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ON REPRODUCTIVE SOLUTIONS OF BOOLEAN EQUATIONS

In this paper we determine a class of general reproductive solutions of arbitrary Boolean equations. We, namely, prove that Deschamps's theorem can be used for the actual finding the general reproductive solutions of Boolean equations.

1. Introduction

Let S be a given non-empty set and $\mathcal{E}(x)$ be any equation (x is an unknown element of S and $\mathcal{E}$ is a given unary relation of S) supposing that there is at least one element x such that $\mathcal{E}(x)$ is true.

DEFINITION 1. Let $h: E \to E$ be a given function. The formula $x = h(t)$ represents a general reproductive solution of x-equation $\mathcal{E}(x)$ if and only if

$$(\forall t)\mathcal{E}(h(t)) \wedge (\forall x)(\mathcal{E}(x) \Rightarrow (\exists t)x = h(t)).$$

DEFINITION 2. Let $h: E \to E$ be a given function. The formula $x = h(t)$ represents a general reproductive solution of x-equation $\mathcal{E}(x)$ if and only if

$$(\forall t)\mathcal{E}(h(t)) \wedge (\forall t)(\mathcal{E}(t) \Rightarrow t = h(t)).$$

DEFINITION 3. The Horn formulas over language L are defined as:

— The elementary Horn formulas are defined as the atomic formulas and the formulas of the form $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_s \Rightarrow \mathcal{D}$, where $\mathcal{D}_1, \ldots, \mathcal{D}_s, \mathcal{D}$ are atomic,

— Every Horn formulas is built from elementary Horn formulas using $\wedge$, $\forall$, $\exists$.

THEOREM 1 (Vaught). *Let $\mathcal{H}$ be a Horn sentence (formula without free variables) in language L_B of Boolean algebras. If $\mathcal{B}_2 \models \mathcal{H}$ then $\mathcal{B} \models \mathcal{H}$.*

See, for instance, [4].

2. Boolean equations

Let n be a natural number and $N = \{0, 1, \ldots, n\}$. We shall also use the notation: $X = (x_1, \ldots, x_n)$ and $T = (t_1, \ldots, t_n)$.

Let $\mathcal{B} = (B, \cup, \cap, ', 0, 1)$ be a Boolean algebra.

THEOREM 2 [7]. *The function $f: B^n \to B$ is Boolean if and only if it can be writen in the canonical disjunctive form*

$$f(x) = \bigcup_A f(A) X^A$$

where $\bigcup_A$ means the union over all $A \in \{0,1\}^n$.

THEOREM 3 [7]. *Let $f: B^n \to B$ be a Boolean function. The equation $f(X) = 0$ is consistent if and only if $\prod_A f(A) = 0$.*

THEOREM 4 [7]. *If $f: B^n \to B$ is a Boolean function then*

$$(\forall X \in B^n) f(X) = 0 \Leftrightarrow (\forall A \in \{0,1\}^n) f(A) = 0\,.$$

THEOREM 5 [3] (Deschamps). *Let f, $g_1, \ldots, g_n: B^n \to B$ be Boolean functions and $G = (g_1, \ldots, g_n)$. The formula $X = G(T)$ i.e.*

$$x_i = g_i(t_i, \ldots, t_n) \quad (i = 1, \ldots, n)$$

represents a general reproductive solution of the consistent Boolean equation $f(X) = 0$ if and only if

$$(\forall X) f(G(X)) = 0 \wedge (\forall X) f(X) = \bigcup_{j=1}^{n} (g_j(X) + x_j)\,.$$

THEOREM 6 [5]. *Let f, $g: B^n \to B$ be Boolean functions and assume that $f(X) = 0$ is consistent. Then the following conditions are equivalent:*

$$(\forall X \in B^n)(f(X) = 0 \Rightarrow g(X) = 0)\,,$$
$$(\forall X \in B^n)(g(X) \le f(X))\,,$$
$$(\forall X \in \{0,1\}^n)(g(X) \le f(X))\,.$$

3. Disjunctive normal forms of general solutions

We prove that every general solution of a given Boolean equation $f(x_1, \ldots, x_n) = 0$ can be written as a disjunctive normal form of n variables and coefficients of that equation.

