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B. P. Pachpatte

ON MULTIVARIATE GREENE TYPE INEQUALITIES

In the present paper we establish a new integral inequality of the Greene type involving functions of several variables. A corresponding inequality on the discrete analogue of the main result is also given.

1. Introduction

In 1977, D. E. Greene [3] proved the following lemma.

LEMMA 1. *Let k_1, k_2 and μ be nonnegative constants and let f, g and h_i be continuous nonnegative functions for all $t \geq 0$ with h_i bounded such that*

$$f(t) \leq k_1 + \int_0^t h_1(s) f(s)\, ds + \int_0^t e^{\mu s} h_2(s) g(s)\, ds,$$

$$g(t) \leq k_2 + \int_0^t e^{-\mu s} h_3(s) f(s)\, ds + \int_0^t h_4(s) g(s)\, ds,$$

for all $t \geq 0$. Then there exist constants c_i and M_i such that

$$f(t) \leq M_1 e^{c_1 t}, \quad g(t) \leq M_2 e^{c_2 t},$$

for all $t \geq 0$.

The proof of these inequalities given in [3] was elementary but long, and much shorter proofs and further generalizations were found in [2], [8], [12]. In 1957, while investigating the boundedness of solutions of certain second order differential equations, Liang Ou–Iang [7] established the following inequality.

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LEMMA 2. *Let u and p be real-valued nonnegative continuous functions defined for all $t \geq 0$. If*

$$u^2(t) \leq c^2 + 2 \int_0^t p(s)u(s)\,ds,$$

for all $t \geq 0$, where $c \geq 0$ is a constant, then

$$u(t) \leq c + \int_0^t p(s)\,ds,$$

for all $t \geq 0$.

The inequality given in Lemma 2 depart somewhat from the structure of the classical Gronwall inequality [1], and has some important applications in the theory of differential equations [4]–[7], [11]. Due to the successful utilization of the inequality given in Lemma 2, it is natural to expect that some new extensions of this inequality similar to those of the inequality given in Lemma 1 would be equally important in certain new applications. Our main objective here is to establish a new integral inequality of the Greene type involving functions of several variables which can be used as handy tool in the study of certain new classes of partial differential and integral equations. The discrete analogue of the main result which can be used in the study of certain new classes of difference and sum-difference equations is also given.

2. Statement of results

Before stating the theorems to be proved in this paper, we summarise some basic notations and definitions which will be used throughout our discussion. Let $R_+ = [0, \infty)$ and define by R_+^n the product $R_+ \times \ldots \times R_+$ (n times). A point $(x_1, \ldots, x_n)$ in R_+^n is denoted by x. For 0, x in R_+^n and some function $p(x)$ defined for $x \in R_+^n$, we set

$$M[x,p] = \int_0^x p(y)\,dy = \int_0^{x_1} \cdots \int_0^{x_n} p(y_1, \ldots, y_n)\,dy_n \ldots dy_1.$$

We denote by $D_i = \frac{\partial}{\partial x_i}$ and $D_n \ldots D_1 = \frac{\partial}{\partial x_n} \cdots \frac{\partial}{\partial x_1}$ for $1 \leq i \leq n$. Let $N_0 = \{0, 1, 2, \ldots\}$ and denote by N_0^n the product $N_0 \times \ldots \times N_0$ (n times). A point $(x_1, \ldots, x_n)$ in N_0^n is denoted by x. For 0, x in N_0^n and some function $p(x)$ defined for $x \in N_0^n$, we set

$$L[x,p] = \sum_{s=0}^{x-1} p(s) = \sum_{s_1=0}^{x_1-1} \cdots \sum_{s_n=0}^{x_n-1} p(s_1, \ldots, s_n).$$

For any function $u(x)$ defined on N_0^n we define the operators

$$\begin{aligned}\Delta_1 u(x) &= u(x_1+1, x_2, \dots, x_n) - u(x), \dots, \\ \Delta_n u(x) &= u(x_1, \dots, x_{n-1}, x_n+1) - u(x), \\ \Delta_2\Delta_1 u(x) &= \Delta_1 u(x_1, x_2+1, x_3, \dots, x_n) - \Delta_1 u(x), \dots, \quad \text{and} \\ \Delta_n\Delta_{n-1}\dots\Delta_1 u(x) &= \Delta_{n-1}\dots\Delta_1 u(x_1, \dots, x_{n-1}, x_n+1) - \\ &\quad - \Delta_{n-1}\dots\Delta_1 u(x).\end{aligned}$$

For all $m > n$, $m, n \in N_0$ and any function $p(s)$ defined on N_0, we use the usual convention $\sum_{s=m}^{n} p(s) = 0$ and $\prod_{s=m}^{n} p(s) = 1$.

