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A POINT OF VIEW ON MEASURES OF NONCOMPACTNESS

1. Introduction

The concept of noncompactness measure is one of the most useful concepts of the general topology. The first measures of noncompactness has been defined by Kuratowski [13] in 1930. Other measures were introduced in [3], [8] and [11]. The use of these measures is discussed for example in [2] and [6]. Banaś and Goebel in their monograph [2] give a review of those measures and characterize the measures of noncompactness in an axiomatic way.

It is of our interest to know whether one can define a measure in such a way that the previous definitions will be contained and advantages of the axiomatic approach will not be wasted. In section 2 we shall give a general scheme of construction of measures of noncompactness in a useful way. This construction was motivated partially by [10]. We shall show the properties of measures and we shall briefly present a discussion for known measures. Section 3 contains an example of application of our measures in the theory of nonlinear differential equations, generalizing the results with the Kuratowski measure of noncompactness [5], [16]. We refer the reader to [3], [7], [9] and [2], [6] with references given there.

2. Measures and their properties

Let $(E, \|.\|)$ denote a Banach space and let B° denote a unit ball of E. Fix some further notation for the families of sets that will be used in the sequel:

$\mathcal{M}_E$ — the family of all nonempty bounded subsets of E,

$\mathcal{N}_E$ — the family of all nonempty and relatively compact subsets of E,

K_E — the family of all convex and bounded neighbourhoods of zero of E.

For fixed E we shall write $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}$ instead $\mathcal{M}_E$, $\mathcal{N}_E$ and $\mathcal{K}_E$.

Let $\mathcal{P} \subseteq \mathcal{N}$ be a family of sets such that:

(A1) $X \in \mathcal{P} \Rightarrow \overline{X} \in \mathcal{P}$,

(A2) $X \in \mathcal{P}, Y \neq \emptyset, Y \subset X \Rightarrow Y \in \mathcal{P}$,

(A3) $X \in \mathcal{P} \Rightarrow \operatorname{conv} X \in \mathcal{P}$,

(A4) The subfamily of all closed sets in $\mathcal{P}$ is closed in the family of all nonempty bounded and closed subsets of E with respect to the Hausdorff topology.

Following Banaś and Goebel [2] we introduce the following notation:

DEFINITION 2.1. The function $\mu : \mathcal{M} \to [0, \infty)$ is said to be a measure of noncompactness with the kernel $\mathcal{P}$ if it is a subject to the following conditions:

(B1) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{P}$,

(B2) $\mu(X) = \mu(\overline{X})$,

(B3) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

(B4) $\mu(\operatorname{conv} X) = \mu(X)$.

Denote by $\mathcal{B}$ a basis of neighbourhoods of zero which is composed of closed convex sets. Let $\mathcal{B}' = \{rB : B \in \mathcal{B},\ r > 0\}$.

Now, we introduce some class of functions $p : \mathcal{B}' \to [0, \infty)$ satisfying the following conditions: $(X, Y \in \mathcal{B}')$

(C1) $X \subset Y \Rightarrow p(X) \leq p(Y)$,

(C2) $p(\operatorname{conv} X) = p(X)$,

(C3) $\forall \varepsilon > 0\ \exists V \in \mathcal{B}'\ p(V) \leq \varepsilon$,

(C4) $p(V) > 0$ whenever $V \notin \mathcal{P}$.

A function p satisfying (C1)–(C4) will be called a p-function.

DEFINITION 2.2. The function $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a $(\mathcal{P}, \mathcal{B}, p)$-measure of noncompactness $[(\mathcal{P}, \mathcal{B}, p) - mnc]$ iff

$$\mu(W) = \inf\{\varepsilon > 0 : \exists H \in \mathcal{P}\ W \subset H + V,\ V \in \mathcal{B}',\ p(V) \leq \varepsilon\},$$

where $W \in \mathcal{M}_E$.

THEOREM 2.1. *Each* $(\mathcal{P}, \mathcal{B}, p) - mnc$ *is a measure of noncompactness (in the sense of Definition* 2.1.*).*

Proof. (B3) $X, Y \in \mathcal{M}$, $X \subset Y$, let $H \in \mathcal{P}$ and $V \in \mathcal{B}'$ be such that $Y \subset H + V$, $p(V) \leq \mu(Y) + \varepsilon$. So $X \subset H + V$ and $\mu(X) \leq \mu(Y) + \varepsilon$. As ε is arbitrary we have $\mu(X) \leq \mu(Y)$.