THEOREM 7. *Let a Boolean equation $f(X) = 0$ be written in the form $h(X, Y) = 0$, where $h: B^n \times B^m \to B$ is a simple Boolean function and $Y \in B^m$. Let $g_1, \ldots, g_n: B^n \to B$ be Boolean functions. If the formulas*

$$x_j = g_j(t_1, \ldots, t_n) \quad (j = 1, \ldots, n)$$

represent a general solution of consistent equation $f(X) = 0$ then there are coefficients $p_{j,A,C} \in B$ such that

$$(\forall T) g_j(T) = \bigcup_A \Big(\bigcup_C p_{j,A,C} Y^C \Big) T^A \qquad (j = 1, \ldots, n), \tag{1}$$

where $\bigcup_A$ and $\bigcup_C$ mean the unions over all $A \in \{0,1\}^n$ and $C \in \{0,1\}^m$, respectively.

Proof. Let P be r-tuple of all $p_{j,A,C}$ occuring in (1), where $r = 2^n 2^m n$. We can write Theorem 7 as follows:

$$(\forall h, g_1, \ldots, g_n)\Big((\exists X) h(X,Y) = 0 \wedge (\forall X) h(X,Y) = \prod_A \bigcup_{j=1}^{n} (g_j(A) + x_j)$$
$$\Rightarrow (\exists P)(\forall T)(\forall j \in N) g_j(T) = \bigcup_A \Big(\bigcup_C p_{j,A,C} Y^C \Big) T^A \Big)$$

(because of Theorem 5)
i.e.

$$(\forall h, g_1, \ldots, g_n)\Big(\prod_A h(A,Y) = 0 \wedge (\forall X) h(X,Y) + \prod_A \bigcup_{j=1}^{n} (g_j(A) + x_j) = 0$$
$$\Rightarrow (\exists P)(\forall T) \bigcup_{j=1}^{n} \Big(g_j(T) + \bigcup_A \Big(\bigcup_C p_{j,A,C} Y^C \Big) T^A \Big) = 0 \Big)$$

(because of $(\forall a, b \in B)(a = 0 \Leftrightarrow a + b = 0)$ and $(\forall a, b \in B)(a = 0 \wedge b = 0 \Leftrightarrow a \cup b = 0)$).

Let $(\forall A \in \{0,1\}^n)$ $h_A = h(A,Y)$ and $(\forall j \in N)(\forall A \in \{0,1\}^n)$ $g_{j,A} = g_j(A)$ i.e.

$$h(X,Y) = \bigcup_A h_A X^A \quad \text{and} \quad g_j(X) = \bigcup_A g_{j,A} X^A \qquad (j = 1, \ldots, n). \tag{2}$$

The 2^n-tuple of all h_A occuring in (2) will be denoted by H_n and the $n \cdot 2^n$-tuple of all $g_{j,A}$ occuring in (2) will be denoted by G_n.

Now, we can write Theorem 7 as

$$(\forall Y)(\forall H_n)(\forall G_n)(\exists P)\Big(\prod_A h_A = 0 \wedge \bigcup_D \Big(h_D + \prod_A \bigcup_{j=1}^{n} (g_{j,A} + (D)_j) \Big) = 0$$
$$\Rightarrow \bigcup_D \bigcup_{j=1}^{n} \Big(g_{j,D} + \bigcup_A \Big(\bigcup_C p_{j,A,C} Y^C \Big) D^A \Big) = 0 \Big)$$

where $\bigcup_D$ means the union over all $D = (d_1, \ldots, d_n) \in \{0,1\}^n$.

Since the latter formula is the Horn sentence it is sufficient to prove Theorem 8 in $\mathcal{B}_2$, because of Theorem 1.

Let $h:\{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ and $g_1,\ldots,g_n:\{0,1\}^n \to \{0,1\}$ be Boolean functions, $h(X,Y)=0$ be consistent and the formulas

$$x_j = g_j(t_1,\ldots,t_n) \quad (j=1,\ldots,n)$$

represent a general solution of $f(X)=0$ in $\mathcal{B}_2$. From $Y \in \{0,1\}^m$ follows

$$\bigcup_A \Big(\bigcup_C p_{j,A,C} Y^C\Big) T^A = \bigcup_A p_{j,A,Y} T^A. \tag{3}$$

If we take

$$(\forall j \in N)(\forall A \in \{0,1\}^m) p_{j,A,Y} = g_{j,i}$$

we get (1) because of (3) and Theorem 4. ∎

4. General reproductive solutions

THEOREM 8. *Let a Boolean equation $f(X)=0$ be written in the form $h(X,Y)=0$, where $h:B^n \times B^m \to B$ is a simple Boolean function and $Y \in B^m$. The formulas*

$$x_j = \bigcup_D \Big(\bigcup_C p_{j,D,C} Y^C\Big) T^D \quad (j=1,\ldots,n) \tag{6}$$

