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ON SYSTEMS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

There have appeared a lot of papers concerning applications of measures of noncompactness to the dfferential equation $x' = f(t,x)$ in Banach spaces (see for example [1]–[4], [10]). In this paper we shall extend these results to finite or infinite systems of differential equations in Banach spaces. More precisely, we shall prove some existence theorems and Aronszajn's type theorems for the systems

$$(1)\qquad \begin{aligned} x_i' &= f_i(t, x_1, x_2, \ldots) \\ x_i(0) &= x_{i0} \end{aligned} \qquad (i = 1, 2, \ldots)$$

and

$$(2)\qquad \begin{aligned} x_i' &= f_i(t, x_1, \ldots, x_m) \\ x_i(0) &= x_{i0} \end{aligned} \qquad (i = 1, \ldots, m).$$

In our proofs an essential role play theorems on systems of differential inequalities from the papers [7] and [6] (see also [8], p. 122 and 360).

1. Infinite systems of differential equations

Assume that $J = \langle 0, a \rangle$ is a compact interval and E_i is a Banach space with a norm $\|\cdot\|_i$ $(i = 1, 2, \ldots)$.

We introduce the following denotations:

$-E = E_1 \times E_2 \times \ldots$ — the Fréchet space of all sequences $x = (x_i)$, $x_i \in E_i$ for $i = 1, 2, \ldots$, with the quasinorm

$$|x| = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x_i\|_i}{1 + \|x_i\|_i};$$

$-C_i = C(J, E_i)$ — the Banach space of all continuous function $u : J \to E_i$ with the norm $\|u\|_{iC} = \sup\{\|u(t)\|_i : t \in J\}$;

$-C = C(J, E)$ — the Fréchet space of all continuous function $u : J \to E$ with the quasinorm $|u|_C = \sup\{|u(t)| : t \in J\}$;

$-\alpha_i, \alpha, \alpha_C$ — the measures of noncompactness in E_i, E, C, respectively.

Assume that for each positive integer i:

I. $(s, x) \to f_i(s, x)$ is a continuous function defined on $J \times E$ with values in E_i;

II. There exists an integrable function $m_i : J \to R_+$ such that

$$\|f_i(s, x)\|_i \le m_i(s) \quad \text{for } s \in J \text{ and } x \in E;$$

III. There exists a function $h = (h_1, h_2, \ldots) : J \times R^\infty \to R^\infty$ such that:

1°. h is continuous in a sequential sense i.e.: if $t_n \in J$, $t_n \to t$ and $y^n = (y_1^n, y_2^n, \ldots) \in R^\infty$, $y_k^n \to y_k^0$ when $n \to \infty$, then for each $j = 1, 2, \ldots h_j(t_n, y_1^n, y_2^n, \ldots) \to h_j(t, y_1^0, y_2^0, \ldots)$ when $n \to \infty$.

2°. If $y_k \le w_k$ for $k \ne i$ $(k = 1, 2, \ldots)$, then

$$h_i(t, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots) \le h_i(t, w_1, \ldots, w_{i-1}, y_i, w_{i+1}, \ldots).$$

3°. There exists a sequence (M_j) of positive numbers such that

$$|h_j(t, y_1, y_2, \ldots)| \le M_j, \quad j = 1, 2, \ldots, \ (t, y_1, y_2, \ldots) \in J \times R^\infty;$$

4°. For each c $(0 < c \le d)$ the function $u = 0$ is the unique solution on $\langle 0, c\rangle$ of the Cauchy problem:

$$u_i' = 2h_i(t, u_1, u_2, \ldots)$$
$$u_i(0) = 0 \quad (i = 1, 2, \ldots).$$

IV. For each $X = X_1 \times X_2 \times \ldots \subset E$ and $t \in J$

$$\alpha_i(f_i(t, X)) \le h_i(t, \alpha_1(X_1), \alpha_2(X_2), \ldots), \quad i = 1, 2, \ldots.$$

THEOREM 1. *Under the above assumptions, the set S of all continuous solutions of* (1) *is a compact R_δ in C, i.e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

