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ON STRICTLY n-CONVEX NORMED LINEAR SPACES

1. Introduction

One of important spaces in functional analysis is a strictly convex normed linear space, by which we mean that each point of the unit surface is an extreme point of the unit ball. For example, inner product spaces and l^p spaces for $1 < p < \infty$ (but l^1 and l^∞ spaces are not). Generally speaking there are three different types of characterizations of a strictly convex space. Firstly by a norm [4], [6]: $\|x+y\| = \|x\|+\|y\|$ for $x, y \neq 0$ implies $x = ay$ for some real $a > 0$, or equivalently, $\frac{1}{2}\|x+y\| = \|x\| = \|y\| = 1$ implies $x = y$. Secondly by a semi-inner-product [1], [5], [9]: $[x, y] = \|x\|\|y\| \neq 0$ implies $x = ay$ for some real $a > 0$, where $[\ ,\ ]$ denotes a semi-inner-product. Thirdly by a duality mapping [2], [5], [8]: A duality mapping J on the space is strictly monotone, or equivalently, $J(x) \cap J(y) \neq \emptyset$ for $x, y \neq 0$ implies $x = y$. In the case of an inner product space more can be said about the norm relation. Indeed, it is known that if $x, y \neq 0$, then the relation $\|x + y\| = \|x\| + \|y\|$ holds if and only if $x = ay$ for some real $a > 0$. The strict convexity has been generalized to the space having Property C[3], by which we mean that if $\|x + y + z\|/3 = \|x\| = \|y\| = \|z\| = 1$, then x, y and z are collinear. A strictly convex space has Property C, but the converse is not generally true ([3], Example 1). It is our object in this paper to define a strictly n-convex normed linear space which is a generalization of the above two types of spaces. This new space will be characterized in terms of a norm, a semi-inner-product and a duality mapping. It is shown that there is a similarity among these three types of characterizations. Finally we shall present relationship between strict $(n-1)$-convexity and strict n-convexity. Indeed, the former implies the latter.

2. Characterizations by a norm

Let X denote a (real or complex) normed linear space throughout this note. We first need the following useful lemma which is essential to the formations of our consequent theorems. It is the author's belief that some of the statements in the lemma are known, but do not seem to have an explicit reference, we include a proof.

LEMMA 1. *The following conditions are equivalent:*

(1) *X is strictly convex;*

(2) $\frac{1}{2}\|x+y\| = \|x\| = \|y\| = 1$ *implies* $x = y$;

(3) $\frac{1}{2}\|x+y\| = \|x\| = \|y\| \neq 0$ *implies* $x = y$;

(4) $\|x+ay\| = 2\|x\| \neq 0$ *for some real* $a > 0$ *implies* $x = ay$, *and* $a = 1$ *if* $\|x\| = \|y\|$;

(5) $\|x-z\| = \|x-y\| + \|y-z\|$ *for* $x, y, z \neq 0$ *implies* $y = (1-b)x + bz$ *for some real* $b > 0$ *and* $0 < b < 1$;

(6) $\|x+y\| = \|x-y\| = \|x\| \neq 0$ *implies* $y = 0$;

(7) $\|z+x\| = \|z+y\| \neq 0$ *for all* $z \in X$ *implies* $x = y$;

(8) $\|x-y\| = |\,\|x\| - \|y\|\,|$ *for* $x, y \neq 0$ *implies* $x = cy$ *for some real* $c > 0$;

(1′) $\|x+y\| = \|x\| + \|y\|$ *for* $x, y \neq 0$ *implies* $\|y\|x = \|x\|y$;

(4′) $\|x+ay\| = 2\|x\| \neq 0$ *for* $a = \|x\|/\|y\|$ *implies* $x = ay$;

(8′) $\|x-y\| = |\|x\| - \|y\||$ *for* $x, y \neq 0$ *implies* $\|y\|x = (\|x-y\| + \|y\|)y$.

