identities (5), (7) and estimate (8) (with $a = A$) give our assertion, immediately.

Remark 1. In the case when I is non-compact finite interval one can reduce the problem to the case of infinite intervals. For example, if $I = [\alpha, \beta)$, we can choose a one-to-one mapping h of $[\alpha, \beta)$ into $[0, \infty)$ and we can apply our theorems to function $w = f \circ h^{-1}$ and operators

$$\widetilde{L}_n[w](y) := \sum_{j\in J_n} w(\eta_{j,n})q_{j,n}(y)\,,$$

where $\eta_{j,n} = h(\xi_{j,n})$, $q_{j,n}(y) = p_{j,n}(h^{-1}(y))$, $y \in [0, \infty)$.

Remark 2. For many well-known operators of the form (1), condition (3) is satisfied and

$$\lim_{n\to\infty} L_n[\mathrm{sgn}_x](x) = \lim_{n\to\infty} L_n[\delta_x](x) = 0$$

at every $x \in \mathrm{Int}\, I$. Consequently, for these operators, the right-hand sides of the inequalities given in Theorems 1 and 2 converge to zero as n tends to infinity (see [10], Remark 1).

3. Corollaries

Consider the class $BV_\Phi(I)$ of all functions of bounded Φ-variation on I. Denote by $V_\Phi(g; Y)$ the total Φ-variation of a function g on an interval $Y \subseteq I$, defined as in [9] or [10]. Proceeding similarly to [10] (pp. 152–153) we get from Theorem 1 the following

COROLLARY 1. *Suppose that conditions* (2) *and* (13) *are satisfied. If* $f \in BV_\Phi(I)$ *then, for every* $x \in \mathrm{Int}\, I$ *and for every* $n \in N$, *we have*

$$|L_n[f](x) - s(x)| \leq P(x)\frac{1}{m} \sum_{k=0}^{m^2-1} \frac{1}{\sqrt{k+1}}\Phi^{-1}\left(\frac{\sqrt{k+1}}{m}V_\Phi(g_x; Y_k)\right) + \\ + |\Delta_n(f, x)| + |s(x)||\varrho_n(x)|\,,$$

where $Y_k := [x - \frac{1}{\sqrt{k}}, x + \frac{1}{\sqrt{k}}] \cap I$ *if* $k = 1, 2, \ldots, m^2 - 1$, $Y_0 = I$, $P(x) := 15\{\varphi_1(x) + 8\varphi_2(x)\}$, $\Delta_n(f; x)$ *and* $\varrho_n(x)$ *have the same meaning as in Theorem* 1.

Analogously, Theorem 2 leads to

COROLLARY 2. *Let I be infinite interval ($I = [0,\infty)$ or $I = (-\infty,\infty)$) and let conditions* (2) *and* (13) *be satisfied. Suppose that $f \in M_{\mathrm{loc}}(I)$ is of bounded Φ-variation on each finite subinterval of I and that it satisfies the growth condition* (14) *with a function ψ for which* (15) *holds true. Then, for every $x \in I$, $x \neq 0$ and for every $n \geq n_0$, we have*

$$|L_n[f](x) - s(x)| \leq P^*(x)\frac{1}{m}\sum_{k=1}^{m^2-1}\frac{1}{\sqrt{k+1}}\Phi^{-1}\Big(\frac{\sqrt{k+1}}{m}V_\Phi(g_x;Y_k^*)\Big)+ \\ + \Lambda^*(x)d_n + |\Delta_n(f,x)| + |s(x)\varrho_n(x)|,$$

where $Y_k^* := [x-\frac{|x|}{\sqrt{k}}, x+\frac{|x|}{\sqrt{k}}]\cap I$ if $k = 1,2,\ldots,m^2-1$, $P^*(x) := 15\{\varphi_1(x)+ 8\varphi_2(x)/x^2\}$, $\Lambda^*(x) := \frac{1}{|x|}(\varphi_3(x)\varphi_2(x))^{1/2} + \frac{1}{2x^2}\psi(x)\varphi_2(x)$, $\Delta_n(f;x)$ and $\varrho_n(x)$ have the same meaning as in Theorem 2.

Now, let us suppose that $f \in C(I)$. Then, $s(x) = f(x)$, the term $\Delta_n(f;x)$ in (5) is equal to zero and Theorems 1, 2 can be applied also at the end-points of the interval I (if they belong to I). Applying the known inequality involving the modulus of variation $v_j(f;\widetilde{Y})$ and the modulus of continuity $\omega(f;\cdot)_{\widetilde{Y}}$ of f on an interval $\widetilde{Y} \subseteq I$ (see e.g. [1] or [9]) we easily deduce from Theorems 1 and 2 the estimates for the rate of uniform convergence of $L_n[f]$, which are given below.