Proof. Let us notice that (1) is equivalent to the equation $x = F(x)$, where $F(x) \coloneqq (F_1(x), F_2(x), \ldots)$ and

$$F_i(x)(t) = x_{i0} + \int_0^t f_i(s, x(s))\, ds \quad \text{for } t \in J, \ x \in C, \ i = 1, 2, \ldots.$$

Fix a positive integer i. As

$$\|F_i(x)(t) - F_i(x)(\tau)\|_i \le \left| \int_i^t \|f_i(s, x(s))\|_i\, ds \right| \le \left| \int_\tau^t m_i(s)\, ds \right|$$

for each $x \in C$, $t, \tau \in J$, $F_i(C)$ is an equicontinuous subset of C_i. Since J is compact, this implies that the set $F_i(C)$ is equi — uniformly continuous, and therefore the numbers

$$\omega_i(d) = \sup\{||u(t) - u(s)||_i : u \in F_i(C); t, s \in J; |t - s| \leq d\} \to 0$$

as $d \to 0_+$. Assume that x^n, $x \in C$ and $\lim_{n\to\infty} |x^n - x|_C = 0$. Then $\lim_{n\to\infty} f_i(s, x^n(s)) = f_i(s, x(s))$ and

$$||f_i(s, x^n(s)) - f_i(s, x(s))||_i \leq 2m_i(s) \quad \text{for } s \in J.$$

Now, by the Lebesque dominated convergence theorem we get

$$\lim_{n\to\infty} \int_0^t ||f_i(s, x^n(s)) - f_i(s, x(s))||_i \, ds = 0, \quad \text{i.e. } \lim_{n\to\infty} F_i(x^n)(t) = F_i(x)(t)$$

for each $t \in J$. Because $F_i(C)$ is equicontinuous, from the above it follows that $\lim_{n\to\infty} ||F_i(x^n) - F_i(x)||_{iC} = 0$. So, $F_i : C \to C_i$ is continuous for any $i \in N$ and therefore $F : C \to C$ is continuous too. Let $\omega(d) = \sup\{|u(t) - u(s)| : u \in F(C); t, s \in J; |t - s| \leq d\}$. Since

$$\omega(d) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\omega_k(d)}{1 + \omega_k(d)} \quad \text{and} \quad \lim_{d\to\infty} \omega_k(d) = 0 \quad \text{for each } k,$$

$\omega(d) \to 0$ as $d \to 0$.

We shall prove that

(3) If $u^n \in C$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} |u^n - F(u^n)|_C = 0$,

then (u^n) has a convergent subsequence. Suppose that $u^n \in C$ $(n = 1, 2, \ldots)$ and

(4) $$\lim_{n\to\infty} |u^n - F(u^n)|_C = 0.$$

Put $V = \{u^n : n = 1, 2, \ldots\}$, $V_i = \{u_i^n : n = 1, 2, \ldots\}$, $V(t) = \{u^n(t) : n = 1, 2, \ldots\}$ and $V_i(t) = \{u_i^n(t) : n = 1, 2, \ldots\}$ $(i = 1, 2, \ldots, t \in J)$. By (4) we infer that $(I - F)(V)$ is an equiuniformly continuous subset of C. Since

(5) $$V \subset (I - F)(V) + F(V)$$

and $F(V)$ is equiuniformly continuous, the set V is equiuniformly continuous too.

Fix i.

From (4) we deduce that $\alpha_i(Z_i(t)) = 0$, where $Z_i(t) = \{u_i^n(t) - F_i(u^n)(t) : n = 1, 2, \ldots\}$. Since $V_i(t) \subset Z_i(t) + F_i(V)(t)$ and $F_i(V)(t) \subset V_i(t) - Z_i(t)$, we have

(6) $$v_i(t) = \alpha_i(V_i(t)) = \alpha_i(F_i(V)(t)) \quad \text{for } t \in J.$$

Next, for any $t, \tau \in J$

$$\begin{aligned}|v_i(t) - v_i(\tau)| &= |\alpha_i(F_i(V)(t)) - \alpha_i(F_i(V)(\tau))| \\ &\le \alpha_i\Big(\Big\{ \int_\tau^t f_i(s, u^n(s))\,ds : n \in N\Big\}\Big) \le 2 \sup \Big\| \int_\tau^t f_i(s, u^n(s))\,ds \Big\|_i \\ &\le 2\Big| \int_\tau^t m_i(s)\,ds \Big| .\end{aligned}$$

