DEMONSTRATIO MATHEMATICA
Vol. XXVI No 3–4 1993

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE SECOND ORDER DIFFERENCE EQUATIONS

Let N denote the set of positive integers, R the set of real numbers. For a function $f : N \to R$ one introduces the difference operator Δ in the following way $\Delta f = f_{n+1} - f_n$ where $f_n = f(n)$ and $\Delta^k f = \Delta^{k-1}(\Delta f)$ for $k > 1, k \in N$.

Next Lemma was presented in [1].

LEMMA 1. *The difference equation*

$$\Delta^2 z_n = a_n z_{n+1}, \quad n \in N \tag{1}$$

where $a : N \to R$, has linearly independent solutions u, v which fulfil the equation

$$\begin{vmatrix} u_n & v_n \\ \Delta u_n & \Delta v_n \end{vmatrix} = -1, \quad \textit{for } n \in N. \tag{2}$$

DEFINITION 1. We will say $B \in B_{\mathrm{F}}$, if $B : N \times R \to R_+$ and B possesses the following properties

1° $0 \le B(n, x_1) \le B(n, x_2)$, for $0 \le x_1 \le x_2$
2° $B(n, kx) \le F(k)B(n, x)$, for $x \ge \varepsilon > 0$

where F is continuous, nondecreasing and positive function.

LEMMA 2. *Let u, v denote linearly independent solutions of the difference equation* (1) *for which* (2) *holds. Moreover, let $a : N \to R$ and function $f : N \times R \to R$ possessing the following properties*

$$|f(n,x)| \le B(n, |x|), \textit{ for every } x \in R \tag{3}$$

where $B \in B_F$, and F fulfil the condition

$$\lim_{x \to 0} \int_0^x \frac{ds}{F(s)} = -\infty \quad (\textit{for every positive constant } \varepsilon). \tag{4}$$

If

$$\sum_{j=1}^{\infty} U_j B(j, U_j) = K < \infty \quad (\textit{for some positive constant } K) \tag{5}$$

$$\textit{where } U_j = \max\{|u_j|, |v_j|, |u_{j+1}|, |v_{j+1}|\} \tag{6}$$

then the solution of the equation

$$\Delta^2 y_n = a_n y_{n+1} + f(n, y_n), \quad n \in N$$

exists and it can be written down in the form

$$y_n = \alpha_n u_n + \beta_n v_n \tag{7}$$

where

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = \beta.$$

Proof. Let us choose two linearly independent solutions u, v of (1) fulfilling the condition (2). Moreover, let us denote

$$A_n = v_n \Delta y_n - y_n \Delta v_n \tag{8}$$

$$B_n = y_n \Delta u_n - u_n \Delta y_n. \tag{9}$$

Then

$$y_n = u_n A_n + v_n B_n. \tag{10}$$

Applying the difference operator Δ to (8) and (9) we obtain

$$\Delta A_n = v_{n+1} \Delta^2 y_n - y_{n+1} \Delta^2 v_n$$
$$\Delta B_n = y_{n+1} \Delta^2 u_n - u_{n+1} \Delta^2 y_n.$$

Using (7) and (1) we have

$$\Delta A_n = v_{n+1} f(n, y_n)$$
$$\Delta B_n = -u_{n+1} f(n, y_n).$$

From (10) we obtain

$$\begin{aligned} \Delta A_j &= v_{j+1} f(j, A_j u_j + B_j v_j) \\ \Delta B_j &= -u_{j+1} f(j, A_j u_j + B_j v_j), \quad j \in N. \end{aligned} \tag{11}$$

Putting from $j = 1$ to $j = n - 1$ and adding obtaining equations one yields

$$\begin{aligned} A_n &= A_1 + \sum_{j=1}^{n-1} v_{j+1} f(j, A_j u_j + B_j v_j) \\ B_n &= B_1 - \sum_{j=1}^{n-1} u_{j+1} f(j, A_j u_j + B_j v_j). \end{aligned} \tag{12}$$

Then

$$|A_n| \le |A_1| + \sum_{j=1}^{n-1} |v_{j+1}||f(j, A_j u_j + B_j v_j)|$$

$$|B_n| \le |B_1| + \sum_{j=1}^{n-1} |u_{j+1}||f(j, A_j u_j + B_j v_j)|.$$

We have

$$|A_n| + |B_n| \le |A_1| + |B_1| + \sum_{j=1}^{n-1} [|v_{j+1}| + |u_{j+1}|]\, |f(j, A_j u_j + B_j v_j)|.$$

