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BVP OF A NON-UNIFORMLY ELLIPTIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

It is well-known that many problems of mathematical physics may be described by the system of partial differential equation of first order. They are frequently related to Boundary Value Problems for elliptic equations. One of the basic problems is the Riemann–Hilbert problem.

The methods of complex functions theory have a wide use in many questions of mathematical analysis and its application. A special statement deals with the application of these methods in the theory of partial differential equations and systems. The possibility and importance of employing complex variable methods in PDE is so wide that it presents a real difficulty to give a survey of them. For a great many special references one may consult for instance the books of Tutschke, W. [LT] and Wendland, W. [WW1]. For the investigation of differential systems of equations with degeneration of the ellipticity, we confine ourselves to a particular case of the following equation

$$w_{\overline{z}} = H(z, w, w_z) \tag{1.1}$$

which is the familiar complex form of the general elliptic nonlinear system of two first order real equations

$$\varphi_j(x, y, u, v, u_x, u_y, v_n, v_y) = 0 \quad (j = 1, 2) \tag{1.2}$$

for the unknown functions $u(x, y), v(x, y)$ of two independent variables x and y (see for instance [BI] or [LT]), where $z = x + iy$ ($\overline{z} = x - iy$), $w = w(z) = u(x, y) + iv(x, y)$ and

$$w_z = \partial w/\partial z = (\partial w/\partial x - i\partial w/\partial y)/2,$$
$$w_{\overline{z}} = \partial w/\partial \overline{z} = (\partial w/\partial x + i\partial w/\partial y)/2.$$

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For the general equation (1.1), ellipticity in the sense of Lavrentiev means that a Lipschitz constant of the right hand $H(z,w,w_z)$ relative to w_z is small enough, i.e.

$$|H(z,w,\eta_1) - H(z,w,\eta_2)| \leq q(z,w)|\eta_1 - \eta_2|, \tag{1.3}$$

$$q(z,w) < 1. \tag{1.4}$$

Equation (1.2) has been elaborated by Bojarski, B. and Iwaniec, T. (see [BI] or [LT]). They have shown that the general nonlinear system (1.2), uniformly elliptic in the sense of Lavrentiev — called by him strong ellipticity in the geometry sense — can be written in the form (1.1), provided that, there exists q_0 such that

$$q(z,w) \leq q_0 < 1, \tag{1.5}$$

$H(z,w,0) = 0$. Equation (1.1) fulfilling the inequality (1.5) is uniformly elliptic in the given domain.

The basic boundary value problem for linear, quasilinear and nonlinear uniformly elliptic systems of the equation has been developed by Begehr, H. [BH], Bojarski, B. V. and Iwaniec, T [BI], Gilbert, R. P. [GR], Hsiao, G. C. and Wendland, W. [HW], Mamourian, A. [MA1], Tutschke, W. [LT], Wen, G. C. [WC] and many others. In a short surway as this it is impossible to bring all features of the uniformly elliptic case. The author apologizes in advance for not mentioning many important papers and results.

Clearly q in (1.3)–(1.4) is assumed to be a real function of complex variable z and complex unknown w. Let us assume that q be a real function of complex variables z, w, η_1 ($\eta_1 = (w_z)_1$), η_2 ($\eta_2 = (w_z)_2$) and in view of this assumption, the function $q(z,w,\eta_1,\eta_2)$ satisfies a particular case of the following inequality (see also [IM])

$$q(z,w,\eta_1,\eta_2) \leq 1. \tag{1.6}$$

In the next section we shall bring the exact conditions on q such that the main boundary value problems are well-posed.