Proof. It is a routine matter to show the implications: (1) ⇒ (2) ⇒ (3) ⇒ (4′) ⇒ (4) ⇒ (2), (1′) ⇒ (1) ⇒ (5), (1) ⇔ (8) and (3) ⇒ (7). Thus it remains to show that (4′) ⇒ (1′), (5) ⇒ (1), (3) ⇔ (6), (7) ⇒ (3) and (1′) ⇒ (8′). That (8′) ⇒ (1′) is clear after we prove (1′) ⇒ (8′).

(4′) ⇒ (1′): Since (3) which implies (4') is symmetric with respect to x and y, we may suppose without loss of generality that $\|x\| \le \|y\|$ and $a = \|x\|/\|y\|$, then $\|x\| + \|y\| = \|x+y\| \le \|x+ay\| + (1-a)\|y\| \le \|x\| + \|y\|$, or $\|x+ay\| = 2\|x\|$. So $x = ay$ by (4'), i.e. $\|y\|x = \|x\|y$.

(5) ⇒ (1): The conclusion of (5) may be rewritten as $y = z = (1-b)(x-z)$. Now in (5) replacing $x-y$ by $x, y-z$ by y, and hence $x-z$ by $x+y$ yields $y = (1-b)(x+y)$, i.e. $x = ay$ with $a = b/(1-b)$.

(3) ⇔ (6): In (3) replacing x by $x+y$, and y by $x-y$ we arrive at (6), and vice versa.

(7) ⇒ (3): Suppose that (3) does not hold, i.e., $\frac{1}{2}\|x+y\| = \|x\| = \|y\| \neq 0$ and $x \neq y$, we have to show that $\|z+x\| = \|z+y\| \neq 0$ for some z's implies $x \neq y$. But this is clear if in the relation $\|z+x\| = \|z+y\|$ we let $z = x$ and $z = y$, respectively.

(1′) ⇒ (8′): We may let $\|x\| \ge \|y\|$ and so $\|x\| = \|x-y\| + \|y\|$. $\|y\|(x-y) = \|x-y\|y$ by (1'), or $\|y\|x = (\|x-y\| + \|y\|)y$, and the proof is complete.

The next definition is motivated by the definitions of strict convexity and the space having Property C, and by the concept of four points being coplanar in vector analysis.

DEFINITION 1. X is said to be strictly n-convex for a positive integer $n \geq 2$ if for a set $\{x_i\}_{i=1}^n$ in X satisfying the relation

$$\Big\|\sum_{i=1}^{n} x_i\Big\|/n = \|x_j\| = 1 \quad \text{for } j = 1, \ldots, n,$$

then the set has Property L, by which we mean that at least two of the vectors in the set are equal, or $\sum_{i=1}^n a_i x_i = 0$ for some nonzero real numbers a_i $(i = 1, \ldots, n)$ such that $\sum_{i=1}^n a_i = 0$.

It is easily seen from the definition that strict convexity is strictly 2-convex, and the space having Property C is strictly 3-convex in our sense. It may be noted that not every normed linear space is strictly n-convex. For example, in the space $\mathbf{R}^n (n \geq 2)$ let a norm be defined by $\|(x_1, \ldots, x_n)\| = \sum_{i=1}^n |x_i|$, then the standard basis $\{e_i\}_{i=1}^n$ satisfies the relation $\|\sum_{i=1}^n e_i\|/n = \|e_j\| = 1$ for $j = 1, \ldots, n$, but clearly it does not have Property L under this norm.

THEOREM 1. *The following conditions are equivalent*:

(1) *X is strictly n-convex*;

(2) *If $\|\sum_{i=1}^n x_i\|/n = \|x_j\| \neq 0$ for $j = 1, \ldots, n$, then the set $\{x_i\}_{i=1}^n$ has Property L*;

(3) *If $\|(\sum_{i=1}^{n-1} b_i x_i) + x_n\| = n\|x_n\| \neq 0$, $x_i \neq 0$ and $b_i = \|x_n\|/\|x_i\|$ for $i = 1, \ldots, n-1$, then the set $\{b_1 x_1, \ldots, b_{n-1} x_{n-1}, x_n\}$ has Property L*;