COROLLARY 3. *Let $f \in C(I)\cap M(I)$. Suppose that conditions* (2) *and* (13) *are satisfied with $\varphi_1 \in M(I)$, $\varphi_2 \in M(I)$ and that $\varrho_n(x) \to 0$ as $n \to \infty$ uniformly in $x \in I$. Then, for every $x \in I$ and every positive integer n, we have*

$$|L_n[f](x) - f(x)| \leq c_1\omega(f;d_n)_I + \|f\|_I\|\varrho_n\|_I,$$

where $\|g\|_I := \sup_{t\in I}|g(t)|$ and c_1 is a positive constant not greater than $23\|\varphi_1\|_I + 170\|\varphi_2\|_I$.

COROLLARY 4. *Let $I = [0,\infty)$ or $I = (-\infty,\infty)$ and let A be an arbitrary positive number. Write $Y = [-A,A]\cap I$ and $2Y = [-2A,2A]\cap I$. Suppose that conditions* (2), (13) *are fulfilled with $\varphi_1,\varphi_2 \in M_{\mathrm{loc}}(I)$ and that a function f of class $C(I)$ satisfies the growth condition as in Theorem* 2 *in which $\varphi_3 \in M_{\mathrm{loc}}(I)$. Then, for every $x \in Y$ and $n \geq n_0$, we have*

$$|L_n[f](x) - f(x)| \leq c_2\omega(f;Ad_n)_{2Y} + c_3d_n + \|f\|_Y\|\varrho_n\|_Y,$$

with positive constants $c_2 \leq 22\{\|\varphi_1\|_Y+8\|\varphi_2\|_YA^{-2}\}$ and $c_3 \leq A^{-1}\|\varphi_2\varphi_3\|_Y^{1/2} + \frac{1}{2}A^{-2}\|\psi\varphi_2\|_Y$.

4. Examples

The class of operators of type (1) is very wide; it includes many well-known discrete operators. We specify here only a few examples for which our results can be applied.

I. The generalized Favard operators F_n are defined for bounded or some unbounded f on $I = R$ by formula (1) in which $J_n = Z$, $\xi_{j,n} = j/n$ and

$$p_{j,n}(x) = p_{j,n}(\gamma; x) = (\sqrt{2\pi} n\gamma_n)^{-1} \exp\left(-\frac{1}{2}\gamma_n^{-2}\left(\frac{j}{n} - x\right)^2\right),$$

where $\gamma = (\gamma_n)_1^\infty$ is a positive null sequence such that

$$n^2\gamma_n^2 \geq \frac{1}{2}\pi^{-2}\log n \quad \text{for } n \geq 2,\ \gamma_1^2 \geq \frac{1}{2}\pi^{-2}\log 2.$$

Applying Lemma 2 of [4] one can get the following estimates concerning the quantities (3) and (4):

$$|\varrho_n(x)| \leq 2 \quad \text{or} \quad |\varrho_n(x)| \leq \frac{4}{n} \leq 7\pi\gamma_n \quad \text{and} \quad |\mu_{2,n}|(x) \leq 51\gamma_n^2 \quad (n \in N)$$

uniformly in $x \in R$. In order to evaluate the quantity $\Delta_n(f; x)$ defined by (6) let us denote by ν the integral part of nx. If $\nu = nx$ then $F_n[\delta_x](x) = p_{\nu,n}(\nu/n) = (\sqrt{2\pi} n\gamma_n)^{-1}$ and $F_n[\mathrm{sgn}_x](x) = 0$. If $\nu \neq nx$, then $F_n[\delta_x](x) = 0$ and

$$|F_n[\mathrm{sgn}_x](x)| = \Big| \sum_{j=\nu+1}^{\infty} p_{j,n}(x) - \sum_{j=-\infty}^{\nu} p_{j,n}(x)\Big| \leq (\sqrt{2\pi} n\gamma_n)^{-1}.$$

Thus, Theorem 1 applies with

$$|\Delta_n(f; x)| \leq \{|r(x)| + |f(x) - s(x)|\}(\sqrt{2\pi} n\gamma_n)^{-1},$$