Our main result is embodied in the following theorem.

THEOREM 1. *Let $u(x)$, $v(x)$, $h_i(x)$, $i = 1,2,3,4$, be real-valued, nonnegative, continuous functions defined for $x \in R_+^n$ and let c_1, c_2 and μ be nonnegative constants such that*

$$u^2(x) \le c_1 + M[x, h_1 u] + M[x, h_2 \overline{v}], \tag{1}$$

$$v^2(x) \le c_2 + M[x, h_2 \overline{u}] + M[x, h_4 v], \tag{2}$$

for $x \in R_+^n$, where

$$\overline{u}(x) = \exp\Big(-2\mu \sum_{i=1}^{n} x_i\Big) u(x) \quad \textit{and} \quad \overline{v}(x) = \exp\Big(2\mu \sum_{i=1}^{n} x_i\Big) v(x)$$

for $x \in R_+^n$. Then

$$u(x) \le \exp\Big(\mu \sum_{i=1}^{n} x_i\Big)[c + M[x, h]], \tag{3}$$

$$v(x) \le c + M[x, h], \tag{4}$$

for $x \in R_+^n$, where $c = \sqrt{2(c_1 + c_2)}$ and

$$h(x) = \max\{[h_1(x) + h_3(x)], [h_2(x) + h_4(x)]\}, \tag{5}$$

for $x \in R_+^n$.

A useful discrete version of Theorem 1 is given in the following theorem.

THEOREM 2. *Let $u(x)$, $v(x)$, $h_i(x)$, $i = 1,2,3,4$, be real-valued, nonnegative functions defined for $x \in N_0^n$ and c_1, c_2 and μ be nonnegative constants such that*

$$u^2(x) \le c_1 + L[x, h_1 u] + L[x, h_2 \overline{v}], \tag{6}$$

$$v^2(x) \le c_2 + L[x, h_3 \overline{u}] + L[x, h_4 v], \tag{7}$$

for $x \in N_0^n$, where

$$\overline{u}(x) = \exp\Big(-2\mu \sum_{i=1}^{n} x_i\Big) u(x) \quad \textit{and} \quad v(x) = \exp\Big(2\mu \sum_{i=1}^{n} x_i\Big) v(x)$$

for $x \in N^n$. Then

$$(8) \qquad u(x) \le \exp\Big(2\mu \sum_{i=1}^{n} x_i\Big)[c + L[x,h]],$$

$$(9) \qquad v(x) \le c + L[x,h],$$

for $x \in N_0^n$, where $c = \sqrt{2(c_1 + c_2)}$ and

$$(10) \qquad h(x) = \max\{[h_1(x) + h_3(x)], [h_2(x) + h_4(x)]\}$$

for $x \in N_0^n$.

3. Proof of Theorem 1

Multiplying (1) by $\exp(-2\mu \sum_{i=1}^{n} x_i)$ we observe that

$$(11) \qquad \Big\{\exp\Big(-\mu \sum_{i=1}^{n} x_i\Big)u(x)\Big\}^2 \le c_1 + M[x, h_1\overline{u}] + M[x, h_2 v].$$

Define

$$(12) \qquad F(x) = \exp\Big(-\mu \sum_{i=1}^{n} x_i\Big)u(x) + v(x).$$

By squaring both sides of (12) and using the elementary inequality $(a+b)^2 \le 2(a^2 + b^2)$, (a, b reals), and by (11), (2), we observe that

$$(13) \qquad F^2(x) \le 2\Big[\Big\{\exp\Big(-\mu \sum_{i=1}^{n} x_i\Big)u(x))\Big\}^2 + v^2(x)\Big] \le$$
$$\le c^2 + 2M[x, (h_1 + h_3)\overline{u}] + 2M[x, (h_2 + h_4)v].$$

Now, by using the fact that $\exp(-2\mu \sum_{i=1}^{n} x_i) \le \exp(-\mu \sum_{i=1}^{n} x_i)$ and by (5), we can rewrite (13) in the form

$$(14) \qquad F^2(x) \le c^2 + 2M[x, hF].$$

In order to obtain the bound on the function $F(x)$ in (14), we first assume that $c > 0$ and define a function $z(x)$ by

$$(15) \qquad z(x) = c^2 + 2M[x, hF].$$

From (15) it is easy to observe that

$$(16) \qquad D_n \dots D_1 z(x) = 2h(x)F(x).$$

Using the fact that $F(x) \le \sqrt{z(x)}$ in (16) we have

$$(17) \qquad D_n \dots D_1 z(x) \le 2h(x)\sqrt{z(x)}.$$

From (17) we observe that

$$\frac{D_n \dots D_1 z(x)}{\sqrt{z(x)}} \le 2h(x) + \frac{[D_{n-1} \dots D_1 z(x)][D_n \sqrt{z(x)}]}{z(x)},$$

i.e.