(B4) $X \subset \operatorname{conv} X$, so by (B3) $\mu(X) \leq \mu(\operatorname{conv} X)$, if $H \in \mathcal{P}$ and $V \in \mathcal{B}'$ be such that $X \subset H + V$, $p(V) \leq \mu(X) + \varepsilon$ then $\operatorname{conv} X \subset \operatorname{conv} H + V$. But $\operatorname{conv} H \in \mathcal{P}$, so $\mu(\operatorname{conv} X) \leq p(V) \leq \mu(X) + \varepsilon$. Finally $\mu(X) = \mu(\operatorname{conv} X)$.

(B2) As above let $X \subset H + V$, $p(V) \leq \mu(X) + \varepsilon$, so $\overline{X} \subset \overline{H} + V$ and $\mu(\overline{X}) \leq p(V) \leq \mu(X) + \varepsilon$. Consequently $\mu(\overline{X}) = \mu(X)$ (by (B3)).

(B1) $\mu(X) = 0 \Rightarrow \mu(\overline{X}) = 0 \Rightarrow \forall V \in \mathcal{B}'\ \exists H \in \mathcal{P}\ \overline{X} \subset H + V$ and by the closedness of $\mathcal{P}$ in the Hausdorff topology $\overline{X} \in \mathcal{P}$. Thus $X \subset \overline{X}$ and by (A2) $X \in \mathcal{P}$.

$X \in \mathcal{P} \Rightarrow X \subset \overline{X} + V$ for each $V \in \mathcal{B}'$. Obviously $\overline{X} \in \mathcal{P}$. But for every $\varepsilon > 0$ there exists $V \in \mathcal{B}'$ such that $p(V) \leq \varepsilon$, so finally $\mu(X) = 0$.

Remarks. If $\mathcal{P} = \mathcal{N}_E$ (so-called full measures) then we can replace the set $H \in \mathcal{P}$ by the finite set. It is necessary to remark that our viewpoint sheds some new light on measures of noncompactness. This concept of construction of mnc gives a general class of noncompactness measures containing the Kuratowski *mnc* α

$$\left(\alpha(W) = \inf\left(\varepsilon > 0 : W \subset \bigcup_{i=1}^{n} A_i,\ \operatorname{diam} A_i \leq \varepsilon,\ i = 1, \dots, n\right),\ W \in M\right),$$

The Hausdorff *mnc* $\beta(\beta(W) = \inf(\varepsilon > 0; W \subset \{x_1, \dots, x_n\} + \varepsilon B^0), W \in M)$ and many others (see [2]). For example: the Kuratowski mnc is the $(\mathcal{N}_E, \mathcal{K}, \operatorname{diam}(.) - mnc$, the Hausdorff *mnc* is the $(\mathcal{N}_E, \mathcal{B}^0, \|.\|) - mnc$ $(\mathcal{B}^0 = \{rB^0 : r > 0\})$, the norm of the set is the $(\{0\}, \mathcal{K}, \|.\|) - mnc$ and the diameter of the set is the $(E_1, \mathcal{K}, \operatorname{diam}(.)) - mnc$, where E_1 is a family of all one-point sets (singeltons). Choosing $\mathcal{P}$, $\mathcal{B}$ and p we can obtain the different measures.

The special cases:

(i) Let V be a bounded closed neighbourhood of zero of E. So $\mathcal{B}^v = \{rV : r > 0\} = \mathcal{B}^{v'}$, is the basis of neighbourhoods of zero of E. If $\mathcal{P} = \mathcal{N}_E$ then by $V - mnc$ we denote the $\{\mathcal{P}, \mathcal{B}^v, p^v\} - mnc$. Compare: the Hausdorff mnc is $\mathcal{B}^0 - mnc$.

(ii) Two-steps procedure (cf. [10]).

$$Q(W) = \{B \in \mathcal{B} : W \subset H + B \text{ for some } H \in \mathcal{P}\}$$

(so-called measure of precompactness)

$$\mu(W) = \inf\{\varepsilon > 0 : p(B) \leq \varepsilon,\ \forall B \in Q(W)\}.$$

We recall that $D(V, \mathcal{P})$ denote the Hausdorff distance between V and $\mathcal{P}$ (cf. [2]).

