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THE DECOMPOSITION OF TWO-DIMENSIONAL TRANSITION RULES INTO ONE-DIMENSIONAL ONES

1. Introduction

The theory of cellular automata was started by John von Neumann [7, 8] and then developed in many fields of study, both theoretical and practical (see for example [2, 10, 11, 12, 13]). Mathematically we deal with transformations of *configurations* by *transition rules*. A configuration is an assignment of elements from certain finite set to each lattice point of n-dimensional grid. A transition rule F assigns a new configuration $F\omega$ to configuration ω in a locally and uniformly determined manner. The studies on such structures are provided in one-, two-, or n-dimensional cases.

The main idea of this paper may be introduced as follows. For a fixed vector $\mathbf{v}$ (its coordinates are integers) we can consider the whole plane of cells (i.e. two-dimensional configuration) as a collection of lines (in direction $\mathbf{v}$) of cells which can be treated as one dimensional configurations. If we transform each such configuration using fixed one-dimensional transition rule F, we actually transform two-dimensional configuration by certain two-dimensional rule denoted by $F_{\mathbf{v}}$. In this paper we are interested in two-dimensional rules (called decomposable) which are equal to the superposition of two or more two-dimensional transition rules arising in the above described way. Section 3 is devoted to main definitions, basic properties and examples concerning this concept.

In Section 4 we try to give reasons for investigating problems connected with such decomposition. One of the first theoretical problem of the cellular automata theory was a question whether or not a given transition rule is onto (see [4, 5]). It was particularly well examined in one-dimensional case (see e.g. [1, 3, 6, 14]). We prove that two-dimensional decomposable transition rule is onto if and only if each of one dimensional components are onto.

In this manner we transform two-dimensional problem into one-dimensional one, with simpler and better developed theory.

2. Preliminaries and basic definitions

Throughout this paper we will use the following notations:

Z — the set of integers,
Z^n — the set of n-tuples of integers,
N — the set of positive integers,
N_k — the set $\{0, 1, \dots, k-1\}$,
N_k^m — the set of m-tuples of elements of N_k,

If $f : X \to Y$ and $V \subset X$ then $f_{|V}$ denotes the restriction of the function f to the set V, that is a function $g : V \to Y$ such that $g(v) = f(v)$ for all $v \in V$.

If $f : X \to Y$ and $g : Y \to Z$ then gf denotes the superposition of those two functions, i.e. $(gf)(x) = g(f(x))$ for all $x \in A$.

Let n and k be fixed positive integers. An *n-dimensional configuration on k symbols* is a mapping $\omega : Z^n \to N_k$. The set of all such configurations will be denoted by $C_k^{(n)}$. Let us define a metric d_n on $C_k^{(n)}$ as follows. For arbitrary $\omega, \gamma \in C_k^{(n)}$ we put $d_n(\omega, \gamma) = 0$ if $\omega = \gamma$; otherwise we put $d_n(\omega, \gamma) = \frac{1}{p}$, where p is the least positive integer such that there exists $\mathbf{x} = (x_1, x_2, \dots, x_n) \in Z^n$ with $\max_{1 \le i \le n} |x_i| = p - 1$ and $\omega(\mathbf{x}) \neq \gamma(\mathbf{x})$.

2.1. Definition. A function $F : C_k^{(n)} \to C_k^{(n)}$ is called a *transition rule* if and only if there exist a non-negative integer m, a function $f : N_k^m \to N_k$ and a sequence of vectors $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m)$ where $\mathbf{r}_i \in Z^n$, such that

$$(1) \qquad f(0, 0, \dots, 0) = 0,$$

$$(2) \qquad (F\omega)(\mathbf{x}) = f(\omega(\mathbf{x} + \mathbf{r}_1), \omega(\mathbf{x} + \mathbf{r}_2), \dots, \omega(\mathbf{x} + \mathbf{r}_m))$$

for all $\mathbf{x} \in Z^n$, $\omega \in C_k^{(n)}$.

2.1. Remark. A transition rule with an initial configuration are called a *cellular automaton*.

2.2. Remark. The assumption (1) assures that if $\{\mathbf{x} \in Z^n : \omega(\mathbf{x}) \neq 0\}$ is finite then $\{\mathbf{x} \in Z^n : (F\omega)(\mathbf{x}) \neq 0\}$ is also finite. We will not make use of (1).

Transition rule on $C_k^{(n)}$ is also called n-dimensional transition rule on k symbols, function f is said to *generate* F, sequence $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m)$ forms the *neighbourhood* for F.