(4) *If $\|\sum_{i=1}^n x_i\| = \sum_{i=1}^n \|x_i\|$, $x_i \neq 0$ for $i = 1, \ldots, n$, then the set $\{b_1 x_1, \ldots, b_{n-1} x_{n-1}, x_n\}$ has Property L for some real $b_i > 0$ such that $\|b_i x_i\| = \|x_n\|$ for $i = 1, \ldots, n-1$.*

Proof. (1) $\Rightarrow$ (2): Let $\|\sum_{i=1}^n x_i\|/n = \|x_j\| = d \neq 0$ for $j = 1, \ldots, n$, then the set $\{x_i/d\}_{i=1}^n$ has Property L, i.e., the set $\{x_i\}_{i=1}^n$ has Property L.

(2) $\Rightarrow$ (3): Since $\|(\sum_{i=1}^{n-1} b_i x_i) + x_n\|/n = \|x_n\| = \|b_i x_i\|$ for $i = 1, \ldots, n-1$, the result follows easily.

(3) $\Rightarrow$ (4): There is clearly no loss of generality in taking $\|x_i\| \geq \|x_n\|$ for $i = 1, \ldots, n-1$. Let $b_i = \|x_n\|/\|x_i\|$ for $i = 1, \ldots, n-1$, then

$$\sum_{i=1}^{n} \|x_i\| = \Big\|\sum_{i=1}^{n} x_i\Big\| \leq \Big\|\Big(\sum_{i=1}^{n-1} b_i x_i\Big) + x_n\Big\| + \sum_{i=1}^{n-1} (1 - b_i)\|x_i\| \leq \sum_{i=1}^{n} \|x_i\|,$$

or $\|(\sum_{i=1}^{n-1} b_i x_i) + x_n\| = n\|x_n\|$ and the result follows.

(4) $\Rightarrow$ (1): If $\|\sum_{i=1}^n x_i\|/n = \|x_j\| = 1$ for $j = 1, \ldots, n$ then $\|\sum_{i=1}^n x_i\| = \sum_{i=1}^n \|x_i\|$ and hence the set $\{b_1 x_1, \ldots, b_{n-1} x_{n-1}, x_n\}$ has Property L.

Next we have to show that the set $\{x_i\}_{i=1}^n$ has Property L. This is clear since $\|b_ix_i\| = \|x_n\| = 1$ for $i = 1, \ldots, n-1$, or $b_i = 1$ for $i = 1, \ldots, n-1$.

3. Characterizations by a semi-inner-product

Recall from functional analysis that a semi-inner-product is a mapping $[\ ,\ ]$ on $X \times X$ into real or complex numbers satisfying the conditions:

(i) $[bx + y, z] = b[x, z] + [y, z]$;
(ii) $[x, x] > 0$ for $x \neq 0$;
(iii) $|[x, y]|^2 \leq [x, x][y, y]$

for all x, y and z in X, b any real or complex number, and the norm in X is given by $\|x\| = [x, x]^{\frac{1}{2}}$.

THEOREM 2. *The following conditions are equivalent*:

(1) *X is strictly n-convex (in the sense of Theorem 1)*;

(2) *If $[\sum_{i=1}^{n-1} x_i, x_n] = (\sum_{i=1}^{n-1} \|x_i\|)\|x_n\|$, $x_i \neq 0$ for $i = 1, \ldots, n$, then the set $\{b_1x_1, \ldots, b_{n-1}x_{n-1}, x_n\}$ has Property L for some real $b_i > 0$ such that $\|b_ix_i\| = \|x_n\|$ for $i = 1, \ldots, n-1$*;

(3) *If $[x_i, x_n] = \|x_i\|^2 = \|x_n\|^2 \neq 0$ for $i = 1, \ldots, n-1$, then the set $\{x_i\}_{i=1}^n$ has Property L*;

(4) *If $[x_i, x_n] = \|x_i\| = \|x_n\| = 1$ for $i = 1, \ldots, n-1$, then the set $\{x_i\}_{i=1}^n$ has Property L*;

(5) *If $[b_ix_i, x_n] = \|x_n\|^2 \neq 0$, $x_i \neq 0$ and $b_i = \|x_n\|/\|x_i\|$ for $i = 1, \ldots, n-1$, then the set $\{b_1x_1, \ldots, b_{n-1}x_{n-1}, x_n\}$ has Property L.*

Proof. The implications that (5) ⇒ (3) ⇒ (4) are obvious.