Let us denote

$$h_n = |A_n| + |B_n|, \quad n \in N. \tag{13}$$

From (6) we have

$$|v_j| \le U_j, \quad |u_j| \le U_j, \quad |v_{j+1}| \le U_j, \quad |u_{j+1}| \le U_j.$$

We can see that

$$|A_j u_j + B_j v_j| \le |A_j||u_j| + |B_j||v_j| \le U_j(|A_j| + |B_j|). \tag{14}$$

Hence by (3) we get

$$|f(j, A_j u_j + B_j v_j)| \le B(j, |A_j u_j + B_j v_j|).$$

Therefore, (13) yields

$$h_n \le h_1 + 2 \sum_{j=1}^{n-1} U_j B(j, |A_j u_j + B_j v_j|).$$

Where, (13) and (14) lead to the following inequality

$$h_n \le h_1 + 2 \sum_{j=1}^{n-1} U_j B(j, U_j h_j).$$

Let

$$b_n = h_1 + 2 \sum_{j=1}^{n-1} U_j B(j, U_j h_j).$$

Then

$$h_i \le b_i, \quad i \in N. \tag{15}$$

From definition of b it follows that

$$\Delta b_i = b_{i+1} - b_i = 2U_i B(i, U_i h_i).$$

From (15) and properties of B we have

$$\Delta b_i \le 2U_i B(i, U_i b_i) \le 2U_i F(b_i) B(i, U_i).$$

This and condition 2° imply

$$\frac{\Delta b_i}{F(b_i)} \le 2U_i B(i, U_i). \tag{16}$$

Since function F is nondecreasing, so the function $\frac{1}{F}$ is nonincreasing. This yields

$$\frac{\Delta b_i}{F(b_i)} \ge \int_{b_i}^{b_{i+1}} \frac{ds}{F(s)}. \tag{17}$$

From (16) and (17) we have

$$\int_{b_i}^{b_{i+1}} \frac{ds}{F(s)} \le 2U_i B(i, U_i), \quad i \in N. \tag{18}$$

Putting from $i = 1$ to $i = n-1$ and adding obtaining equations one yields

$$\int_{b_1}^{b_n} \frac{ds}{F(s)} \le 2 \sum_{i=1}^{n-1} U_i B(i, U_i). \tag{19}$$

Denoting

$$\int_{\varepsilon}^{X} \frac{ds}{F(s)} = G(x), \quad \text{where } \varepsilon \text{ is a positive constant} \tag{20}$$

we obtain that

$$\int_{b_1}^{b_n} \frac{ds}{F(s)} = G(b_n) - G(b_1).$$

From this and (19) we can observe

$$G(b_n) \le G(b_1) + 2 \sum_{i=1}^{n-1} U_i B(i, U_i). \tag{21}$$

Function G is increasing from (20) and properties of function F. Then,

$$G^{-1} \text{ is increasing.} \tag{22}$$

We have two possibilities:

(i) $\lim_{x\to\infty} G(x) = \infty$. Then $G(b_1) + 2\sum_{i=1}^{n-1} U_i B(i, U_i)$ belongs to the domain of function G^{-1}, for every $n \in N$.

(ii) $\lim_{x\to\infty} G(x) = g < \infty$. We can take, because of condition (4), $b_1 = |A_1| + |B_1|$ which implies

$$G(b_1) + 2\sum_{i=1}^{\infty} U_i B(i, U_i) < g.$$

Then $G(b_1) + 2\sum_{i=1}^{\infty} U_i B(i, U_i)$ belongs to domain of function G^{-1} in this case, too.