It is obvious that we can find different generating functions and neighbourhoods for the same transition rule. But we may choose the generating function and the neighbourhood, so that f depends on each variable. We call such neighbourhood *minimal*. There exists one set of vectors which creates minimal neighbourhood for given transition rule.

The set of all n-dimensional transition rules on k symbols is denoted by $\Phi_k^{(n)}$.

For $\mathbf{v} \in Z^n$, $\sigma_{\mathbf{v}}$ denotes an operation $\sigma_{\mathbf{v}} : C_k^{(n)} \to C_k^{(n)}$ such that for all $\mathbf{x} \in Z^n$, $\omega \in C_k^{(n)}$ we have $(\sigma_{\mathbf{v}}\omega)(\mathbf{x}) = \omega(\mathbf{x}-\mathbf{v})$. It is easy to verify that each transition rule is continuous (with respect to the metric d_n) and commutes with $\sigma_{\mathbf{v}}$ for arbitrary $\mathbf{v} \in Z^n$.

Throughout the remainder of the paper we deal only with two- or one-dimensional cases. One-dimensional configurations will be denoted by letters b, c while elements from $C_k^{(2)}$ by ω, γ, α. $(0,0) \in Z^2$ will be denoted by $\mathbf{0}$.

3. The decomposition of two-dimensional rules

In this section we introduce the concept of the decomposition of two-dimensional rules into one-dimensional ones. We give some examples and we prove that each transition rule can be extended to the decomposable one.

3.1. Definition. Let F be a one-dimensional transition rule on k symbols with the generating function f and the neighbourhood $(r_1, r_2, \dots, r_m)$, $r_i \in Z$. Let $\mathbf{v} \in Z^2$. Two-dimensional transition rule $F_{\mathbf{v}}$ is defined by the equality

$$(F_{\mathbf{v}}\omega)(\mathbf{x}) = f(\omega(\mathbf{x}+r_1\mathbf{v}), \omega(\mathbf{x}+r_2\mathbf{v}), \dots, \omega(\mathbf{x}+r_m\mathbf{v}))$$
$$\text{for each } \omega \in C_k^{(2)},\ \mathbf{x} \in Z^2 .$$

Assume that $\mathbf{v} = (p, q)$ where p and q are relatively prime. We see that activity of $F_{\mathbf{v}}$ on the whole plane of cells is equal to the parallel activity of F on each line $\mathbf{x} + t\mathbf{v}, (t \in Z)$ for every $\mathbf{x} \in Z^2$, treated as cells of one-dimensional configuration. It can also be noticed that if $F \in \Phi_k^{(1)}$ is defined by neighbourhood $(r_1, r_2, \dots, r_m)$ and the generating function f then $F_{\mathbf{v}}$ is obtained by the same generating function with the neighbourhood $(r_1\mathbf{v}, r_2\mathbf{v}, \dots, r_m\mathbf{v})$.

3.1. Proposition. *Let F be a one-dimensional transition rule on k symbols and let $\mathbf{v}$ be an element of Z^2. For each $\omega \in C_k^{(2)}$, $\mathbf{x} \in Z^2$ we define $b_{\mathbf{x}} \in C_k^{(1)}$ by equality $b_{\mathbf{x}}(i) = \omega(\mathbf{x} + i\mathbf{v})$. Then*

$$(F_{\mathbf{v}}\omega)(\mathbf{x}) = (Fb_{\mathbf{x}})(0) .$$

Proof. Let f be the generating function for F and let $(r_1, r_2, \dots, r_m)$ be the neighbourhood for F. We have

$$(Fb_{\mathbf{x}})(0) = f(b_{\mathbf{x}}(r_1), b_{\mathbf{x}}(r_2), \dots, b_{\mathbf{x}}(r_m)) = \\ f(\omega(\mathbf{x} + r_1\mathbf{v}), \omega(\mathbf{x} + r_2\mathbf{v}), \dots, \omega(\mathbf{x} + r_m\mathbf{v}))$$

and by Definition 3.1 we have $(Fb_{\mathbf{x}})(0) = (F_{\mathbf{v}}\omega)(\mathbf{x})$. ∎

Now we specify the idea which is the core of this paper.