Thus, the function $t \to v_i(t)$ is absolutely continuous on J. Moreover $v_i(0) = \alpha_i(F_i(V)(0)) = \alpha_i(\{x_{i0}\}) = 0$. Let $W_i = \{w_i^n = f_i(\cdot, u^n) : n \in N\}$. It is clear that $w_i^n \in C_i$ and $\|w_i^n(t)\|_i \le m_i(t)$ for $n \in N$ and $t \in J$. Next, let us take any $t \in J$ and $r > 0$ such that $t + r \in J$. As

$$F_i(u^n)(t+r) = F_i(u^n)(t) + \int_t^{t+r} f_i(s, u^n(s))\,ds \quad \text{for } n \in N ,$$

we have

$$F_i(V)(t+r) \subset F_i(V)(t) + \Big\{ \int_t^{t+r} f_i(s, u^n(s)\,ds : n \in N \Big\} .$$

Since W_i satisfies the assumptions of Heinz's Theorem [5], by condition IV we infer that

$$\begin{aligned}\alpha_i(F_i(V)(t+r)) &\le \alpha_i(F_i(V)(t)) + \alpha_i\Big(\Big\{ \int_t^{t+r} f_i(s, u^n(s))\,ds : n \in N\Big\}\Big) \\ &\le \alpha_i(F_i(V)(t)) + 2 \int_t^{t+r} \alpha_i(f_i(s, V(s)))\,ds \\ &\le \alpha_i(F_i(V)(t)) + 2 \int_t^{t+r} h_i(s, v(s))\,ds, \quad \text{where } v(s) = (v_1(s), v_2(s), \ldots) .\end{aligned}$$

From the above and (6) we get

$$\frac{v_i(t+r) - v_i(t)}{r} \le \frac{2 \int_t^{t+r} h_i(s, v(s))\,ds}{r} .$$

By the continuity of the function $s \to h_i(s, v(s))$, this implies

$$D^+ v_i(t) \le 2h_i(t, v(t)) \qquad \text{for } t \in J ,$$

where $D^+ v_i$ is a right upper Dini's derivative ($i = 1, 2, \ldots$). Applying now the following theorem on infinite systems of differential inequalities:

"Assume that a function $h : J \times R^\infty \to R^\infty$ satisfies the conditions III 1°–3°. Suppose also that there exists a continuous function $\psi = (\psi_1, \psi_2, \ldots) : J \to R^\infty$ such that $D^+\psi_i(t) \le h_i(t, \psi(t))$ for $x \in J$ and $i = 1, 2, \ldots$. Then there exists a solution $y = (y_1, y_2, \ldots) : J \to R$ of the equation $y' = h(t, y)$ such that $y(0) = \psi(0)$ and $\psi_i(t) \le y_i(t)$ for $t \in J$, $i = 1, 2, \ldots$". (cf. [6] or [8], p. 360), we conclude that $v_i(t) = 0$ for $t \in J$ and $i = 1, 2, \ldots$. On the other hand

$$\alpha(A_1 \times A_2 \times \ldots) \le \sum_{i=1}^{\infty} \frac{1}{2^1} \, \frac{\alpha_i(A_i)}{1 + \alpha_i(A_i)}$$

for any sequence of bounded sets $A_i \subset E_i$ $(i = 1, 2, \ldots)$. (For the proof see ([9], p. 191). Therefore

$$\alpha(V(t)) \le \sum_{i=1}^{\infty} \frac{1}{2^1} \, \frac{v_i(t)}{1 + v_i(t)} = 0 \qquad \text{for } t \in J\,.$$

By Ascoli's theorem this proves that the set V is relatively compact in C. Hence the sequence (u^n) has a convergent subsequence. Let

$$F^n(x)(t) = \begin{cases} x_0 & \text{for } 0 \le t \le a_n \\ F(x)(t - a_n) & \text{for } a_n \le t \le a, \end{cases}$$

for each $x \in C$, $n = 1, 2, \ldots$, where $x_0 = (x_{10}, x_{20}, \ldots)$ and $a_n = \frac{a}{n}$. Obviously, F^n is a continuous mapping $C \to C$ and

$$|F^n(x) - F(x)|_C \le \omega(a_n) \qquad \text{for } x \in C\,.$$

Put $T = I - F$ and $T^n = I - F^n$. Then T, T^n are continuous mappings $C \to C$ and $\lim_{n\to\infty} |T^n(x) - T(x)|_C = 0$ uniformly on C. Fix n and assume that $y \in C$.