COROLLARY 2.1. ([2] th. 3.1.1.): *For arbitrary kernel* $\mathcal{P}$ $(\mathcal{P}, \mathcal{K}, D(., \mathcal{P}) - mnc$ *is the measure of noncompactness.*

For the proof it suffices to check the properties of $D(., \mathcal{P})$.

Now, we can give the properties of $(\mathcal{P}, \mathcal{B}, p)$-measures. Our lemmas are parallel to those given in [2], [6] and [12]. We omit the proofs on acount of its clearity.

LEMMA 2.1. *If* $E_\mu = \{x \in E : \{x\} \in \mathcal{P}\}$ *and for every* $A \in \mathcal{P}$ *and* $x \in E_\mu$ *we have* $A \cup \{x\} \in \mathcal{P}$ *then each* $(\mathcal{P}, \mathcal{B}, p) - mnc$ *has the following property*:
(B6) $\mu(A \cup \{x\}) = \mu(A), \quad x \in E_\mu,\ A \in \mathcal{M}_E$.

LEMMA 2.2. *Each of the full* $(\mathcal{P}, \mathcal{B}, p) - mnc$ μ *has the following property*:
(B7) *If* $A_n \in \mathcal{M}_E, A_n = \overline{A}_n, A_{n+1} \subset A_n$, $n = 1, 2, \ldots$ *and* $\lim_{n\to\infty} \mu(A_n) = 0$ *then* $A_\infty = \bigcap_{n=1}^\infty A_n \neq \emptyset$ *and* $A_\infty \in \mathcal{P}$.

LEMMA 2.3. *Each* $(\mathcal{P}, \mathcal{B}, p) - mnc$ *satisfies*:
(B8) $\mu(A \cap B) \le \min(\mu(A), \mu(B)),\ A, B \in \mathcal{M}_E,\ A \cap B \neq 0$.

LEMMA 2.4. *Assume that*:
(A6) $A, B \in \mathcal{P} \Rightarrow A \cup B \in \mathcal{P}$,
(C5) $\forall U, V \in \mathcal{B}'\ \exists W \in \mathcal{B}'\ W \supset V, U$ *and* $p(W) = p(U)$ *or* $p(W) = p(V)$.
Under the above assumptions $(\mathcal{P}, \mathcal{B}, p) - mnc$ *has the maximum property*:
(B9) $\mu(A \cup B) = \max(\mu(A), \mu(B))$.

Remark. Naturally $\mathcal{B}^{0\prime}, \mathcal{B}^{v\prime}$ and $\mathcal{K}$ satisfy (C5) for each p-function $p(.)$.

EXAMPLE 1. Let $P = \{0\}$.
Let $A = \{(x, y) \in \mathbb{R}^2 : 3x - 3 \le y \le 3x + 3,\ -3x - 3 \le y \le -3x + 3\}$ and let $B = \{(x, y) \in \mathbb{R}^2 : \frac{1}{2}x - 1 \le y \le \frac{1}{2}x + 1,\ -\frac{1}{2}x - 1 \le y \le -\frac{1}{2}x + 1\}$. Put $\mathcal{B} = \{rA, rB : r > 0\} = \mathcal{B}'$ and $p(rA) = p(rB) = r$. Thus $(\mathcal{B}, p)$ does not satisfy (C5) and we see that: $\mu(A) = \mu(B) = 1 (A \subset \{0\} + A, B \subset \{0\} + B)$, but $A \cup B \subset \{0\} + 3B$ or $A \cup B \subset \{0\} + 2A$. It is clear that $\mu(A \cup B) = 2 > \max(\mu(A), \mu(B)) = 1$. This $(\mathcal{P}, \mathcal{B}, p) - mnc$ not satisfy (B9).

LEMMA 2.5. *Assume that*:
(A7) $A \in \mathcal{P},\ k > 0 \Rightarrow kA \in \mathcal{P}$,
(C6) $p(kV) = kp(V),\ V \in \mathcal{B}',\ k > 0$.
Under the above assumptions $(\mathcal{P}, \mathcal{B}, p) - mnc\,\mu$ *has the following property*:
(B10) $\mu(kA) = k\mu(A),\ k > 0,\ A \in \mathcal{M}_E$.