($\bigcup_D$ and $\bigcup_C$ mean the unions over all $D \in \{0,1\}^n$ and $C \in \{0,1\}^m$ resp.) represents a general reproductive solution of the consistent equation $f(X) = 0$ if and only if

$$(\forall Y \in V)(\forall X \in \{0,1\}^n)\Big(\bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{n,X,Y}^{e_n}) = 0 \\ \wedge\, h(X,Y) = \bigcup_{j=1}^{n} (p_{j,X,Y} + x_j)\Big) \tag{7}$$

where $V = \{Y \mid Y \in \{0,1\}^m \wedge (\exists X) h(X,Y) = 0\}$ and $\bigcup_E$ means the union over all $E = (e_1,\ldots,e_n) \in \{0,1\}^n$.

COMMENT. If we solve the system (7) we get the general solution (6).

Proof. Bearing in mind Theorem 3 and Theorem 6 we have

$$(\forall Y \in B^m)\Big((\exists X) h(X,Y) = 0 \Rightarrow \wedge\, (\forall X \in B^n) G(f(X)) = 0 \\ \wedge\, (\forall X \in B^n) h(X,Y) = \bigcup_{j=1}^{n} \bigcup_D \Big(\bigcup_C p_{j,D,C} Y^C\Big) X^D + x_j\Big)$$

$$\Leftrightarrow (\forall Y \in B^m)\Big(\prod_A h(A,Y) = 0 \\ \Rightarrow (\forall X \in B^n) \bigcup_A \Big(\bigcup_E f(E)(g_1(A))^{e_1} \ldots (g_n(A))^{e_n}\Big) X^A = 0$$

$$\wedge\ (\forall X \in \{0,1\}^n) h(X,Y) = \bigcup_{j=1}^{n} \bigcup_{D} \Big(\bigcup_{C} p_{j,D,C} Y^C \Big) X^D + x_j \Big)$$

$$\Leftrightarrow (\forall Y \in B^m)(\forall X \in B^n)\Big(\prod_{A} h(A,Y) = 0$$

$$\Rightarrow \bigcup_{A} \Big(\bigcup_{E} f(E)(g_1(A))^{e_1} \ldots (g_n(A))^{e_n} \Big) X^A = 0$$

$$\wedge\ h(X,Y) = \bigcup_{j=1}^{n} \bigcup_{D} \Big(\bigcup_{C} p_{j,D,C} Y^C \Big) X^D + x_j \Big)$$

$$\Leftrightarrow (\forall Y \in B^m)(\forall X \in B^n)\Big(\prod_{A} h(A,Y) = 0$$

$$\Rightarrow \bigcup_{A} \Big(\bigcup_{E} f(E)(g_1(A))^{e_1} \ldots (g_n(A))^{e_n} \Big) X^A$$

$$\cup\ h(X,Y) + \bigcup_{j=1}^{n} \bigcup_{D} \Big(\bigcup_{C} p_{j,D,C} Y^C \Big) X^D + x_j = 0 \Big)$$

$$\Leftrightarrow (\forall Y \in \{0,1\}^m)(\forall X \in \{0,1\}^n)\Big(\bigcup_{A} \Big(\bigcup_{E} f(E)(g_1(A))^{e_1} \ldots (g_n(A))^{e_n} \Big) X^A.$$

$$\cup\ h(X,Y) + \bigcup_{j=1}^{n} \bigcup_{D} \Big(\bigcup_{C} p_{j,D,C} Y^C \Big) X^D + x_j \leq \prod_{A} h(A,Y) \Big)$$

(by Theorem 6)

$$\Leftrightarrow (\forall Y \in \{0,1\}^m)(\forall X \in \{0,1\}^n)\Big(\prod_{A} h(A,Y) = 0$$

$$\Rightarrow \bigcup_{A} \Big(\bigcup_{E} f(E)(g_1(A))^{e_1} \ldots (g_n(A))^{e_n} \Big) X^A$$

$$\cup\ h(X,Y) + \bigcup_{j=1}^{n} \bigcup_{D} \Big(\bigcup_{C} p_{j,D,C} Y^C \Big) X^D + x_j = 0 \Big)$$