We define a finite sequence (x^k), $k = 1, 2, \ldots, n$, of continuous functions in the following way

$$x^1(t) = x_0 + y(t) \quad \text{for } t \in J,$$

$$x^{k+1}(t) = \begin{cases} x^k(t) & \text{for } 0 \le t \le ka_n \\ y(t) + F(x^k)(t - a_n) & \text{for } ka_n \le t \le a. \end{cases}$$

It is easy to show that

$$x^k(t) = y(t) + F^n(x^k)(t) \qquad \text{for } 0 \le t \le ka_n,\ k = 1, 2, \ldots, n\,,$$

and consequently $T^n(x^n) = y$. Conversely, if $T^n(x) = y$ and $x \in C$ then $x(t) = x^k(t)$ for $0 \le t \le ka_n$, $k = 1, \ldots, n$, and therefore $x = x^n$. This proves that T^n is a bijection $C \to C$.

Now, assume that $\lim_{j\to\infty} |T^n(u^j) - T^n(u)|_C = 0$, where u^n, $u \in C$. Since $u^j(t) = T^n(u^j)(t) + x_0$ and $u(t) = T^n(u)(t) + x_0$ for $0 \le t \le a_n$, $\lim_{j\to\infty} u^j(t) =$

$u(t)$ uniformly on $\langle 0, a_n \rangle$. Further

$$u^j(t) = T^n(u^j)(t) + F(u^j)(t - a_n) \quad \text{and} \quad u(t) = T^n(u)(t) + F(u)(t - a_n)$$

for $a_n \leq t \leq 2a_n$, and

$$\lim_{j \to \infty} F(u^j)(t - a_n) = F(u)(t - a_n) \quad \text{uniformly on } \langle a_n, 2a_n \rangle .$$

Thus $\lim_{j \to \infty} u^j(t) = u(t)$ uniformly on $\langle 0, 2a_n \rangle$. Repeating this argument we get $\lim_{j \to \infty} u^j(t) = u(t)$ uniformly on $\langle 0, ka_n \rangle$ for $k = 1, \ldots, n$, i.e., $\lim_{j \to \infty} u^j = u$ in C. This shows the continuity of $(T^n)^{-1}$. So, T^n is a homeomorphism $C \to C$.

Now, by Th. 2.4 from [11], we conclude that the set $T^{-1}(0)$ is a compact R_δ. It is clear that $S = T^{-1}(0)$. This ends the proof.

2. Finite systems of differential equations

Assume that $J = \langle 0, a \rangle$ and E_i is a Banach space with a norm $|| \cdot ||_i$ ($i = 1, \ldots, m$). In this section we study the existence of a solution of the problem (2). Let $B_i = \{x \in E_i : ||x||_i \leq b\}$ for $i = 1, \ldots, m$, and $B = B_1 \times \ldots \times B_m$. In contrast to Section 1, now we assume that functions f_i satisfy only the Caratheodory conditions:

1° for each $x \in B$ the function $t \to f_i(t, x)$ is strongly measurable on J;

2° for each $t \in J$ the function $x \to f_i(t, x)$ is continuous on B;

3° there exists an integrable function $p_i : J \to R_+$ such that

$$||f_i(t, x)||_i \leq p_i(t) \qquad \text{for } (t, x) \in J \times B .$$

Let $P_i(t) = \int_0^t p_i(s)\, ds$ for $t \in J$ ($i = 1, \ldots, m$) and $I = \langle 0, d \rangle$, where $0 < d \leq a$ and $P_i(d) \leq b$ for $i = 1, \ldots, m$.

Let us recall some definitions from [8].