$$D_n \left\{ \frac{D_{n-1} \dots D_1 z(x)}{\sqrt{z(x)}} \right\} \le 2h(x). \tag{18}$$

By keeping $x_1, \dots, x_{n-1}$ fixed in (18), we set $x_n = y_n$ and then integrating with respect to y_n from 0 to x_n, we have

$$\frac{D_{n-1} \dots D_1 z(x)}{\sqrt{z(x)}\cdot} \le 2 \int_0^{x_n} h(x_1, \dots, x_{n-1}, y_n)\, dy_n. \tag{19}$$

Again as above from (19) we get

$$D_{n-1} \left\{ \frac{D_{n-2} \dots D_1 z(x)}{\sqrt{z(x)}} \right\} \le \int_0^{x_n} (x_1, \dots, x_{n-1}, y_n)\, dy_n.$$

By keeping $x_1, \dots, x_{n-2}$ and x_n fixed in the above inequality, set $x_{n-1} = y_{n-1}$ and then integrating with respect to y_{n-1} from 0 to x_{n-1}, we have

$$\frac{D_{n-2} \dots D_1 z(x)}{\sqrt{z(x)}} \le 2 \int_0^{x_{n-1}} \int_0^{x_n} h(x_1, \dots, x_{n-2}, y_{n-1}, y_n)\, dy_n\, dy_{n-1}.$$

Continuing in this way, we obtain

$$\frac{D_1 z(x)}{\sqrt{z(x)}} \le 2 \int_0^{x_2} \dots \int_0^{x_n} h(x_1, y_2, \dots, y_n)\, dy_n \dots dy_2. \tag{20}$$

Now keeping $x_2, \dots, x_n$ fixed in the above inequality, we set $x_1 = y_1$ and then integrating with respect to y_1 from 0 to x_1, we have

$$\sqrt{z(x)} \le c + M[x, h]. \tag{21}$$

By using the fact that $F(x) \le \sqrt{z(x)}$ in (21) we get

$$F(x) \le c + M[x, h]. \tag{22}$$

Now suppose that $c = 0$. Then from (14) we see that the inequality

$$F^2(x) \le \varepsilon^2 + 2M[x, hF]$$

holds for every arbitrary positive number ε and $x \in R_+^n$, which by the above argument yields the estimate

$$F(x) \le \varepsilon + M[x, h]. \tag{23}$$

Since $F(x) \geq 0$ and $\varepsilon > 0$ is arbitrary number independent of $x \in R_+^n$, then as $\varepsilon \to 0$, it follows from (23) that

$$F(x) \leq M[x, h].$$

This shows that (22) gives the upper bound on $F(x)$ for all $c \geq 0$. The desired inequalities in (3), (4) follow by using (22) in (12) and splitting. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Multiplying (6) by $\exp(-2\mu \sum_{i=1}^n x_i)$, we observe that

$$\left\{\exp\Big(-\mu \sum_{i=1}^{n} x_i\Big) u(x)\right\}^2 \leq c_1 + L[x, h_1\overline{u}] + L[x, h_2 v]. \tag{24}$$

Define

$$G(x) = \exp\Big(-\mu \sum_{i=1}^{n} x_i\Big) u(x) + v(x). \tag{25}$$

By squaring both sides of (25) and using the elementary inequality $(a+b)^2 \leq 2(a^2 + b^2)$, ($a$, b reals), (24), (7) we observe that

$$\begin{aligned} G^2(x) &\leq 2\Big[\Big\{\exp\Big(-\mu \sum_{i=1}^{n} x_i\Big) u(x)\Big\}^2 + v^2(x)\Big] \leq \\ &\leq c^2 + 2L[x, (h_1 + h_3)\overline{u}] + 2L[x, (h_2 + h_4)v]. \end{aligned} \tag{26}$$