(1) ⇒ (2): Let $[\sum_{i=1}^{n-1} x_i, x_n] = (\sum_{i=1}^{n-1} \|x_i\|)\|x_n\|$, then

$$\Big(\sum_{i=1}^{n} \|x_i\|\Big)\|x_n\| = \Big[\sum_{i=1}^{n} x_i, x_n\Big] \leq \Big\|\sum_{i=1}^{n} x_i\Big\|\|x_n\| \leq \Big(\sum_{i=1}^{n} \|x_i\|\Big)\|x_n\|,$$

or $\|\sum_{i=1}^n x_i\| = \sum_{i=1}^n \|x_i\|$, and the result follows from (4) in Theorem 1.

(2) ⇒ (3): Let $[x_i, x_n] = \|x_i\|^2 = \|x_n\|^2$ for $i = 1, \ldots, n-1$, then $[\sum_{i=1}^{n-1} x_i, x_n] = (\sum_{i=1}^{n-1} \|x_i\|)\|x_n\|$, and hence the set $\{b_1x_1, \ldots, b_{n-1}x_{n-1}, x_n\}$ has Property L by (2) for some real $b_i > 0$ such that $\|b_ix_i\| = \|x_n\|$ for $i = 1, \ldots, n-1$. It follows that $b_i = 1$ for $i = 1, \ldots, n-1$.

(4) ⇒ (1): Given a set $\{x_i\}_{i=1}^n$ such that $[x_i, x_n] = \|x_i\| = \|x_n\| = 1$ for $i = 1, \ldots, n-1$, then

$$n\|x_n\| = \Big(\sum_{i=1}^{n} \|x_i\|\Big)\|x_n\| = \Big[\sum_{i=1}^{n} x_i, x_n\Big] \leq \Big\|\sum_{i=1}^{n} x_i\Big\|\|x_n\| \leq n\|x_n\|,$$

or $\| \sum_{i=1}^{n} x_i\|/n = \|x_j\| = 1$ for $j = 1, \dots, n$. Therefore, by contrapositive we see that if the set $\{x_i\}_{i=1}^{n}$ does not have Property L under the latter relation, it does not have Property L under the former relation neither.

(3) $\Rightarrow$ (5): Assumption in (5) implies that $[b_i x_i, x_n] = \|b_i x_i\|^2 = \|x_n\|^2$ for $i = 1, \dots, n-1$.

4. Characterizations by duality mappings

Let X^* denote the conjugate space of X, the two commonly known duality mappings are:

$$J(x) = \{f \in X^* : f(x) = \|f\|\|x\|,\ \|f\| = \|x\|\} \quad \text{(see [2], [5], [8])}$$

and

$$I(x) = \{f \in X^* : f(x) = \|f\|\|x\|\} \quad \text{(see [5])}$$

with duality mappings $J, I : X \to 2^{X^*}$.

Evidently, with regard to such mappings the following assertions are true: (a) $J(x) \subseteq I(x)$. (b) $x = 0$ if and only if $I(x) = X^*$. (c) $I(x) = I(cx) = cI(x)$ for any real $c > 0$. (d) $0 \neq f \in I(x)$ for $x \neq 0$ implies $f \in J(cx)$ for some real $c > 0$. (e) If $x \neq 0$, then there exists an $f \in J(x)$ such that $f \neq 0$ (by the Hahn–Banach theorem).

DEFINITION 2. Let $I'(x)$ be the same as $I(x)$ which has an additional property that $\|x\| \geq \|y\|$ if and only if $\|f\| \geq \|g\|$ for $x, y \neq 0$, $f \in I(x)$ and $g \in I(y)$.

It follows easily from the definition that $f \in I'(x) \cap I'(y)$ for $x, y \neq 0$ if and only if $f(x) = \|f\|\|x\|$, $f(y) = \|f\|\|y\|$ and $\|x\| = \|y\|$.