3.2. Definition. Let F be two-dimensional transition rule on k symbols. We say that F is decomposable into one-dimensional rules if and only if there exist $G^{(1)}, G^{(2)}, \dots, G^{(p)} \in \Phi_k^{(1)}$ and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p \in Z^2$ such that

$$F = G^{(1)}_{\mathbf{v}_1} G^{(2)}_{\mathbf{v}_2} \dots G^{(p)}_{\mathbf{v}_p}.$$

We also say that F is *composed of* one-dimensional rules $G^{(1)}, G^{(2)}, \dots$ $\dots, G^{(p)}$ with vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$.

3.1. Example. Let $G^{(1)}$ and $G^{(2)}$ (from $\Phi_2^{(1)}$) be given by equalities

$$(G^{(1)}b)(i) = b(i) + b(i+1) \pmod 2$$
$$(G^{(2)}b)(i) = b(i) \cdot b(i+2),$$

for all $b \in \mathcal{C}_2^{(1)}$ and $i \in Z$. If F is composed of $G^{(1)}$ and $G^{(2)}$ with certain vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then for all $\mathbf{x} \in Z^2$ and $\omega, \gamma \in \mathcal{C}_2^{(2)}$ we have

$$(G^{(1)}_{\mathbf{v}_1}\gamma)(\mathbf{x}) = (G^{(1)}b_{\mathbf{x}})(0) \quad \text{where } b_{\mathbf{x}}(i) = \gamma(\mathbf{x} + i\mathbf{v}_1) \text{ and}$$
$$(G^{(2)}_{\mathbf{v}_2}\omega)(\mathbf{x}) = (G^{(2)}b'_{\mathbf{x}})(0) \quad \text{where } b'_{\mathbf{x}}(i) = \omega(\mathbf{x} + i\mathbf{v}_1).$$

Therefore

$$(G^{(1)}_{\mathbf{v}_1}\gamma)(\mathbf{x}) = b_{\mathbf{x}}(0) + b_{\mathbf{x}}(1) \pmod 2 = \gamma(\mathbf{x}) + \gamma(\mathbf{x} + \mathbf{v}_1) \pmod 2$$

and

$$(G^{(2)}_{\mathbf{v}_2}\omega)(\mathbf{x}) = b'_{\mathbf{x}}(0) \cdot b'_{\mathbf{x}}(2) = \omega(\mathbf{x}) \cdot \omega(\mathbf{x} + 2\mathbf{v}_2).$$

Finally

$$(F\omega)(\mathbf{x}) = (G^{(1)}_{\mathbf{v}_1} G^{(2)}_{\mathbf{v}_2}\omega)(\mathbf{x}) = (G^{(1)}_{\mathbf{v}_1}(G^{(2)}_{\mathbf{v}_2}\omega))(\mathbf{x}) = \\ (G^{(2)}_{\mathbf{v}_2}\omega)(\mathbf{x}) + (G^{(2)}_{\mathbf{v}_2}\omega)(\mathbf{x} + \mathbf{v}_1) \pmod 2 = \\ \omega(\mathbf{x}) \cdot \omega(\mathbf{x} + 2\mathbf{v}_2) + \omega(\mathbf{x} + \mathbf{v}_1) \cdot \omega(\mathbf{x} + \mathbf{v}_1 + 2\mathbf{v}_2) \pmod 2.$$

In our considerations the following result is very useful.

3.2. Proposition. *Let $F \in \Phi_2^{(2)}$ and let $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m)$ be its minimal neighbourhood. If F is decomposable into two one-dimensional rules then there exist $P, Q \subset Z$ and vectors $\mathbf{u}, \mathbf{v} \in Z^2$ such that*

$$\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\} = \{i\mathbf{u} + j\mathbf{v} : i \in P \text{ and } j \in Q\}.$$

Proof. Let us assume that F is decomposable into two one dimensional transition rules, thus $F = G_{\mathbf{u}}H_{\mathbf{v}}$ for certain $G, H \in \Phi_2^{(1)}$ and $\mathbf{u}, \mathbf{v} \in Z^2$. Let

$P, Q \subset Z$ be the sets of all minimal neighbourhood elements for G and H, respectively. Obviously

$$\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\} \subset \{i\mathbf{u} + j\mathbf{v} : i \in P \text{ and } j \in Q\}$$

(for $(G_\mathbf{u} H_\mathbf{v} \omega)(\mathbf{x})$ does not depend on values other than $\omega(\mathbf{x}+i\mathbf{u}+j\mathbf{v})$ where $i \in P$ and $j \in Q$). We will prove that

$$\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\} \supset \{i\mathbf{u} + j\mathbf{v} : i \in P \text{ and } j \in Q\}.$$