Replacing (A7) by
(A7)′ $A \in \mathcal{P},\ k \in \mathbb{R} \Rightarrow kA \in \mathcal{P}$
and (C6) by
(C6)′ $kV \in \mathcal{B}', p(kV) = |k| p(V), \quad V \in \mathcal{B}', k \in \mathbb{R}$
we obtain
(B10)′ $\mu(kV) = |k|\mu(V), \quad V \in \mathcal{B}', k \in \mathbb{R}$.
Now we shall investigate the subadditivity of the $(\mathcal{P}, \mathcal{B}, p) - mnc$:

LEMMA 2.6. *Assuming*:
(A8) $A, B \in \mathcal{P} \Rightarrow A + B \in \mathcal{P}$,
(C7) $V, U \in \mathcal{B}' \Rightarrow V + U \in \mathcal{B}'$ *and* $p(V + U) \le p(V) + p(U)$, *we obtain the* $(\mathcal{P}, \mathcal{B}, p) - mnc\,\mu$ *satisfying*:
(B11) $\mu(A + B) \le \mu(A) + \mu(B),\ A, B \in \mathcal{M}_E$.

COROLLARY 2.2. *Let* $\{0\} \in E_\mu$. *If we assume* (A7) *and* (C6) *then:*

$$\Big[\Big(\sum_{i=1}^{n} t_i = T,\ t_i \geq 0,\ A_i \in \mathcal{M}_E,\ i = 1,2,\ldots,n\Big) \Rightarrow \mu\Big(\sum_{i=1}^{n} t_i A_i\Big) \leq \sum_{i=1}^{n} t_i \mu(A_i)\Big].$$

Proof. If $T > 0$ then

$$\mu\Big(\sum_{i=1}^{n} t_i A_i\Big) = \mu\Big(\sum_{i=1}^{n}\Big(T \cdot \frac{t_i}{T} \cdot A_i\Big)\Big) = T \cdot \mu\Big(\sum_{i=1}^{n} \frac{t_i}{T} A_i\Big) \leq T \cdot \sum_{i=1}^{n} \frac{t_i}{T} \cdot \mu(A_i) = \sum_{i=1}^{n} t_i \mu(A_i).$$

The case when $T = 0$ is obvious.

COROLLARY 2.3. $[\{0\} \in E_\mu, \text{(B10)}] \Rightarrow \text{(B11)}$.

The proof is obvious.

The most general scheme of construction of measures in [2] which is described in paragraph 12.2 can be obtained by taking

$$\mathcal{P} = \{M \in \mathcal{M}_E : \limsup_{n\to\infty} \sup_{x\in M} f_n(x) = 0\},$$
$$\mathcal{B} = \mathcal{K} \text{ and } p(V) = \|V\|,\ V \in \mathcal{B}.$$

Here $(f_n)_{n\in\mathbb{N}}$ is a sequence of real nonnegative functionals defined on E which are lower semicontinuous, equibounded on each bounded set, homogeneous and subadditive, and which are subject to the following condition:

$$M \in \mathcal{M}_E \Rightarrow (\limsup_{n\to\infty} \sup_{x\in M} f_n(x) = 0 \Rightarrow M \in \mathcal{N}_E).$$

Certainly, this is well-defined $(\mathcal{P},\mathcal{B},p) - mnc$. It suffices to verify our assumptions on $\mathcal{P}, \mathcal{B}$ and p. We see, that this measure has the maximum property (B9).

The choice of pair $(\mathcal{B}, p)$ is ambiguous. For example, let $A \in \mathcal{K}, M \in \mathcal{M}_E$. Put $\mathcal{B}_1 = \mathcal{K}$ and $p_A(M) = \sup_{x\in M}(\inf_{r>0} x \in rA)$ and $\mathcal{B}_z = \mathcal{B}^A, p(rA) = r$.

For each fixed kernel $\mathcal{P}$ $(\mathcal{P},\mathcal{B}_1,p_A) - mnc$ and $(\mathcal{P},\mathcal{B}_z,p) - mnc$ are the same. Clearly this measure is positively homogeneous, but it is homogeneous only for balanced sets A.

The other properties of $(\mathcal{P},\mathcal{B},p) - mnc$ will be given in the next section.

3. Example of application

An application of $(\mathcal{P},\mathcal{B},p) - mnc$ in the theory of differential equations is immediate.