(by Theorem 6)

$$\Leftrightarrow (\forall Y \in \{0,1\}^m)(\forall X \in \{0,1\}^n)\Big(\prod_{A} h(A,Y) = 0$$

$$\Rightarrow \bigcup_{A} \Big(\bigcup_{E} f(E)(g_1(A))^{e_1} \ldots (g_n(A))^{e_n} \Big) X^A = 0$$

$$\wedge\ h(X,Y) + \bigcup_{j=1}^{n} \bigcup_{D} \Big(\bigcup_{C} p_{j,D,C} Y^C \Big) X^D + x_j = 0 \Big)$$

$$\Leftrightarrow (\forall Y \in V)(\forall X \in \{0,1\}^n)\Big(\bigcup_A \Big(\bigcup_E f(E)(g_1(A))^{e_1} \dots (g_n(A))^{e_n}\Big) X^A = 0$$

$$\wedge\ h(X,Y) + \bigcup_{j=1}^{n} \bigcup_D \Big(\bigcup_C p_{j,D,C} Y^C\Big) X^D + x_j = 0\Big)$$

$$(V = \{Y \mid Y \in \{0,1\}^m \wedge (\exists X) h(X,Y) = 0\})$$

$$\Leftrightarrow (\forall Y \in V)(\forall X \in \{0,1\}^n)\Big(\bigcup_E f(E)(g_1(X))^{e_1} \dots (g_n(X))^{e_n} = 0$$

$$\wedge\ h(X,Y) + \bigcup_{j=1}^{n} \bigcup_D \Big(\bigcup_C p_{j,D,C} Y^C\Big) X^D + x_j = 0\Big)$$

$$\Leftrightarrow (\forall Y \in V)(\forall X \in \{0,1\}^n)\Big(\bigcup_E f(E)\Big(\bigcup_D p_{1,X,D} Y^D\Big)^{e_1} \dots$$

$$\dots \Big(\bigcup_D p_{n,X,D} Y^D\Big)^{e_n} = 0 \wedge h(X,Y) + \bigcup_{j=1}^{n} \Big(\bigcup_C p_{j,X,C} Y^C + x_j\Big) = 0\Big)$$

$$\Leftrightarrow (\forall Y \in V)(\forall X \in \{0,1\}^n)\Big(\bigcup_E f(E)\Big(\bigcup_D p_{1,X,D}^{e_1} \cdots p_{n,X,D}^{e_n} Y^D\Big) = 0$$

$$\wedge\ h(X,Y) + \bigcup_{j=1}^{n} \Big(\bigcup_C p_{j,X,C} Y^C + x_j\Big) = 0\Big)$$

$$\Leftrightarrow (\forall Y \in V)(\forall X \in \{0,1\}^n)\Big(\bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{n,X,Y}^{e_n}) = 0$$

$$\wedge\ h(X,Y) = \bigcup_{j=1}^{n} (p_{j,X,Y} + x_j) = 0\Big) \ \blacksquare$$

Remark 1. If $Y \in V$, $X \in \{0,1\}^n$ and $h(X,Y) = 0$ then the conjunction

$$\bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{n,X,Y}^{e_n}) = 0 \wedge h(X,Y) = \bigcup_{j=1}^{n} (p_{j,X,Y} + x_j) \tag{8}$$

reduces to

$$(\forall j \in N) p_{i,X,Y} = x_j\,. \tag{9}$$

Proof. If $h(X,Y) = 0$ then $h(X,Y) = \bigcup_{j=1}^n (p_{j,X,Y} + x_j)$ is equivalent to $(\forall j \in N) p_{j,X,Y} = x_j$. Since

$$h(p_{1,X,Y}, \dots, p_{n,X,Y}, Y) = \bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{n,X,Y}^{e_n})$$

(9) implies that the equality $\bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{j,X,Y}^{e_n}) = 0$ is fulfilled.

Remark 2. If $Y \in V$, $X \in \{0,1\}^n$ and $h(X,Y) = 1$ then the conjunction (8) reduces to

$$\bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{n,X,Y}^{e_n}) = 0 .$$

Proof. It is easy to proof that the equality $\bigcup_{j=1}^n (p_{j,X,Y} + x_j) = 1$ is equivalent to $p_{1,X,Y}^{x_1} \cdots p_{n,X,Y}^{x_n} = 0$. In the other hand $f(X) = h(X,Y) = 1$ implies that the union $\bigcup_E f(E)(p_{1,X,Y}^{e_1} \cdots p_{n,X,Y}^{e_n})$ contains the product $p_{1,X,Y}^{x_1} \cdots p_{n,X,Y}^{x_n}$.