LEMMA 2. *If* $0 \neq f \in I'(x)$ *and* $0 \neq g \in I'(y)$ *for* $x, y \neq 0$, *then*

(1) $\mathrm{Re}(f - g)(x - y) \geq 0$;

(2) $\mathrm{Re}(f - g)(x - y) = 0$ *if and only if* $f(y) = \|f\|\|y\|$, $g(x) = \|g\|\|x\|$ *and* $\|x\| = \|y\|$;

(3) $\mathrm{Re}(f - g)(x - y) = 0$ *if and only if* $f, g \in I'(x) \cap I'(y)$.

Proof. The proof of (1) and (2) appeared as part of the proof in [5, Corollary 8] except for an obvious change in there. Indeed, it can be shown that $\mathrm{Re}(f - g)(x - y) = [(\|f\| - \|g\|)(\|x\| - \|y\|)] + [\|f\|\|y\| - \mathrm{Re}\, f(y)] + [\|g\|\|x\| - \mathrm{Re}\, g(x)] \geq 0$. Also $\mathrm{Re}\, f(y) = \|f\|\|y\|$ if and only if $f(y) = \|f\|\|y\|$, and similarly $\mathrm{Re}\, g(x) = \|g\|\|x\|$ if and only if $g(x) = \|g\|\|x\|$. (3) is consequences of assumptions, (2) and a remark above.

THEOREM 3. *The following conditions are equivalent:*

(1) X *is strictly* n*-convex* (*in the sense of Theorem* 1);

(2) *If* $\bigcap_{i=1}^n I(x_i) \neq \emptyset$, $x_i \neq 0$ *for* $i = 1,\ldots,n$, *then the set* $\{b_1x_1,\ldots, b_{n-1}x_{n-1}, x_n\}$ *has Property L for some real* $b_i > 0$ *such that* $\|b_ix_i\| = \|x_n\|$ *for* $i = 1,\ldots,n-1$.

(3) *If* $\bigcap_{i=1}^n I'(x_i) \neq \emptyset$ *and* $x_i \neq 0$ *for* $i = 1,\ldots,n$, *then the set* $\{x_i\}_{i=1}^n$ *has Property L;*

(4) *If* $\bigcap_{i=1}^n J(x_i) \neq \emptyset$ *and* $x_i \neq 0$ *for* $i = 1,\ldots,n$, *then the set* $\{x_i\}_{i=1}^n$ *has Property L;*

(5) *If* $0 \neq f_i \in I'(x_i)$ *and* $x_i \neq 0$ *for* $i = 1,\ldots,n$ *and the set* $\{x_i\}_{i=1}^n$ *does not have Property L, then* $\mathrm{Re}(f_i - f_k)(x_i - x_k) > 0$ *for* $i = 1,\ldots,n$ *and* $i \neq k \in \{1,\ldots,n\}$.

(6) *If* $0 \neq f_i \in J(x_i)$ *and* $x_i \neq 0$ *for* $i = 1,\ldots,n$ *and the set* $\{x_i\}_{i=1}^n$ *does not have Property L, then* $\mathrm{Re}(f_i - f_k)(x_i - x_k) > 0$ *for* $i = 1,\ldots,n$ *and* $i \neq k \in \{1,\ldots,n\}$.

Proof. It is trivial that (3) $\Rightarrow$ (4), and (6) is a special case of (5).

(1) $\Rightarrow$ (2): Let $0 \neq f \in \bigcap_{i=1}^n I(x_i) = (\bigcap_{i=1}^{n-1} I(b_ix_i)) \cap I(x_n)$ for $b_i = \|x_n\|/\|x_i\|$, $i = 1,\ldots,n-1$, then

$$\|f\|\Big\|\Big(\sum_{i=1}^{n-1} b_ix_i\Big) + x_n\Big\| \geq f\Big(\Big(\sum_{i=1}^{n-1} b_ix_i\Big) + x_n\Big) = n\|f\|\,\|x_n\|$$
$$\geq \|f\|\Big\|\Big(\sum_{i=1}^{n-1} b_ix_i\Big) + x_n\Big\|,$$

or $\|(\sum_{i=1}^{n-1} b_ix_i) + x_n\| = n\|x_n\|$, and we may apply (3) in Theorem 1.