We shall need the several lemmas

LEMMA 3.1. *Let $L : E \to E$ be a continuous linear mapping from E to E, V — a bounded balanced neighbourhood of zero of E. So*

$$LV \subset |L| \cdot V.$$

Proof. LV is a bounded neighbourhood of zero, so there exists a constant $t > 0$ such that $LV \subset t \cdot V$. We shall show that the constant $|L|$ satisfies this condition. Fix arbitrary $y \in LV$. Let Z denote a linear span of $\{0, y\}$, $A = V \cap Z$, $B = LA$, $r = \|A\|$.

If there exists $x \in A$ such that $\|x\| = r$ then by B_r we shall understand a closed ball with center at zero and with radius r, if not then by B_r we shall understand an open ball. Thus $A = B_r \cap Z$, since V is a balanced set.

But $LB_r \subset |L| \cdot B_r$ (since $\|Lx\| \le |L| \cdot \|x\|$), and

$$\begin{aligned} B = LA = L(V \cap Z) &= L(B_r \cap Z) \subset LB_r \cap LZ \subset |L| \cdot B_r \cap Z \\ &= |L| \cdot (B_r \cap (Z/|L|)) = |L| \cdot (B_r \cap Z) \\ &= |L| \cdot (V \cap Z) \subset |L| \cdot V. \end{aligned}$$

Therefore $y \in |L| \cdot V$ and finally $LV \subset |L| \cdot V$.

LEMMA 3.2. *Denote by μ a $(\mathcal{P}, \mathcal{B}, p)$ – mnc with:*

(i) *kernel $\mathcal{P}$ is closed with respect to continuous linear operations,*

(ii) *$\mathcal{B}$ is composed of balanced sets,*

(iii) *p satisfies* (C6).

Thus for each bounded subset W of E and for each $L \in L(E)$ we have:

$$\mu(LW) \le |L| \cdot \mu(W).$$

Proof. Fix arbitrary $\varepsilon > 0$. Let $H \in \mathcal{P}$ and $V \in \mathcal{B}'$ be such that $W \subset H + V$ with $p(V) = \mu(W) + \varepsilon$. Therefore $LW \subset LM + LV \subset H' + |L| \cdot V$, where $H' = LH \in \mathcal{P}$.

Moreover $p(|L| \cdot V) = |L| \cdot p(V) = |L| \cdot (\mu(W) + \varepsilon) = |L| \cdot \mu(W) + |L| \cdot \varepsilon$. As ε is arbitrary we obtain our assertion.

Remark. If $\mathcal{B}$ is composed of balanced sets then (C6)$\equiv$(C6)$'$. Here $\mathcal{P} = \mathcal{N}$, $\mathcal{P} = \{0\}$ and $\mathcal{P} = E_1$ satisfies (i). It suffices that the basis $\mathcal{B}$ contains a subbasis $\mathcal{B}_1$ composed of balanced sets such that:

(D1) $\quad \forall k \ge 0 \ \exists W \in \mathcal{B}_1, W \in p^{-1}(k)$.

So Kuratowski measure of noncompactness satisfies the assertion of this lemma too.

LEMMA 3.3. *Assume the closedness of $\mathcal{P}$ with respect to continuous linear operations and let $\mathcal{P}$ satisfy* (A6). *Let the basis $\mathcal{B}$ contain a subbasis composed of balanced neighbourhoods as in* (D1) *and let p satisfy* (C6). *If K*

is a continuous mapping from a compact interval I of $\mathbb{R}$ to $L(E)$ and W is a bounded subset of E then

$$\mu\Big(\bigcup_{t\in I} K(t)W\Big) \le \sup_{t\in I} |K(t)| \cdot \mu(W).$$

Proof. As W is bounded there exists $b > 0$ such that $\|W\| \le b$. Fix arbitrary $\varepsilon > 0$. Let $W \subset p+V$ for some $P \in \mathcal{P}$ and $V \in \mathcal{B}'$, $p(V) = \mu(W)+\varepsilon$. Put $U = \varepsilon \cdot V$, so $U \in \mathcal{B}'$. Let $\delta > 0$ be such that $B(0,\delta) \subset U$. Divide the interval $I := [t_0, T]$ in such a way that $t_1 < t_2 < \ldots < t_n = T$ with $|K(t_i) - K(t_{i-1})| < \delta/n$ (by continuity of K). For $t \in [t_{i-1}, t_i]$, denoting $K(t)W \div K(t_i)W := (K(t) - K(t_i))W = \{K(t)w - K(t_i)w : w \in W\}$ we obtain