$\varphi_1(x) = 3$, $\varphi_2(x) = 51\kappa^2$ and $d_n = \gamma_n/\kappa$, where $\kappa := \max\{1, 2\sup_{\nu\in N}\gamma_\nu\}$. Also, Theorem 2 can be applied for functions $f \in M_{\mathrm{loc}}(R)$ satisfying the growth condition (14) with $\psi(x) = \exp(\sigma x^2)$, $\sigma > 0$. Indeed, assuming $\sigma\gamma_n^2 \leq 3/32$ we get

$$\sum_{j=-\infty}^{\infty} \psi^2\left(\frac{j}{n}\right) p_{j,n}(x) =$$

$$= \frac{\exp(4\sigma x^2)}{\sqrt{2\pi} n\gamma_n} \sum_{j=-\infty}^{\infty} \exp\left(2\sigma\left(\frac{j}{n}\right)^2 - 4\sigma x^2\right) \exp\left(-\frac{1}{2}\gamma_n^{-2}\left(\frac{j}{n} - x\right)^2\right) \leq$$

$$\leq \frac{\exp(4\sigma x^2)}{\sqrt{2\pi} n\gamma_n} \sum_{j=-\infty}^{\infty} \exp\left(\left(4\sigma - \frac{1}{2}\gamma_n^{-2}\right)\left(\frac{j}{n} - x\right)^2\right) \leq$$

$$\leq 2\{1 + \varrho_n(2\gamma; x)\} \exp(4\sigma x^2) \leq 6\exp(4\sigma x^2);$$

whence condition (15) holds with $\varphi_3(x) = 6\exp(4\sigma x^2)$.

Note that the above results remain valid for classical Favard operators for which $\gamma_n = \sqrt{c}/\sqrt{2n}$, $n \in N$, $c = \text{const.} > 0$. However, using Lemmas 2.1 and 2.2 of [2] we verify that, in this case,

$$|\varrho_n(x)| \le (3cen)^{-2}, \quad |\mu_{2,n}|(x) \le \left(\frac{1}{2}c + \frac{2}{3}(ce)^{-2}\right)n^{-1} \quad (n \in N)$$

uniformly in $x \in R$. Therefore, conditions (2) and (13) are satisfied with $\varphi_1(x) = (1+3ce)^{-2}$, $\varphi_2(x) = \frac{1}{2}c + \frac{2}{3}(ce)^{-2}$, $d_n = n^{-1/2}$. If $|f(x)| \le \exp(\sigma x^2)$, $\sigma > 0$, then condition (15) holds for $n > 8\sigma c$, with $\varphi_3(x) = \sqrt{2}(1 + (6ce)^{-2})\exp(4\sigma x^2)$.

II. Let $G := \{p_{j,1}(x), x \in I\}$ be a lattice distribution concentrated on some set $J_1 \subset Z \cap I$, with expectation equal to x and finite variance $\sigma^2(x)$. Put $\xi_{j,n} = j/n$ and suppose that $\{p_{j,n}(x)\}$ is the n-fold convolution of G. Taking into account quantities (3) and (4) we have

$$\varrho_n(x) = 0, \quad |\mu_{2,n}|(x) = \sigma^2(x)/n$$

for every $x \in I$ and every $n \in N$. Hence, conditions (2) and (13) are fulfilled with $\varphi_1(x) = 1$, $\varphi_2(x) = \sigma^2(x)$ and $d_n = n^{-1/2}$. Further, consider a point $x \in I$ at which $\sigma^2(x) > 0$ and

$$|\mu_{3,1}|(x) := \sum_{j \in J_1} |j - x|^3 p_{j,1}(x) < \infty\,.$$

In view of the Berry–Esséen Theorem [3, p. 515],

$$\Big|\sum_{j - nx \le t\sigma(x)\sqrt{n}} p_{j,n}(x) - \mathfrak{N}(t)\Big| \le \frac{\tau|\mu_{3,1}|(x)}{\sqrt{n}\sigma^3(x)} \quad (n \in N, t \in R),$$

where

$$\mathfrak{N}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-u^2/2)\, du$$

and the positive number τ is not greater than 0.82 (see [5], p. 93). From this it follows that

$$|L_n[\operatorname{sgn}_x](x)| = \Big|\sum_{j > nx} p_{j,n}(x) - \sum_{j < nx} p_{j,n}(x)\Big| \le \frac{5\tau|\mu_{3,1}|(x)}{\sqrt{n}\sigma^3(x)}$$

and

$$|L_n[\delta_x](x)| \le \frac{3\tau|\mu_{3,1}|(x)}{\sqrt{n}\sigma^3(x)}\,.$$

Consequently, for the expression $\Delta_n(f;x)$ defined by (6), we have the estimate

$$|\Delta_n(f;x)| \le 4\{|r(x)| + 0.6|f(x) - s(x)|\}|\mu_{3,1}|(x)/(\sqrt{n}\sigma^3(x)) \quad (n \in N)$$

at every $x \in I$ at which $\sigma^2(x) > 0$ and $|\mu_{3,1}|(x) < \infty$.

In particular, from these inequalities and Theorems 1, 2 one can get estimates for the rate of pointwise convergence of the Baskakov operators and, via Remark 1, for the Meyer–König and Zeller operators.