Above, (21) and (22) are followed by

$$b_n \leq G^{-1}\Big\{G(b_1) + 2\sum_{i=1}^{n-1} U_i B(i, U_i)\Big\}.$$

Applying (15) we have

$$h_n \leq G^{-1}\Big\{G(b_1) + 2\sum_{i=1}^{n-1} U_i B(i, U_i)\Big\}.$$

From (5) and (13) we have

$$|A_n| + |B_n| \leq G^{-1}\{|A_1| + |B_1| + 2K\} = C_1 < \infty, \quad \text{where } n \in N.$$

Properties of function B and (3) give the following inequalities

$$\begin{aligned} |v_{j+1}||f(j, A_j u_j + B_j v_j)| &\leq U_j B(j, |A_j u_j + B_j v_j|) \\ &\leq U_j B\,(j, U_j(|A_j| + |B_j|)\,) \\ &\leq U_j B(j, U_j C_1) \leq F(C_1) U_j B(j, U_j) = F(C_1)K. \end{aligned}$$

This means that the series

$$\sum_{j=1}^{\infty} v_{j+1} f(j, A_j u_j + B_j v_j)$$

is absolute convergent. By (12) finite limit $\lim_{n\to\infty} A_n = \alpha$ exists. Analogously $\lim_{n\to\infty} B_n = \beta < \infty$ exists.

Therefore the assertion of Lemma 2 follows from (12).

THEOREM 1. *Let u, v denote linearly independent solutions of the difference equation* (1). *Moreover, let $a : N \to R$ and function $f : N \times R \to R$ possessing the following properties*

$$|f(n, x)| \leq B(n, |x|), \quad \textit{for every } x \in R$$

where $B \in B_F$ and F fulfil the condition

$$\lim_{x\to 0} \int_{\varepsilon}^{x} \frac{ds}{F(s)} = -\infty \quad \textit{for a positive constant } \varepsilon.$$

If

$$\sum_{j=1}^{\infty} U_j B(j, U_j) = K < \infty$$

(for some positive constant K) where

$$U_j = \max\{|u_j|, |v_j|, |u_{j+1}|, |v_{j+1}|\}$$

then solution of the equation exists

$$\Delta^2 y_n = a_n y_{n+1} + f(n, y_n), \quad n \in N$$

which can be written down in the form

$$y_n = \alpha_n u_n + \beta_n v_n$$

where $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$.

Proof. Let us choose two linearly independent solutions $\overline{u}$, $\overline{v}$ of (1). Let u and v be two linearly independent solutions of (1) fulfilling condition (2).

Then for some constant c_i, $i = 1,2,3,4$ such that

$$\begin{vmatrix} c_1 & c_2 \\ c_3 & c_4 \end{vmatrix} \neq 0,$$

we have

$$u = c_1\overline{u} + c_2\overline{v}, \quad v = c_3\overline{u} + c_4\overline{v}. \tag{23}$$

By Lemma 2 solution of (7) exists and

$$y_n = \alpha_n u_n + \beta_n v_n \tag{24}$$

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = \beta. \tag{25}$$

Using (23) in (24) we get the following result

$$y_n = \overline{u}_n(\alpha_n c_1 + \beta_n c_3) + \overline{v}_n(\alpha_n c_2 + \beta_n c_4).$$

From (25) we can observe that

$$\lim_{n\to\infty}(\alpha_n c_1 + \beta_n c_3) = \overline{\alpha} < \infty \quad \text{and} \quad \lim_{n\to\infty}(\alpha_n c_2 + \beta_n c_4) = \overline{\beta} < \infty$$

exist so the theorem is proved.

EXAMPLE 1. The next equation

$$\Delta^2 y_n = -2y_{n+1} + \frac{1}{n^2} y_n^m, \quad m \geq 1 \tag{$*$}$$

is considered. The following sequences

$$u = \{1, 0, -1, 0, 1, 0, -1, \ldots\} \quad \text{and} \quad v = \{0, 1, 0, -1, 0, 1, 0, \ldots\}$$

are linearly independent solutions of the equation

$$\Delta^2 z = -2z_{n+1}.$$

Let $B(n,x) = \frac{1}{n^2}x^m$ and $F(k) = k^m$. Then assumptions of the Theorem 1 hold and the solution of equation (*) exists. It can be written down in the form

$$y_n = \alpha_n u_n + \beta_n v_n$$

where

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = \beta.$$

THEOREM 2. *Let function $f : N \times R \to R$ fulfil conditions* (3) *and* (4) *of Lemma* 2. *If*

$$\sum_{j=1}^{\infty} jB(j,j) = k < \infty \quad (\textit{for some positive constant } k) \tag{26}$$