$$K(t)W \subset (K(t)W \div K(t_i)W) + K(t_i)W.$$

But $\|K(t)W \div K(t_i)W\| \le (\delta/b)\cdot b = \delta$, so $K(t)W \div K(t_i)W \subset B(0,\delta) \subset U$. Since $W \subset P + W$ thus by lemma 3.2. $K(t_i)W \subset K(t_i)P + K(t_i)V \subset K(t_i)P + \sup_{t\in I} |K(t)| \cdot V$.

Now

$$\begin{aligned}
\bigcup_{t\in I} K(t)W &= \bigcup_{i=1}^{n} \bigcup_{t\in[t_{i-1},t_i]} K(t)W \\
&\subset \bigcup_{i=1}^{n} [(K(t)W \div K(t_i)W) + K(t_i)W] \\
&\subset \bigcup_{i=1}^{n} [U + K(t_i)P + \sup_{t\in I} |K(t)| \cdot V] \\
&\subset U + \sup_{t\in I} |K(t)| \cdot V + \bigcup_{i=1}^{n} K(t_i)P \\
&= \mathcal{P}' + (\varepsilon + \sup_{t\in I} |K(t)|) \cdot V \text{ (by convexity of } V),
\end{aligned}$$

where $P' = \bigcup_{i=1}^{n} K(t_i)P$.

We have $p((\varepsilon + \sup_{t\in I} |K(t)|) \cdot V) = (\varepsilon + \sup_{t\in I} |K(t)|) \cdot p(V) = (\varepsilon + \sup_{t\in I} |K(t)|) \cdot (\mu(W) + \varepsilon)$ and since ε is arbitrary $\mu(\bigcup_{t\in I} K(t)W) \le \sup_{t\in I} |K(t)| \cdot \mu(W)$.

LEMMA 3.4. *Denote by $C(\mathbb{R}, E)$ the space of all continuous functions from $\mathbb{R}$ to E equipped with the topology of almost uniform convergence. Let μ be a full $(\mathcal{P}, \mathcal{B}, p)$ – mnc on E and let W be a bounded equicontinuous subset of $C(\mathbb{R}, E)$. For any subset X of W put*

$$\gamma(X) = \sup_{t\in\mathbb{R}} \mu(X(t)).$$

Then the index γ has the following properties:

(i) $X \subset Y \Rightarrow \gamma(X) \le \gamma(Y)$,

(ii) $\gamma(\overline{\text{conv}}X) = \gamma(X)$, $X \in \mathcal{M}_E$,

(iii) $\gamma(X \cup \{x\}) = \gamma(X)$, $X \in \mathcal{M}_E$, $x \in E_\mu$,

(iv) *if* $\gamma(X) = 0$ *then* X *is relatively compact in* $C(\mathbb{R}, E)$.

Proof. In view of corresponding properties of μ (i), (ii) and (iii) are obvious. Since $C(\mathbb{R}, E)$ is taken with the topology of almost uniform convergence then Ascoli's theorem proves that for any subset X of W we have:

$\gamma(X) = 0 \Leftrightarrow \mu(X(t)) = 0$ for every $t \in \mathbb{R} \Rightarrow X(t)$ is relatively compact in E for every $t \in \mathbb{R} \Leftrightarrow X$ is relatively compact in $C(\mathbb{R}, E)$.

Now, we need a fixed point theorem of Schauder type.

LEMMA 3.5. *Let W be a bounded closed and convex subset of $C(\mathbb{R}, E)$. Let $\gamma : 2^W \to [0, \infty)$ satisfies the conditions* (i)–(iv) *of Lemma* 3.4. *Assume $F : W \to W$ is a continuous mapping satisfying*

$$\gamma(F(X)) < \gamma(X)$$

for arbitrary $X \subset W$ with $\gamma(X) > 0$. Then F has a fixed point in W.

Let $A : \mathbb{R} \to L(E)$ be strongly measurable and Bochner integrable on every finite subinterval of $\mathbb{R}$. Suppose that linear differential equation $x'(t) = A(t)x(t)$ admits a regular exponential dichotomy ([4], [15]). Denote by G the main Green function for this equation.