(2) $\Rightarrow$ (3): Obvious, as (3) implies $\|x_i\| = \|x_n\|$ for $i = 1,\ldots,n-1$, and so $b_i = 1$ for $i = 1,\ldots,n-1$.

(4) $\Rightarrow$ (1): Given a set $\{x_i\}_{i=1}^n$ such that $0 \neq f \in \bigcap_{i=1}^n J(x_i)$ (hence $\|x_1\| = \|x_i\|$ for $i = 2,\ldots,n$), then

$$n\|f\|\,\|x_1\| \geq \|f\|\Big\|\sum_{i=1}^n x_i\Big\| \geq f\Big(\sum_{i=1}^n x_i\Big)$$
$$= \|f\|\Big(\sum_{i=1}^n \|x_i\|\Big) = n\|f\|\,\|x_1\|,$$

or $\|\sum_{i=1}^n x_i\|/n = \|x_j\|$ for $j = 1,\ldots,n$. This is precisely the condition (2) in Theorem 1. Thus, by contrapositive the rest of the proof is the same as (4) $\Rightarrow$ (1) in Theorem 2.

(3) $\Rightarrow$ (5): If $0 \neq f_i \in I'(x_i)$ for $i = 1,\ldots,n$, and the set $\{x_i\}_{i=1}^n$ does not have Property L, and suppose that $0 = \mathrm{Re}(f_i - f_k)(x_i - x_k)$ for $i = 1,\ldots,n$ and $i \neq k \in \{1,\ldots,n\}$, i.e., the negation of (5), then $f_k \in \bigcap_{i=1}^n I'(x_i)$ by

Lemma 2. Hence $\bigcap_{i=1}^{n} I'(x_i) \neq \emptyset$ and the set $\{x_i\}_{i=1}^{n}$ does not have Property L. In other words, (3) does not hold.

(5) $\Rightarrow$ (3): If $0 \neq f \in \bigcap_{i=1}^{n} I'(x_i)$ and suppose that the set $\{x_i\}_{i=1}^{n}$ does not have Property L, then $0 = \mathrm{Re}(f - f)(x_i - x_k) > 0$ by (5) yielding a contradiction.

In view of the definition of a duality mapping (see [2], [5], [8]) and statements (5) and (6) in Theorem 3, we may say that X is strictly n-convex if and only if I' or J is strictly monotone.

5. Strict $(n-1)$-convexity and n-convexity

In this final section we shall present relationship between strict $(n-1)$-convexity and n-convexity.

THEOREM 4. *For $n \geq 3$ a strictly $(n-1)$-convex space X is strictly n-convex.*

Proof. If $x_1, \dots, x_n \in X$ are nonzero vectors, X is strictly $(n-1)$-convex and if $\bigcap_{i=1}^{n} J(x_i) \neq \emptyset$, then the set A consisting of any $n-1$ vectors from the set $\{x_i\}_{i=1}^{n}$ has Property L by (4) in Theorem 3. We want to show that the set $\{x_i\}_{i=1}^{n}$ has Property L. If at least two vectors in A are equal, we are finished. Otherwise we may pick up two series $\sum_{i=1}^{n} a_i x_i$ and $\sum_{i=1}^{n} b_i x_i$ such that

$$\Big(\sum_{i=1}^{n} a_i x_i\Big) - a_j x_j = 0 \quad \text{with} \quad \Big(\sum_{i=1}^{n} a_i\Big) - a_j = 0\,,$$

$$\Big(\sum_{i=1}^{n} b_i x_i\Big) - b_h x_h \quad \text{with} \quad \Big(\sum_{i=1}^{n} b_i\Big) - b_h = 0$$

for some $j \neq h$, $1 \leq j, h \leq n$, some nonzero real numbers a_i $(i \neq j)$ and $b_i (i \neq h)$.