We shall assume that $P \neq \emptyset$, $Q \neq \emptyset$ and $\mathbf{u}$, $\mathbf{v}$ are linearly independent (otherwise the proposition is obvious). Let us fix $i \in P$ and $j \in Q$. There exist $b, b' \in C_2^{(1)}$ such that $b(k) = b'(k)$ for $k \neq i, b(i) = 1 + b'(i) \pmod 2$ and $(Gb)(0) \neq (Gb')(0)$ (because P is the set of all minimal neighbourhood elements for G). For the similar reasons, there exist configurations $c, c' \in C_2^{(1)}$ such that $c(k) = c'(k)$ for $k \neq j$, $c(j) = 1 + c'(j) \pmod 2$ for which we have $(Hc)(0) = 0$, $(Hc')(0) = 1$. Let us define $\omega \in C_2^{(2)}$ in the following way:

$$\omega(k\mathbf{u} + l\mathbf{v}) = \begin{cases} c(l) & \text{if } b(k) = 0 \\ c'(l) & \text{if } b(k) = 1 \end{cases}$$

and $\omega(\mathbf{x}) = 0$ if $\mathbf{x} \neq k\mathbf{u} + l\mathbf{v}$ for $k, l \in Z$. We also define $\omega' \in C_2^{(2)}$ by equalities $\omega'(\mathbf{x}) = \omega(\mathbf{x})$ for $\mathbf{x} \neq i\mathbf{u} + j\mathbf{v}$ and $\omega'(i\mathbf{u}+j\mathbf{v}) = \omega(i\mathbf{u}+j\mathbf{v}) + 1 \pmod 2$. One can easily check that $(F\omega)(\mathbf{0}) = (Gb)(0)$ and $(F\omega')(\mathbf{0}) = (Gb')(0)$ therefore $(F\omega)(\mathbf{0}) \neq (F\omega')(\mathbf{0})$. That means $i\mathbf{u} + j\mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$. ∎

The following example shows that the above proposition is not true if $\Phi_2^{(2)}$ is replaced by $\Phi_k^{(2)}$ for $k > 2$.

3.2. EXAMPLE. Let G and $H \in \Phi_3^{(1)}$ have generating functions $g, h : N_3^2 \to N_3$, respectively, such that

$$g(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) = (1, 0) \text{ or } (x_1, x_2) = (1, 1) \\ 1 & \text{if } (x_1, x_2) = (1, 2) \\ 2 & \text{otherwise,} \end{cases}$$

$$h(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) = (1, 0) \text{ or } (x_1, x_2) = (1, 1) \\ 1 & \text{if } (x_1, x_2) = (1, 2) \\ 2 & \text{otherwise} \end{cases}$$

and the neighbourhood is $(0, 1)$ for both. Given two linearly independent vectors $\mathbf{u}$, $\mathbf{v}$ one can easily check that the minimal vectors for $F = G_\mathbf{u} H_\mathbf{v}$ are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$ only. The set $\{\mathbf{0}, \mathbf{u}, \mathbf{v}\}$ is not equal to $\{i\mathbf{w}_1 + j\mathbf{w}_2 : i \in P \text{ and } j \in Q\}$ for any $P, Q \subset Z, \mathbf{w}_1, \mathbf{w}_2 \in Z^2$.

The next example explains why we consider decomposition into (possibly) more than two factors.

3.3. Example. Let $G \in \Phi_2^{(1)}$ be given by equality $(Gb)(i) = b(i) + b(i+1) \pmod 2$. Let $\mathbf{u} = (1,0), \mathbf{v} = (0,1)$, $\mathbf{w} = (1,1)$ and let $F \in \Phi_2^{(2)}$ be composed of G with vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. After the computations similar to those in Example 3.1 we obtain

$$\begin{aligned}(F\omega)(\mathbf{x}) = (G_{\mathbf{u}}G_{\mathbf{v}}G_{\mathbf{w}}\omega)(x) &= \omega(\mathbf{x}) + \omega(\mathbf{x}+\mathbf{u}) + \omega(\mathbf{x}+\mathbf{v}) + \\ &+ \omega(\mathbf{x}+\mathbf{w}) + \omega(\mathbf{x}+\mathbf{u}+\mathbf{v}) + \omega(\mathbf{x}+\mathbf{u}+\mathbf{w}) + \omega(\mathbf{x}+\mathbf{v}+\mathbf{w}) + \\ &+ \omega(\mathbf{x}+\mathbf{u}+\mathbf{v}+\mathbf{w}) \pmod 2 .\end{aligned}$$