2. BVP of a degenerate elliptic system

Let $\Gamma = \Gamma_0 + \Gamma_1 + \ldots + \Gamma_m$ be the boundary contours of an $m+1$-connected Liapunov region D where Γ_0 contains all contours $\Gamma_j, j \geq 1$. Consider the equation

$$w_{\bar{z}} = H(z,w,w_z) = \widetilde{H}(w_z) + A(z)w + F(z) \tag{2.1}$$

in D, where the right-hand side of equation (2.1) fulfills the conditions:

$$|H(z,w,\eta_1) - H(z,w,\eta_2)| \leq q(z,w,\eta_1,\eta_2)|\eta_1 - \eta_2| \tag{2.2}$$

$$q(z,w,\eta_1,\eta_2) \leq \widetilde{q}(|\eta_1 - \eta_2|) \leq 1, \tag{2.3}$$

with the boundary condition

$$Re[\overline{a(t)}w(t)] = \gamma(t) \tag{2.4}$$

on Γ; a, γ are given function on Γ. In respect to $\widetilde{q}$ we assume:

(I) $\widetilde{q}(\alpha)$ as a real function of $\alpha = |\eta_1 - \eta_2|$ is continuous in $[0, \infty]$; if $\alpha \in (0, \infty]$, then $\widetilde{q}(\alpha) < 1$; the function $\alpha\widetilde{q}^2(\alpha)$ is increasing and concave.

Concerning the coefficients of the boundary conditions (2.4), we shall make the usual assumptions for uniformly elliptic case, i.e.

(II) The complex function $A(z), F(z)$ assumed to be measurable belonging to the class $L_p(D)$, for some $p > 2$, the complex function $a(t)$ and real function $\gamma(t)$ are Hölder continuous on Γ, with respect to β, where $0 < \beta \leq 1(a, \gamma \in H_\beta(\Gamma),\ 0 < \beta \leq 1)$. The solution w will be sought in the Sobolev space $W_p^1(D), p > 2$.

Similar to the boundary value problems for uniformly elliptic system of equations, we introduce the following notation. Let

$$n = \frac{1}{2\pi}\Delta_\Gamma ARGa(t)$$

then n will be called the index corresponding to the boundary value problem (2.1)–(2.4).

The coefficient of ellipticity corresponding to the equation (2.1) is defined by

$$\widetilde{q}_0 = \lim_{\alpha\to\infty} \sup(\widetilde{q}(\alpha)) < 1$$

which is of crucial importance in the studying of the existence and regularity problem for (2.1)–(2.4).

If $\widetilde{H} = A = 0$ in (2.1), the non-homogeneous boundary value problem (2.1)–(2.4) will be called problem P_0. In the case when $A = 0$, the boundary value problem (2.1)–(2.4) is called problem P_1.

LEMMA 1. *Under hypothesis* (I) *and* (II). *If the index* $n \leq 0$, *then the necessary and sufficient condition for solvability of the non-homogeneous boundary value problem* P_0 *will be as follows*

$$1/2i\sum_{k=0}^{m} \int_{\Gamma_k} a(t)\psi(t)\gamma(t)\,dt - Re\Big\{ \int_D \psi(z)F(z)\,do_z\Big\} = 0, \tag{2.6}$$

where ψ *is an arbitrary solution of the homogeneous boundary value problem adjoint to the problem* P_0 (see for instance [LT] pp. 98–101).

Let us recall that in the classical boundary value problems of the type (2.4), relative to the systems of equations with uniformly ellipticity, the solution is sought in the space $W_p^1(D)$, for some $p > 2$. Regarding equation (2.1), with the condition of non-uniformly ellipticity (2.2), we shall not make use directly from the L_p-theory for the proof of existence of the solution. Therefore the formulation of the problem (2.1)–(2.4) contains the weak boundary conditions (see also [IM]).

We shall also bring here in example for non-uniformly case of (2.1), which fulfills the conditions (I):

Let

$$(2.7)\qquad w_{\overline{z}} = H(z, w, w_z) = \frac{1 + |w_z|^2}{1 + 2|w_z| + 4|w_z|^2} + A(z)w + F(z)$$

then by some calculations, we can observe that (2.7) satisfies the inequalities (2.2), (2.3) where

$$\widetilde{q}(\alpha) = \frac{1 + \alpha^2}{1 + 2\alpha^2},$$

the coefficient of ellipticity corresponding to the equation (2.7) equal to 1/2, i.e. $\widetilde{q}_0 = 1/2$, and the function $\alpha\widetilde{q}^2(\sqrt{\alpha}), \alpha > 0$, is concave.