Now by using the fact that $\exp(-2\mu \sum_{i=1}^n x_i) \leq \exp(-\mu \sum_{i=1}^n x_i)$ and by (10) we can rewrite (26) in the form

$$G^2(x) \leq c^2 + 2L[x, hG]. \tag{27}$$

In order to obtain the bound on the function $G(x)$ in (27), we first assume that $c > 0$ and define a function $z(x)$ by

$$z(x) = c^2 + 2L[x, hG]. \tag{28}$$

From (28) it is easy to observe that

$$\Delta_n \dots \Delta_1 z(x) = 2h(x)G(x). \tag{29}$$

Using the facts that $G(x) \leq \sqrt{z(x)}$ and $\sqrt{z(x)} \leq \sqrt{z(x_1, \dots, x_{n-1}, x_n + 1)}$ in (29) we have

$$\begin{aligned} \Delta_n \dots \Delta_1 z(x) &\leq 2h(x)\sqrt{z(x)} \leq \\ &\leq 2h(x)\sqrt{z(x_1, \dots, x_{n-1}, x_n + 1)}. \end{aligned} \tag{30}$$

From (30) we observe that

$$(31)\quad \frac{\Delta_{n-1}\dots\Delta_1 z(x_1,\dots,x_{n-1},x_n+1)}{\sqrt{z(x_1,\dots,x_{n-1},x_n+1)}} - \frac{\Delta_{n-1}\dots\Delta_1 z(x_1,\dots,x_{n-1},x_n)}{\sqrt{z(x_1,\dots,x_{n-1},x_n)}} \le 2h(x_1,\dots,x_{n-1},x_n).$$

Keeping $x_1,\dots,x_{n-1}$ fixed in (31), we set $x_n = s_n$ and sum over $s_n = 0,1,2,\dots,x_n-1$ to obtain the estimate

$$(32)\quad \frac{\Delta_{n-1}\dots\Delta_1 z(x)}{\sqrt{z(x)}} \le 2\sum_{s_n=0}^{x_n-1} h(x_1,\dots,x_{n-1},s_n).$$

From (32) and in view of the facts that

$$\sqrt{z(x)} \le \sqrt{z(x_1,\dots,x_{n-2},x_{n-1}+1,x_n)}$$

for all $x_i \in N_0$, $1 \le i \le n$, we observe that

$$(33)\quad \frac{\Delta_{n-2}\dots\Delta_1 z(x_1,\dots,x_{n-2},x_{n-1}+1,x_n)}{\sqrt{z(x_1,\dots,x_{n-2},x_{n-1}+1,x_n)}} - \frac{\Delta_{n-2}\dots\Delta_1 z(x_1,\dots,x_{n-2},x_{n-1},x_n)}{\sqrt{z(x_1,\dots,x_{n-2},x_{n-1},x_n)}} \le$$
$$\le 2\sum_{s_n=0}^{x_n-1} h(x_1,\dots,x_{n-2},x_{n-1},s_n).$$

Keeping $x_1,\dots,x_{n-2},x_n$ fixed in (33), we set $x_{n-1} = s_{n-1}$ and sum over $s_{n-1} = 0,1,2,\dots,x_{n-1}-1$ to obtain the estimate

$$\frac{\Delta_{n-2}\dots\Delta_1 z(x)}{\sqrt{z(x)}} \le 2\sum_{s_{n-1}=0}^{x_{n-1}-1}\sum_{s_n=0}^{x_n-1} h(x_1,\dots,x_{n-2},,s_{n-1},s_n).$$

Proceeding in this way, we obtain the estimate

$$(34)\quad \frac{\Delta_1 z(x)}{\sqrt{z(x)}} \le 2\sum_{s_2=0}^{x_2-1}\dots\sum_{s_n=0}^{x_n-1} h(x_1,s_2,\dots,s_n).$$

Now we observe that

$$(35)\quad \Delta_1\sqrt{z(x)} = \sqrt{z(x_1+1,x_2,\dots,x_n)} - \sqrt{z(x)} = \frac{z(x_1+1,x_2,\dots,x_n) - z(x)}{\sqrt{z(x_1+1,x_2,\dots,x_n)} + \sqrt{z(x)}} \le \frac{\Delta_1 z(x)}{2\sqrt{z(x)}}.$$

Here in the last step we have used the fact that

$$\sqrt{z(x)} \le \sqrt{z(x_1+1,x_2,\dots,x_n)}.$$

Using (34) in (35), we get

$$\Delta_1\sqrt{z(x)} \le \sum_{s_2=0}^{x_2-1} \cdots \sum_{s_n=0}^{x_n-1} h(x_1, s_2, \dots, s_n)\,. \tag{36}$$

Now, keeping $x_2, \dots, x_n$ fixed in (36), we set $x_1 = s_1$ and sum over $s_1 = 0, 1, 2, \dots, x_1 - 1$ to obtain the estimate

$$\sqrt{z(x)} \le c + L[x, h]\,. \tag{37}$$

By using the fact that $G(x) \le \sqrt{z(x)}$ in (37), we get

$$G(x) \le c + L[x, h]\,. \tag{38}$$

The proof of the case when $c = 0$ can be completed by following the arguments as in the proof of Theorem 1 with suitable modifications. This shows that (38) gives the upper bound on $G(x)$ for all $c \ge 0$. The required inequalities in (8), (9) follow by using (38) in (25) and splitting. The proof is complete.