Multiply a suitable constant $c \neq 0$ to b_i, if necessary, to make sure that $a_i + cb_i \neq 0$ for $i \neq j, h$, and then by adding the above two series we have

$$\Big[\sum_{i=1}^{n} (a_i + cb_i) x_i\Big] - (a_j x_j + cb_h x_h) = 0 \quad \text{with} \quad \Big[\sum_{i=1}^{n} (a_i + cb_i)\Big] - (a_j + cb_h) = 0\,.$$

This shows that $\{x_i\}_{i=1}^{n}$ has Property L and hence X is strictly n-convex.

Although we do not expect the converse of Theorem 4 to be true (a strictly 3-convex space is not necessarily 2-convex [3, Example 1]), we have the following result.

THEOREM 5. *If X is strictly n-convex for $n \geq 6$, $\{x_i\}_{i=1}^{n-1}$ is a set of nonzero vectors in X such that $\bigcap_{i=1}^{n-1} J(x_i) \neq \emptyset$ and $\|x_k\| = \|2x_k - x_m\|$*

for some $k \neq m$, $1 \leq k, m \leq n-1$. Then at least one of the following five statements is fulfilled.

(1) *The set $\{x_i\}_{i=1}^{n-1}$ has Property L;*

(2) *The set $\{x_i\}_{i=1}^{n-1} \setminus \{x_k\}$ has Property L;*

(3) *The set $\{x_i\}_{i=1}^{n-1} \setminus \{x_m\}$ has Property L;*

(4) *The set $\{x_i\}_{i=1}^{n-1} \setminus \{x_k, x_m\}$ has Property L;*

(5) *The set $\{x_k, x_m, x_s\}$ has Property L, where $s \neq k, m$ and $1 \leq s \leq n-1$.*

Proof. Consider the set $\{x_i\}_{i=1}^{n-1} \cup \{2x_k - x_m\}$, and let $f \in \bigcap_{i=1}^{n-1} J(x_i)$ (hence $\|x_k\| = \|x_m\| = \|f\|$). Then $f(2x_k - x_m) = 2f(x_k) - f(x_m) = \|f\|\|x_k\| = \|f\|\|2x_k - x_m\|$, and thus $f \in \bigcap_{i=1}^{n-1} J(x_i) \cap J(2x_k - x_m)$. It follows that the set $\{x_i\}_{i=1}^{n-1} \cup \{2x_k - x_m\}$ has Property L by (4) in Theorem 3 as X is strictly n-convex. If any two vectors in the set $\{x_i\}_{i=1}^{n-1}$ are equal, or $2x_k - x_m = x_k$, or $2x_k - x_m = x_m$, then we have (1). If $2x_k - x_m = x_s$ for $s \neq k, m$ and $1 \leq s \leq n-1$, then we have (5). Otherwise we may let $(\sum_{i=1}^{n-1} a_i x_i) + a(2x_k - x_m) = 0$ with $(\sum_{i=1}^{n-1} a_i) + a = 0$ for some real $a, a_i \neq 0$, $i = 1, \ldots, n-1$, then

$$\sum_{i=1}^{n-1} b_i x_i = 0 \quad \text{with} \quad \sum_{i=1}^{n-1} b_i = 0,$$

where $b_i = a_i$ if $i \neq k, m$, $b_k = a_k + 2a$ and $b_m = a_m - a$. Therefore, if $b_k, b_m \neq 0$, then (1) holds. $b_k = 0$ and $b_m \neq 0$ imply (2). $b_k \neq 0$ and $b_m = 0$ imply (3). If $b_k = b_m = 0$, then (4) holds and the proof is complete.

COROLLARY 1. *Assumptions as in Theorem 5 and let the relation $\|x_k\| = \|2x_k - x_m\|$ be replaced by $\|x_s\| = \|cx_k + (1-c)x_m\|$ for $s \neq k, m$, $1 \leq s \leq n-1$ and some real $c \neq 0, 1$, then the same conclusions in Theorem 5 hold.*

Proof. Let $f \in \bigcap_{i=1}^{n-1} J(x_i)$, then we can easily show that $f(cx_k + (1-c)x_m) = \|f\|\|cx_k + (1-c)x_m\|$, i.e., $f \in \bigcap_{i=1}^{n-1} J(x_i) \cap J(cx_k + (1-c)x_m)$. Hence the set $\{x_i\}_{i=1}^{n-1} \cup \{cx_k + (1-c)x_m\}$ has Property L, and the desired result follows by a similar discussion as in Theorem 5.