Let $f : \mathbb{R} \cdot E \to E$ be continuous with $\|f(t,x)\| \le m(t)$ for $t \in \mathbb{R}$ and $x \in E$, where m is a locally integrable function on $\mathbb{R}$ with

$$\sup \int_t^{t+1} m(s)\, ds \le M < \infty \text{ (cf. [15], [16])}.$$

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with

$$L = \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} |G(t,s)| g(s)\, ds < \infty$$

and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing such that $L \cdot h(t) < t$ for $t > 0$.

If in addition we assume that $\mu(f(I \times X)) \le \sup_{t \in I} g(t) \cdot h(\mu(X))$ for any compact subset I of $\mathbb{R}$, for each $X \in \mathcal{M}_E$ and for the full $(\mathcal{P}, \mathcal{B}, p) - mnc\, \mu$ such that p satisfies (C6).

THEOREM 3.1. *Under the conditions stated above, the differential equation $x'(t) = A(t)x(t) + f(t, x(t))$ has a bounded solution on $\mathbb{R}$.*

Proof. We begin by defining a mapping $F : C(\mathbb{R}, E) \to C(\mathbb{R}, E)$ as follows

$$F(x)(t) = \int_{\mathbb{R}} G(t,s)f(s,x(s))\,ds\,.$$

We see that is suffices to find a fixed point of F.

By properties of G [4] there exists a constant $N \geq 1$ and a constant $k > 0$ such that $|G(t,s)| \leq N \cdot e^{-k|t-s|}$ for each $t, s \in \mathbb{R}$.

Let W be a subset of $C(\mathbb{R}, E)$ such that $x \in W$ iff

$$\|x(t)\| \leq K = 2NM(1-e^{-k})^{-1} \text{ and}$$
$$\|x(t) - x(\tau)\| \leq K \cdot \int_t^\tau |A(s)|\,ds + \int_t^\tau m(s)\,ds$$

for $t, \tau \in \mathbb{R}$ and $t < \tau$. So W is a bounded closed equicontinuous and convex subset of $C(\mathbb{R}, E)$. It remains to prove that F is a continuous mapping from W into W. Let $x \in W$. We can estimate: $\|F(x)(t)\| \leq K$. Therefore the function $F(x)$ is a solution of the differential equation $y'(t) = A(t)x(t) + f(t, x(t))$ on $\mathbb{R}$ (see [4]). Hence $\|F(x)(t) - F(x)(\tau)\| \leq K \cdot \int_t^\tau |A(s)|ds + \int_t^\tau m(s)ds$ whenever $t < \tau$. Thus $F(x) \in W$ whenever $x \in W$. If f is continuous then the operator $x(.) \to f(., x(.))$ is continuous from $C(\mathbb{R}, E)$ into itself.

Fix arbitrary $\varepsilon > 0$. Let $x, y \in W, t \in \mathbb{R}$ and $a > 0$ be such that $K \cdot e^{-k\alpha} < \varepsilon$.

$$\begin{aligned}
&\|F(x)(t) - F(y)(t)\| \\
&\quad \leq N \cdot \Big(\int_{-\infty}^{t-\alpha} \int_{t-\alpha}^{t+\alpha} \int_{t+\alpha}^{\infty} \Big) e^{-k|t-s|} \cdot \|f(s,x(s)) - f(s,y))\|\,ds \\
&\quad \leq N \cdot \sup_{t-\alpha \leq s \leq t+\alpha} \|f(s,x(s)) - f(s,y(s))\| \cdot \int_{t-\alpha}^{t+\alpha} e^{-k|t-s|}\,ds \\
&\qquad + 2N \cdot \Big(\int_{-\infty}^{t-\alpha} + \int_{t+\alpha}^{\infty} \Big) e^{-k|t-s|} m(s)\,ds \\
&\quad \leq 2Nk^{-1}(1-e^{-k\alpha}) \cdot \sup_{t-\alpha \leq s \leq t+\alpha} \|f(s,x(s)) - f(s,y(s))\| + K \cdot e^{-k\alpha}\,.
\end{aligned}$$

Thus F is continuous. Fix arbitrary $\varepsilon > 0$. By (C4) there exists $V \in \mathcal{B}'$ such that $p(V) \leq \varepsilon$. Let us denote by δ a positive constant such that $B(0,\delta) \subset V$ and by l such a constant that $K \cdot e^{-kl} < \delta$. So

$$\Big\| \Big\{ \int_{-\infty}^{t-l} G(t,s)f(s,x(s))\,ds : x \in X \Big\} \Big\| \leq K \cdot e^{-kl} < \delta$$

and $W \subset \{0\} + B(0,\delta) \subset \{0\} + V$.