PROPOSITION 1. *Let the conditions* (I), (II), (2.6) *hold. If the index* $n = 0$ *(m-arbitrary finite), then there exists a solution (in* W_p^1*; for some* $p > 2$*) of the boundary value problem* P_1.

The proof will be carried out through the following representation formula for the solution w of the boundary value problem P_1

$$(2.8)\qquad w = w(z) = T(\varrho) + \chi(z)$$

where in the case of unit disc domain, the operator T has the following form

$$T(\varrho) = (T\varrho)_{(z)} = -\frac{1}{\pi}\int_D \left(\frac{\varrho(t)}{t - z} + \frac{z\overline{\varrho(t)}}{1 - z\overline{t}}\right) d\sigma_t;$$

we shall not bring here the explicit form of $T(\varrho)$ for the case of multiply-connected domains (see [BH]), since this would involve extremely lengthy expressions. Making use of the Green's function, this operator has been represented by Begehr, H. [BH], $\varrho \in L_p(D)$, $p \geq 2$ and $\chi(z)$ is the solution of the boundary value problem P_0. As it is well-known, problem P_0 has been studied by many authors and brought to a rather satisfactory state (see for instance [LT] pp. 98–101).

The operator T has the following properties

$$(2.9)\qquad Re[\overline{a(t)}T(\varrho)] = 0$$

on the boundary Γ. In other words, when $z \to t$ $(z \in D,\ t \in \Gamma)T(\varrho)$ satisfies the homogeneous boundary condition (2.4), moreover the generalized derivatives of $T(\varrho)$ relative to $\overline{z}$ and z are

$$(2.10)\qquad \partial T(\varrho)/\partial\overline{z} = \varrho(z)$$

and

$$\partial T(\varrho)/\partial z = S(\varrho),$$

then since the index $n = 0$, the L_2-norm of S is equal to one.

Remark. Actually, in problem (2.1)–(2.4), it is assumed that, the index $n = 0$ and m arbitrary finite. In fact, in this case, through an appropriate transformation, the boundary condition (2.4) can be written in the form $\{Re\, w(t)\} = \gamma(t)$ (see for instance [MA2].

Let us recall that S is a bounded operator from $L_p(D), p > 1$ into itself, and the well-known Riesz–Thorin convexity theorem assures the continuity of the norm of $S(\|S\|_p)$ with respect to $p > 1$.

Henceforth, in view of (2.9), (2.10), ϱ fulfills the singular integral equation

$$\varrho = \widetilde{H}(S(\varrho) + \chi'), \tag{2.11}$$

$(\chi' = \partial\chi/\partial z)$ which can be solved through a successive approximation method.

For the proof of the existence; let us assume

$$\varrho_{k+1} = \widetilde{H}(S(\varrho_k) + \chi') \quad (\varrho_0 = 0,\ k = 0, 1, \ldots). \tag{2.12}$$

At first, let us prove the L_2 convergence of the sequence ϱ_k. According to (2.3)

$$|\varrho_{k+1} - \varrho_{j+1}| \le \widetilde{q}(|S(\varrho_k - \varrho_j)|)|S(\varrho_k - \varrho_j)|, \quad (k, j = 0, 1, \ldots) \tag{2.13}$$

then in view of (2.13), it is clear that

$$|\varrho_{k+1} - \varrho_{j+1}|^2 \le \widetilde{q}^2(|S(\varrho_k - \varrho_j)|)|S(\varrho_k - \varrho_j)|^2. \tag{2.14}$$

Note:

$$\|\varrho\|_{L_2(D)} = \left(\frac{1}{|D|}\int_D |\varrho(z)|^2 d\sigma_z\right)^{1/2}.$$