COROLLARY 2. *Let $n = 5$ in Theorem 5, then at least one of the four statements (1), (2), (3) and (5) in Theorem 5 is fulfilled.*

Proof. Consider the set $\{x_i\}_{i=1}^{4}$ in Theorem 5, then the set, say, $\{x_1, x_2, x_3, x_4, 2x_1 - x_2\}$ has Property L by the proof of Theorem 5. Let $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5(2x_1 - x_2) = 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 = 0$ for some real $a_i \neq 0$, $i = 1, \ldots, 5$. Then $(a_1 + 2a_5)x_1 + (a_2 - a_5)x_2 + a_3x_3 + a_4x_4 = 0$.

If $a_1 + 2a_5 = a_2 - a_5 = 0$, then $x_3 = x_4$, i.e., (1) holds. All other cases are similar as in Theorem 5.

COROLLARY 3. *Let $n = 5$ in Theorem 5 and replace the relation $\|x_k\| = \|2x_k - x_m\|$ by $\|x_s\| = \|cx_k + (1-c)x_m\|$ for $s \neq k, m$, $1 \leq s \leq n-1$ and some real $c \neq 0, 1$, then at least one of the four statements (1), (2), (3) and (5) in Theorem 5 is fulfilled.*

Proof. Similar to the proof in Corollary 2.

THEOREM 6. *The following conditions are equivalent:*

(1) X is strictly 3-convex;

(2) X is strictly 4-convex, and if $\bigcap_{i=1}^{3} J(x_i) \neq \emptyset$ for nonzero vectors x_i, $i = 1, 2, 3$, then $\|x_k\| = \|2x_k - x_m\|$ for some $k \neq m$, $1 \leq k, m \leq 3$, or $\|x_3\| = \|cx_1 + (1-c)x_2\|$ for some real $c \neq 0, 1$.

Proof. (1) $\Rightarrow$ (2): A special case of Theorem 4 says that a strictly 3-convex space is strictly 4-convex. If $\bigcap_{i=1}^{3} J(x_i) \neq \emptyset$, then the set $\{x_1, x_2, x_3\}$ has Property L by (4) in Theorem 3. If any two of them are equal, say, $x_k = x_m$, then $\|x_k\| = \|2x_k - x_m\|$. Otherwise we have $a_1x_1 + a_2x_2 + a_3x_3 = 0$ with $a_1 + a_2 + a_3 = 0$ for some real $a_i \neq 0$, $i = 1, 2, 3$. Thus, $x_3 = cx_1 + (1-c)x_2$, where $c = -a_1/a_3$.

(2) $\Rightarrow$ (1): Consider the first case first, i.e., $\bigcap_{i=1}^{n} J(x_i) \neq \emptyset$ and, say, $\|x_1\| = \|2x_1 - x_2\|$. Then as in the proof of Theorem 5 a simple calculation shows that the set $\{x_1, x_2, x_3, 2x_1 - x_2\}$ has Property L. Therefore, at least two of the vectors in the set $\{x_1, x_2, x_3\}$ are equal, or $2x_1 - x_2 - x_3 = 0$, or $b_1x_1 + b_2x_2 + b_3x_3 = 0$ for some real $b_i \neq 0$, $i = 1, 2, 3$ and $b_1 + b_2 + b_3 = 0$. In any way X is strictly 3-convex.

In the second case we have $\|x_3\| = \|cx_1 + (1-c)x_2\|$ and $\bigcap_{i=1}^{3} J(x_i) \neq \emptyset$. Then as in Corollary 1 we have the same conclusions as the first case, except that the relation $2x_1 - x_2 - x_3 = 0$ should be replaced by $cx_1 + (1-c)x_2 - x_3 = 0$ and the proof is complete.

Remark that Theorem 6 indicates the relationship between "three points being collinear" and "four points being coplanar" for certain points in the space. We may similarly discuss strict 2 and 3-convexity.

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