By definition of μ we have $\mu(\{\int_{-\infty}^{t-1} G(t,s)f(s,x(s)\},ds : x \in V) \leq p(V) \leq \varepsilon$. Analogously $\mu(\{\int_{t+l}^{\infty} G(t,s)f(s,x(s))\,ds : x \in X\} \leq \varepsilon$. It suffices to show that $\mu(\{\int_{t-l}^{t+l} G(t,s)f(s,x(s))ds : x \in X\} \leq h(\mu(X([t-1, t+1]))) \cdot \int_{\mathbb{R}} |G(t,s)|g(s)\,ds$.

For arbitrary $\varepsilon > 0$, by continuity of g and $G(t,.)$ we can find a $\delta > 0$ such that $|s'-s''| < \delta \Rightarrow |g(s')-g(s'')| < \varepsilon$ and $|G(t,s')-G(t,s'')| < \varepsilon$ with $s', s'' \in [t-1,t]$ or $s', s'' \in [t, t+1]$. Let $t-1 = t_0 < t_1 < \ldots < t_m = t < t_{m+1} < \ldots < t_{2m} = t+1$ with $t_{i-1} - t_i < \delta$. Let I_i denote an interval $[t_{i-1}, t_i]$ $(i = 1, 2, \ldots, 2m)$ and I an interval $[t-1, t+1]$. Let $a_i, b_i \in I_i$ be such that $|G(t,a_i)| = \sup_{s\in I_i} |G(t,s)|$, $g(b_i) = \sup_{s \in I_i} g(s)$ and let c_1, c_2 are equal respectively $\sup_{s\in I} |G(t,s)|$ and $\sup_{s\in I} g(s)$.

By the mean value theorem we have

$$\Big\{ \int_{t-l}^{t+l} G(t,s)f(s,x(s))\,ds : x \in X \Big\} \subset \sum_{i=1}^{2m} (t_i - t_{i-1} \overline{\operatorname{conv}}\Big(\bigcup_{s\in I_i} G(t,s)f(I_i \times X(I))\Big)$$

and by our Lemma 3.3. and the corresponding properties of μ from this we obtain

$$\begin{aligned}
&\mu\Big\{ \int_{t-l}^{t+l} G(t,s)f(s,x(s))\,ds : x \in X \Big\} \\
&\leq \mu\Big(\sum_{i=1}^{2m} (t_i - t_{i-1}) \overline{\operatorname{conv}}\Big(\bigcup_{s\in I_i} G(t,s)f(I_i \times X(I))\Big) \\
&\leq \sum_{i=1}^{2m} (t_i - t_{i-1}) \cdot \sup\{|G(t,s)| : s \in I_i\} \cdot \sup\{g(s) : s \in I_i\} \cdot h(\mu(X(I))) \\
&\leq h(\mu(X(I))) \cdot \sum_{i=1}^{2m} (t_i - t_{i-1}) |G(t,a_i)| \cdot g(b_i) \\
&\leq h(\mu(X(I))) \cdot \Big(2l(c_1 + c_2)\varepsilon + \int_{t-l}^{t+l} |G(t,s)| g(s)\,ds \Big).
\end{aligned}$$

As ε is arbitrary then the above inequality proves our claim.

Let γ be as in Lemma 3.4. So $\mu(X(I)) \leq \gamma(X)$ and $\mu(F(X)(t)) \leq 2\varepsilon + h(\mu(X(I))) \cdot \int_{\mathbb{R}} |G(t,s)|g(s)ds$. By our assumptions $\mu(F(X)(t)) \leq L \cdot h(\gamma(X)) < \gamma(X)$ $(\gamma(X) > 0)$. Consequently $\gamma(F(X)) < \gamma(X)$ whenever $\gamma(X) > 0$. Using Lemma 3.5. we obtain a fixed point of F which ends the proof.

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