A function $h = (h_1, \ldots, h_m) : I \times R_+^m \to R_+^m$ is said to have the property W_+ if for each $(t, \widetilde{r})$, $(t, r) \in I \times R_+^m$ the following implication holds:

$$r \overset{i}{\leq} \widetilde{r} \Rightarrow h_i(t, r) \leq h_i(t, \widetilde{r}) \quad \text{for } i = 1, \ldots, m, \quad \text{where}$$

$$r \overset{i}{\leq} \widetilde{r} \Leftrightarrow [r_k \leq \widetilde{r}_k \quad \text{for } k = 1, \ldots, m \quad \text{and} \quad r_i = \widetilde{r}_i] .$$

A nonnegative function $h = (h_1, \ldots, h_m) : I \times R_+^m \to R_+^m$, which is measurable in $t \in I$, continuous in $r \in R_+^m$ and satisfies the property W_+, is said to have:

1) a property W_1 ($h \in W_1$), if for each c ($0 < c \leq d$) the function $u = 0$ is the only absolutely continuous function on $\langle 0, c \rangle$ which satisfies almost everywhere the equality $u' = h(t, u)$ and such that $u(0) = 0$;

2) a property W_2 ($h \in W_2$), if for every bounded subset Z of $I \times R_+^m$ there exists a function $w_Z = (w_Z^1, \ldots, w_Z^m) : (0, d\rangle \to R_+^m$ such that $h(t,r) \leq w_Z(t)$ for $(t,r) \in Z$ and for every small $c > 0$ w_Z is integrable on $\langle c, d\rangle$; for each $c \in (0,d)$ the function $u = 0$ is the only absolutely continuous function on $\langle 0, c\rangle$ which satisfies everywhere the equality $u' = h(t,u)$ and such that $D_+u(0) = 0$ and $u(0) = 0$.

The following theorem is well known (cf. [7] or [8], p. 122):

Assume that

– $h : I \times R_+^m \to R_+^m$ satisfies the Caratheodory conditions and the property W_+;

– $y : I \to R_+^m$ is a maximal solution of the Cauchy problem $y' = h(t,y)$ and $y(0) = 0$.

If $\psi : I \to R_+^m$ is an absolutely continuous function such that $\psi(0) \leq 0$ and $\psi'(t) \leq h(t, \psi(t))$ for almost every $t \in J$, then $\psi(t) \leq y(t)$ for $t \in I$.

Let $E = E_1 \times \ldots \times E_m$ and $C = C(J, E)$ be the Banach space of continuous functions $J \to E$. Using this theorem and applying similar method of provinf as in Theorem 1, we get the following

THEOREM 2. *If there exists a function $h = (h_1, \ldots, h_m)$ such that $2h \in W_1$ and for each $i = 1, \ldots, m$*

$$\alpha_i(f_i(t, X_1, \times \ldots \times X_m)) \leq h_i(t, \alpha_1(X_1), \ldots, \alpha_m(X_m))$$

for almost every $t \in I$ and for each $X_1 \times \ldots \times X_m \subset B$, then there exists at least one solution of (2) *defined on I. Moreover, the set of all such solutions is a compact R_δ in C.*

To complete our considerations, let us notice that combining the proofs of Theorem 1 and Theorems 2, 3 from [10], we can prove the next existence theorems for (2):

THEOREM 3. *Assume that there exists a function $h \in W_1$ such that for any $\varepsilon > 0$ and $X_1 \times \ldots \times X_m \subset B$ there exists a closed subset $I_\varepsilon \subset I$ such that* $\operatorname{mes}(I - I_\varepsilon) < \varepsilon$ *and*

$$\alpha_i(f_i(T \times X_1 \times \ldots \times X_m)) \leq \sup_{s \in T} h_i(s, \alpha_1(X_1), \ldots, \alpha_m(X_m))$$

for each compact subset $T \subset I_\varepsilon$ and $i = 1, \ldots, m$.

Then there exists at least one solution of (2) defined on I and the set of all such solutions is a compact R_δ in C.

THEOREM 4. *Assume that the functions f_i are bounded and continuous for $i = 1, \ldots, m$. Then Theorem 3 is true also for $h \in W_2$.*

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