5. Appendix

Choosing $I = [0,\infty)$ or $I = (-\infty,\infty)$, let us consider the Lupaş type modification of operators (1), defined by

$$(16)\qquad L_{n,t}[f](x) := \sum_{j\in J_n} f(t+\xi_{j,n})p_{j,n}(x) \quad (x\in I,\ t\in I,\ n\in N).$$

For bounded continuous functions f on $[0,\infty)$ the above modifications of the Baskakov operators and the Szász–Mirakyan operators were investigated in [7] and [11].

Clearly, all our results given in Section 2 and 3 can be transferred to operators (16). In particular, the analogue of Theorem 2 and Corollary 4 can be stated as follows.

THEOREM 3. *Let assumptions* (2), (3), (13) *hold. Suppose that a function f of class $M_{\rm loc}(I)$ satisfies the growth condition* (14) *with $\psi\in C(I)$ such that*

$$\sum_{j\in J_n} \psi^2(t+\xi_{j,n})|p_{j,n}(x)| \le \varphi_4(x,t) \quad (x,t\in I,\ n\ge n_0),$$

φ_4 being a bivariate positive function on $I\times I$. If $t\in I$, $x\in \operatorname{Int} I$, $|x|\le A$ $(A>0)$ and if both the limits $f(t+x\pm 0)$ exist then, for every $n\ge n_0$, we have

$$|L_{n,t}[f](x) - s(t+x)| \le$$
$$\le P_A(x)\left\{\sum_{j=1}^{m-1}\frac{1}{j^3}v_j(g_{t+x};I_j^A(t+x)) + \frac{1}{m^2}v_m(g_{t+x};I^A(t+x))\right\}+$$
$$+\left\{\frac{1}{A}(\varphi_4(x,t)\varphi_2(x))^{1/2} + \frac{1}{2A^2}\psi(t+x)\varphi_2(x)\right\}d_n+$$
$$+|\Delta_{n,t}(f;x)| + |s(t+x)\varrho_n(x)|,$$

where $\Delta_{n,t}(f;x)$ is of the form (6) with $r(x)$, $f(x)$, $s(x)$ replaced by $r(x+t)$, $f(x+t)$, $s(x+t)$, respectively. If, in addition, f is continuous on I and if φ_1, φ_2, $\varphi_4(\cdot,t)\in M_{\rm loc}(I)$ for $t\in I$, then

$$\sup_{x\in Y}|L_{n,t}[f](x) - f(t+x)| \le c_2\omega(f;Ad_n)_{2Y} + c_4(t)d_n + \|f(\cdot+t)\|_Y\|\varrho_n\|_Y,$$

where $Y=[-A,A]\cap I$, the constant c_2 is the same as in Corollary 4 and $c_4(t)\le A^{-1}\|\varphi_4(\cdot,t)\varphi_2\|_Y^{1/2} + \frac{1}{2}A^{-1}\|\psi(t+\cdot)\varphi_2\|_Y$.

Acknowledgement. I would like to express my warmest thanks to Professor Paulina Pych-Taberska for suggesting the problem and many stimulating conversations.

References

[1] З. А. Чантурия, *Модуль изменения функции и его приложения в теории рядов Фурье*, Dokl. Akad. Nauk SSSR 214 (1974), 63–66.

[2] M. Becker, P. L. Butzer, R. J. Nessel, *Saturation for Favard operators in weighted function spaces*, Studia Math. 59 (1976), 139–153.

[3] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, New York 1966.

[4] W. Gawronski, U. Stadtmüller, *Approximation of continuous functions by generalized Favard operators*, J. Approx. Theory 34 (1982), 384–395.

[5] S. Guo, M. K. Khan, *On the rate of convergence of some operators on functions of bounded variation*, J. Approx. Theory 58 (1989), 90–101.

[6] W. Kratz, U. Stadtmüller, *On the uniform modulus of continuity of certain discrete approximation operators*, J. Approx. Theory 54 (1988), 326–337.

[7] G. C. Papanicolau, *Some Bernstein-type operators*, Amer. Math. Month. 82 (1975), 674–677.

[8] M. Powierska, P. Pych-Taberska, *Approximation of continuous functions by certain discrete operators in Hölder's norms*, Funct. Approx. Comment. Math. 21 (1992), 75–83.

[9] P. Pych-Taberska, *On the rate of pointwise convergence of Bernstein and Kantorovich polynomials*, Funct. Approximatio Comment. Math. 16 (1988), 63–76.

[10] P. Pych-Taberska, *On the rate of convergence of the Feller operators*, Comment. Math. Prace Mat. 31 (1991), 147–156.

[11] S. P. Singh, O. P. Varshney, *On Szász type operators*, Rend. Mat., (7) 2 (1982), 565–571.

[12] R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics, Vol. 293, Springer-Verlag, New York, 1972.

INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY
J. Matejki 48/49
60-769 POZNAŃ, POLAND

Received May 7, 1992.