But $\omega(\mathbf{x}+\mathbf{u}+\mathbf{v}) + \omega(\mathbf{x}+\mathbf{w}) \pmod 2 = 0$ so

$$\begin{aligned}(F\omega)(\mathbf{x}) &= \omega(\mathbf{x}) + \omega(\mathbf{x}+\mathbf{u}) + \omega(\mathbf{x}+\mathbf{v}) + \omega(\mathbf{x}+\mathbf{u}+\mathbf{w}) + \omega(\mathbf{x}+\mathbf{v}+\mathbf{w}) + \\ &+ \omega(\mathbf{x}+\mathbf{u}+\mathbf{v}+\mathbf{w}) \pmod 2 ,\end{aligned}$$

and finally $\{\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is the set of all minimal neighbourhood vectors. By dint of Proposition 3.2 we infer that F is not decomposable into two one-dimensional transition rules; but it is clear that F *is* decomposable.

Throughout the remainder of this section we concentrate on the decomposition into two one-dimensional transition rules.

3.4. Example. Let $F \in \Phi_2^{(2)}$ be given by neighbourhood $((1,0)$, $(0,1)$, $(1,1))$ and the generating function $f : N_2^3 \to N_2$ such that

$$f(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } (x_1, x_2, x_3) = (1,1,1) \\ 0 & \text{otherwise.} \end{cases}$$

Applying Proposition 3.2 we can prove that F is not decomposable into two one-dimensional transition rules. Let $H \in \Phi_2^{(1)}$ be given by the neighbourhood $(0,1)$ and the generating function $h : N_3^2 \to N_3$ such that

$$h(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) = (0,1) \text{ or } (x_1, x_2) = (2,1) \\ 1 & \text{if } (x_1, x_2) = (1,1) \\ 2 & \text{otherwise} \end{cases}$$

and let $G \in \Phi_3^{(1)}$ be given by neighbourhood $(0,1)$ and the generating function $g : N_3^2 \to N_3$ such that

$$g(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1 = 0 \text{ or } x_1 = 1) \text{ and } x_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{u} = (1,0), \mathbf{v} = (0,1)$. If $\omega \in C_2^{(2)}$ then it is easy to check that $F\omega = G_{\mathbf{u}}H_{\mathbf{v}}\omega$. Thus F can be extended to transition rule from $\Phi_2^{(3)}$ which is decomposable into two one-dimensional transition rules.

The property suggested in the above example is true in all cases. This fact is expressed in Theorem 3.2. But first we have to prove the following theorem.

3.1. THEOREM. *Let F be the two-dimensional transition rule on k symbols with minimal neighbourhood $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m)$. Let $\mathbf{u}, \mathbf{v} \in Z^2$. The following statements are equivalent:*

(a) $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\} \subset \{i\mathbf{u} + j\mathbf{v} : i, j \in Z\}$,

(b) *there exist positive integer p, one-dimensional transition rules on p symbols G and H such that if $F' = G_{\mathbf{u}} H_{\mathbf{v}}$ then $F'_{|C_k^{(2)}} = F$.*

Proof. Suppose that (a) holds. If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent then $F = G_{\mathbf{v}}$ for some $G \in \Phi_k^{(1)}$ and the proof is completed. Thus we can assume $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Let q be the least non-negative integer such that $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\} \subset \{i\mathbf{u} + j\mathbf{v} : |i| \leq q \text{and} |j| \leq q\}$ and let $p = k^{2q+1}$. We define a function $d : N_k^{2q+1} \to N_p$ by equality

$$d(x_{-q}, x_{-q+1}, \ldots, x_q) = x_{-q} + x_{-q+1} \cdot k + \ldots + x_q \cdot k^{2q} .$$

It is obvious that d is injective and onto, thus function d^{-1} exists and transforms N_p onto N_k^{2q+1}. If $d^{-1}(y) = (x_{-q}, x_{-q+1}, \ldots, x_q)$ then by $d_j^{-1}(y)$ we denote x_j for $-q \leq j \leq q$. For each $b \in C_p^{(1)}$ and each $i \in Z$ we define $H \in \Phi_p^{(1)}$ by

$$(Hb)(i) = \begin{cases} d(b(i-q), b(i-q+1), \ldots, b(i+q)) & \text{if } b(j) < k \text{ for } \\ & i - q \leq j \leq i + q \\ 0 & \text{otherwise.} \end{cases}$$