then the solution of the equation

$$\Delta^2 y_n = f(n, y_n), \quad n \in N \tag{27}$$

exists. It can be written down in the form

$$y_n = an + b + \phi_n \quad \textit{where} \lim_{n\to\infty} \phi_n = 0.$$

Proof. Equation

$$\Delta^2 z_n = 0$$

has two linearly independent solutions $u_n = n$, $v_n = 1$, $n \in N$. These solutions satisfy condition (2) of Lemma 1. We will prove that condition (5) is also satisfied. From Lemma 1, U_j is equal $j+1$. It is worth noticing that condition (26) implies condition (5) of Lemma 2. Indeed,

$$\begin{aligned}\sum_{j=1}^{\infty} U_j B(j,U_j) &= \sum_{j=1}^{\infty}(j+1)B(j,j+1) \le \sum_{j=1}^{\infty}(j+j)B(j,j+j) \\ &= \sum_{j=1}^{\infty}(2j)B(j,2j) \le 2F(2)\sum_{j=1}^{\infty} jB(j,j) = 2F(2)k < \infty.\end{aligned}$$

So, we have all assumptions of Lemma 2 are fulfilled and it can be useful for our problem. Solution of equation (27) is

$$y_n = A_n n + B_n \tag{28}$$

where A_n and B_n are defined by (8) and (9) and finite limits of sequences $\{A_n\}$, $\{B_n\}$ exist. Let

$$\lim_{n\to\infty} A_n = a, \quad \lim_{n\to\infty} B_n = b. \tag{29}$$

From (12) we get

$$A_n = A_1 + \sum_{j=1}^{n-1} f(j, jA_j + B_j).$$

From this and (29) we obtain

$$a = A_1 + \sum_{j=1}^{\infty} f(j, jA_j + B_j).$$

Using properties of function f we have

$$|A_n - a| = |\sum_{j=n}^{\infty} f(j, jA_j + B_j)| \le \sum_{j=n}^{\infty} B(j, |jA_j + B_j|) \le \sum_{j=n}^{\infty} B(j, j|A_j| + |B_j|).$$

It is followed by

$$n|A_n - a| \le \sum_{j=n}^{\infty} jB(j, j|A_j| + |B_j|).$$

From (29) constant C exists and

$$|A_n| + |B_n| \le C \quad \text{for } n \in N.$$

Then

$$n|A_n - a| \le \sum_{j=n}^{\infty} jB(j, jC) \le F(C) \sum_{j=n}^{\infty} jB(j, j)$$

and by (26) we have

$$\lim_{n \to \infty} F(C) \sum_{j=n}^{\infty} jB(j, j) = 0$$

what gives

$$\lim_{n \to \infty} n|A_n - a| = 0. \tag{30}$$

The solution (28) of equation (27) can be written in the form

$$y_n = an + b + (A_n - a)n + (B_n - b).$$

This and (30) give us the conclusion:

$$y_n = an + b + \phi(n),$$

where $\phi_n = (A_n - a)n + (B_n - b)$ and $\lim_{n \to \infty} \phi_n = 0$. The theorem is proved.

EXAMPLE 2. The next equation

$$\Delta^2 y_n = \frac{1}{n^5} y_n^2$$

is considered. Let $B(n,x) = \frac{1}{n^5}x^2$ and $F(k) = k^2$. Then the assumptions of the Theorem 2 hold and the solution of this equation exists. It can be written down in the form

$$y_n = an + b + \phi_n \quad \text{where } \lim_{n\to\infty} \phi_n = 0.$$

Remark. We get theorems proved by A. Drozdowicz in [1] as special cases of our results. To get theorems contained therin we take $f(n,x) = g_n x^{2m+1}$ in (6) and (8). In the proof of Lemma 2 estimations like in paper [3] are applied. The second order difference equations were studied also in [1], [4], [5].

References

[1] A. Drozdowicz, *On the asymptotic behaviour of solutions of the second order difference equations*, Glasnik Matematicki 22 (1987), 327–333.

[2] J. W. Hooker, W. T. Patula, *A second-order nonlinear difference equation: oscillation and asymptotic behaviour*, J. Math. Anal. Appl. 91 (1983), 9–23.

[3] J. Popenda, *On the discrete analogy of Gronwall lemma*, Demonstratio Math. 14 (1983), 11–15.

[4] J. Popenda, Werbowski, *On the asymptotic behaviour of the solutions of the difference equations of second order*, Comm. Math. 22 (1980), 135–142.

[5] Z. Szafranski, *On some ocsillation criteria for difference equation of second order*, Fasc. Math. 14 (1985), 91–99.

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Received August 19, 1991; revised version September 15, 1992.