5. Some applications

In this section we indicate some applications of our results to obtain the bounds on the solutions of certain partial differential and integral equations and partial difference and sum-difference equations for which the earlier inequalities do not apply directly (see [1]–[12]). For example consider the following system of sum-difference equations

$$u^2(x) = f(x) + \sum_{s=0}^{x-1} A[y, u(y), v(y)]\,, \tag{39}$$

$$v^2(x) = g(x) + \sum_{s=0}^{x-1} B[y, u(y), v(y)]\,, \tag{40}$$

for $x \in N_0^n$, where $f, g : N_0^n \to R$; $A, B : N_0^n \times R \times R \to R$, in which R denotes the set of real numbers. We assume that

$$|f(x)| \le c_1, \quad |g(x)| \le c_2\,, \tag{41}$$

$$|A[x, u, v]| \le h_1(x)|u| + \exp\Big(2\mu \sum_{i=1}^{n} x_i\Big) h_2(x)|v|\,, \tag{42}$$

$$|B[x, u, v]| \le \exp\Big(-2\mu \sum_{i=1}^{n} x_i\Big) h_3(x)|u| + h_4(x)|v|\,, \tag{43}$$

where h_i, $i = 1, 2, 3, 4$, and c_1, c_2, μ are as defined in Theorem 2. From (39)–(43) we observe that

$$|u(x)|^2 \le c_1 + L[x, h_1|u|] + L[x, h_2|\overline{v}|]\,, \tag{44}$$

$$|v(x)|^2 \le c_2 + L[x, h_3|\overline{u}|] + L[x, h_4|v|], \tag{45}$$

where $\overline{u}$ and $\overline{v}$ are as defined in Theorem 2. Now an application of Theorem 2 yields

$$|u(x)| \le \exp\Big(-\mu\sum_{i=1}^{n} x_i\Big)[c + L[x, h]], \tag{46}$$

$$|v(x)| \le c + L[x, h], \tag{47}$$

where c, h, L are as defined in Theorem 2. The inequalities in (46), (47) give the bounds on the solution (u, v) of the equations (39)–(40).

We finally note that the inequality given in Theorem 1 can be used to obtain the bound on the solution of the following system of integral equations

$$u^2(x) = f(x) + \int_0^x A[y, u(y), v(y)]\, dy, \tag{48}$$

$$v^2(x) = g(x) + \int_0^x B[y, u(y), v(y)]\, dy, \tag{49}$$

under some suitable conditions on the functions f, g, A, B involved in (48), (49). Various other applications of these inequalities will appear elsewhere.

References

[1] R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1953.

[2] K. M. Das, *A note on an inequality due to Greene*, Proc. Amer. Math. Soc., 77 (1979), 424–425.

[3] D. E. Greene, *An inequality for a class of integral systems*, Proc. Amer. Math. Soc., 62 (1977), 101–104.

[4] A. Haraux, *Nonlinear evolution equations: global behavior of solutions*, Lecture Notes in Mathematics No. 841, Springer-Verlag-Berlin-New York, 1981.

[5] R. Ikehata, N. Okazawa, *Yosida approximation and nonlinear hyperbolic equation*, Nonlinear Anal. TMA, 15 (1990), 479–495.

[6] S. N. Olekhnik, *Boundedness and unboundedness of solutions of some systems of ordinary differential equations*, Vestnik Moskov. Univ. Mat., 27 (1972), 34–44.

[7] Liang Ou-Iang, *The boundedness of solutions of linear differential equations $y'' + A(t)y = 0$*, Shuxue Jinzhan, 3 (1957), 409–415.

[8] B. G. Pachpatte, *A note on Greene's inequality*, Tamkang J. Math., 15 (1984), 49–54.

[9] B. G. Pachpatte, *On some partial integral inequalities in n independent variables*, J. Math. Anal. Appl., 79(1981), 256–272.

[10] B. G. Pachpatte, *On multidimensional discrete inequalities and their applications*, Tamkang J. Math., 21 (1990), 111–122.

[11] M. Tsutsumi, I. Fukunda, *On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem*, Funkcial. Ekvac., 23 (1980), 259–277.
[12] C. L. Wang, *A short proof of a Greene theorem*, Proc. Amer. Math. Soc., 69 (1978), 357–358.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MARATHWADA UNIVERSITY
AURANGABAD 431004
MAHARASHTRA, INDIA

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