EXAMPLE. Determine a general reproductive solution of

$$ax'y' \cup bxy = 0$$

in arbitrary Boolean algebra.

We shall determine a general reproductive solution in the form

$$\begin{aligned} x = {} & (p_{0,0}a'b' \cup p_{0,1}a'b \cup p_{0,2}ab' \cup p_{0,3}ab)t_1't_2' \\ & \cup (p_{1,0}a'b' \cup p_{1,1}a'b \cup p_{1,2}ab' \cup p_{1,3}ab)t_1't_2 \\ & \cup (p_{2,0}a'b' \cup p_{2,1}a'b \cup p_{2,2}ab' \cup p_{2,3}ab)t_1t_2' \\ & \cup (p_{3,0}a'b' \cup p_{3,1}a'b \cup p_{3,2}ab' \cup p_{3,3}ab)t_1t_2 , \\ y = {} & (q_{0,0}a'b' \cup q_{0,1}a'b \cup q_{0,2}ab' \cup q_{0,3}ab)t_1't_2' \\ & \cup (q_{1,0}a'b' \cup q_{1,1}a'b \cup q_{1,2}ab' \cup q_{1,3}ab)t_1't_2 \\ & \cup (q_{2,0}a'b' \cup q_{2,1}a'b \cup q_{2,2}ab' \cup q_{2,3}ab)t_1t_2' \\ & \cup (q_{3,0}a'b' \cup q_{3,1}a'b \cup q_{3,2}ab' \cup q_{3,3}ab)t_1t_2 . \end{aligned}$$

Note that $V = \{0,1\}^2$. If $(x,y,a,b) \in \{0,1\}^4$ it is easy to test the statement

$$h(x,y,a,b) = 1 \Leftrightarrow (x,y,a,b) \in \{(0,0,1,0),(0,0,1,1),(1,1,0,1),(1,1,1,1)\} .$$

Bearing in mind Remark 1 and Remark 2 we get

$$\begin{aligned} & (p_{0,0}, q_{0,0}) = (p_{0,1}, q_{0,1}) = (0,0) , \\ & (p_{1,0}, q_{1,0}) = (p_{1,1}, q_{1,1}) = (p_{1,2}, q_{1,2}) = (p_{1,3}, q_{1,3}) = (0,1) , \\ & (p_{2,0}, q_{2,0}) = (p_{2,1}, q_{2,1}) = (p_{2,2}, q_{2,2}) = (p_{2,3}, q_{2,3}) = (1,0) , \\ & (p_{3,0}, q_{3,0}) = (p_{3,2}, q_{3,2}) = (1,1) , \\ & p_{0,2}'q_{0,2}' = 0, \quad p_{0,3}'q_{0,3}' \cup p_{0,3}q_{0,3} = 0, \quad p_{3,1}q_{3,1} = 0, \quad p_{3,3}'q_{3,3}' \cup p_{3,3}q_{3,3} = 0 . \end{aligned}$$

From the latter line we get, for instance,

$$(p_{0,2}, q_{0,2}) = (p_{0,3}, q_{0,3}) = (p_{3,1}, q_{3,1}) = (p_{3,3}, q_{3,3}) = (0,1)$$

(any missing term $p_k^\alpha q_k^\beta$ indicates that (α, β) is a solution of the corresponding equation).

Finaly, we have

$$x = (a'b' \cup a'b \cup ab' \cup ab)t_1t_2' \cup a'b't_1t_2$$

$$y = (a'b \cup ab' \cup ab)t_1't_2' \cup (a'b' \cup a'b \cup ab' \cup ab)t_1't_2 \cup (a'b' \cup ab)t_1t_2 .$$

References

[1] D. Banković, *Certain Boolean equations*, Discrete Appl. Math., 35 (1992), 21–27.
[2] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland 1973.
[3] J. P. Deschamps, *Parametric solution of Boolean equations*, Discrete Math. 3 (1972), 334–342.
[4] Ž. Mijajlović, *Some remarks on Boolean terms — model theoretic approach*, Publ. Inst. Math. Beograd, 21 (35) 1977, 135–140.
[5] S. Prešić, *Une méthode de résolution des equations dont toutes les solutions appartiennent à un ensemble finni donné*, C. R. Acad. Sci. Paris, Ser. A 272 (1971), 654–657.
[6] S. Prešić, *All reproductive solutions of finite equations*, Publ. Inst. Math. Beograd, 44 (58) (1988), 3–7.
[7] S. Rudeanu, *Boolean Functions and Equations*, North-Holland 1974.

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