By integrating both sides of (2.14), in view of the concavity assumption of $t\widetilde{q}^2(t)$, and the familiar Jenssen inequality,

$$\|\varrho_{k+1} - \varrho_{j+1}\|^2 \le \widetilde{q}^2(\|S(\varrho_k - \varrho_j)\|)\|S(\varrho_k - \varrho_j)\|^2. \tag{2.15}$$

Since S is an isometry in $L_2(D)$, we have

$$\|\varrho_{k+1} - \varrho_{j+1}\| \le \widetilde{q}(\|\varrho_k - \varrho_j\|)\|\varrho_k - \varrho_j\| \quad (k, j = 0, 1, \ldots). \tag{2.16}$$

Now, if we assume that

$$e_n = \|\varrho_{n+1} - \varrho_n\|_{L_2(D)}, \tag{2.17}$$

we obtain

$$e_{n+1} \le \widetilde{q}(e_n) \cdot e_n \le e_n. \tag{2.18}$$

But inequalities (2.18) show that e_n is increasing and converges to a non-negative number. In view of continuity of $\tilde{q}$ and the assumption (I) on $\tilde{q}$, we observe that e_n converges to zero.

Making use of the above results, the Cauchy condition for sequence ϱ_k in the topology of $L_2(D)$ can be proved.

Let ε be an arbitrary positive number, then it is clear that $\delta(\varepsilon) = \varepsilon(1 - \tilde{q}(\varepsilon))$ is positive, and for sufficiently large M, we have

$$\|\varrho_{i+1} - \varrho_i\| \leq \delta(\varepsilon), \quad i > M \tag{2.19}$$

by induction (with respect to $i > M$), we can prove that

$$\|\varrho_k - \varrho_j\| < \varepsilon, \quad k, j > M. \tag{2.20}$$

Clearly, inequality (2.20) holds, when $k = j$. In view of (2.16) and (2.20) and the following triangle inequality

$$\|\varrho_{k+1} - \varrho_j\| \leq \|\varrho_{k+1} - \varrho_{j+1}\|\|\varrho_{j+1} - \varrho_j\|,$$

we obtain

$$\|\varrho_{k+1} - \varrho_j\| \leq \tilde{q}(\|\varrho_k - \varrho_j\|)\|\varrho_k - \varrho_j\| + \delta(\varepsilon) \leq \varepsilon.$$

Since $\tilde{H}$ is continuous with respect to $S(\varrho) + \chi'$, the function $\varrho = \lim_{k\to\infty} \varrho_k$ fulfills the equation (2.12), and the existence of the solution has been proved.

Now we can prove the existence of the solution of the boundary value problem in the Sobolev space $W_p^1(D)$, for some $p > 2$. Let p satisfy the following inequality

$$\tilde{q}_0\|S\|_p < 1, \tag{2.21}$$

where $\tilde{q}_0$ is the coefficient of ellipticity corresponding to the problem P_1, then there exists ϱ belonging to $L_p(D)$, and the solution w of the problem P_1 belongs to the space $W_p^1(D)$, $p > 2$. For the proof is similar to the previous work jointed with T. Iwaniec (see [IM]).

Proposition 2. *Let the conditions* (I), (II), *and* (2.6) *hold, the index* $n = 0$ (*m-arbitrary finite*), *then if there exists a solution in* $W_p'(D), p > 2$ *for the boundary value problem* (2.1)–(2.4), *then it is unique.*

Let ϱ_i and ϱ_2 be the solutions of singular integral equation corresponding to the boundary value problem (2.1)–(2.4), then from concavity property of the function $t\tilde{q}^2(t)$ and the familiar Jenssen inequality, we have

$$\|\varrho_1 - \varrho_2\| \leq \tilde{q}(\|\varrho_1 - \varrho_2\|)\|\varrho_1 - \varrho_2\| \tag{2.22}$$

which in view of the condition (I) (see assumptions concerning the function $\tilde{q}(t)$, we conclude that $\varrho_1 = \varrho_2$ almost everywhere.

This method can be extended to the non-uniformly elliptic Boundary Value Problem with non zero index.

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