$G \in \Phi_p^{(1)}$ is defined as follows. Let $b \in C_p^{(1)}, t \in Z$. Let $\omega_{b,t} \in C_k^{(2)}$ be a configuration such that for all $\mathbf{x} \in Z^2$

$$\omega_{b,t}(\mathbf{x}) = \begin{cases} d_j^{-1}(b(t+i)) & \text{if } \mathbf{x} = i\mathbf{u} + j\mathbf{v} \text{ for some } i, j \in Z \\ & \text{such that } |i| \leq q \text{ and } |j| \leq q \\ 0 & \text{otherwise.} \end{cases}$$

We put $(Gb)(t) = (F\omega_{b,t})(\mathbf{0})$. Direct calculation shows that $G_{\mathbf{u}} H_{\mathbf{v}} \omega = F\omega$ for $\omega \in C_k^{(2)}$. Thus (b) holds.

Now we assume that (a) does not hold . Our task is to prove that (b) does not hold either. Without the loss of generality we may assume that $\mathbf{r}_1 \neq i\mathbf{u} + j\mathbf{v}$ for all $i, j \in Z$. As $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m)$ is the minimal neighbourhood, we infer that there exist two configurations ω and ω' such that $\omega(\mathbf{x}) = \omega'(\mathbf{x})$ for all $\mathbf{x} \in Z^2$ except $\mathbf{x} = \mathbf{r}_1$ and $(F\omega)(\mathbf{0}) \neq (F\omega')(\mathbf{0})$. Let $p \in N$, and let G, H be one-dimensional rules on p symbols with neighbourhoods $(i_1, i_2, \ldots, i_r)$, $(j_1, j_2, \ldots, j_s)$, respectively. It is obvious that $(G_{\mathbf{u}} H_{\mathbf{v}} \omega)(\mathbf{0})$ may depend only on values $\omega(i\mathbf{u} + j\mathbf{v})$ where $i \in \{i_1, i_2, \ldots, i_r\}$ and $j \in \{j_1, j_2, \ldots, j_s\}$, thus $(G_{\mathbf{u}} H_{\mathbf{v}} \omega)(\mathbf{0})$ does not depend on $\omega(\mathbf{r}_1)$. That means $(F\omega)(\mathbf{0}) = (F\omega')(\mathbf{0})$. Since p, G, H are arbitrary, (b) does not hold. ∎

As a conclusion we obtain the following theorem.

3.2. THEOREM. *For every two-dimensional transition rule $F \in \Phi_k^{(2)}$ there exist $p \in N$ and $F' \in \Phi_p^{(2)}$ such that $F'_{|C_k^{(2)}} = F$ and F' is decomposable into two one-dimensional transition rules.*

Proof. If we take $\mathbf{u} = (1,0)$, $\mathbf{v} = (0,1)$ then (a) in Theorem 3.1 is true thus (b) holds. ■

4. Properties of decomposable transition rules

In this section we show some motivations for searching for the decomposition defined in the previous section. The main result is Theorem 4.2 and the general idea of its proof is due to [3]. First some lemmas and theorems should be presented.

We will deal with configurations on k symbols, for certain fixed positive integer k. Let p, q be fixed positive integers and let $\mathcal{M}_{p,q}$ denote the set of all matrices with p rows and q columns and of entries from N_k. We say that $A = (a_{ij}) \in \mathcal{M}_{p,q}$ *appears* in matrix $B = (b_{ij}) \in \mathcal{M}_{r,s}$ if and only if there exist $g, h \in Z$ such that $a_{ij} = b_{g+i,h+j}$ for $1 \le i \le p$, $1 \le j \le q$. Similarly, if $\omega \in C_k^{(2)}$ then A *appears* in ω if and only if for some $g, h \in Z$ we have $a_{ij} = \omega(g+i, h+j)$. If $C = (c_{ij}) \in \mathcal{M}_{p,q}$ and $D = (d_{ij}) \in \mathcal{M}_{p,r}$ then CD denotes the matrix $E = (e_{ij}) \in \mathcal{M}_{p,q+r}$ such that $e_{ij} = c_{ij}$, $1 \le j \le p$ and $e_{ij} = d_{i,j-p}$, $p < j \le q + r$.

4.1. LEMMA. *Let $p \in N$, $A \in \mathcal{M}_{p,p}$. For $q \ge p$ let $N(A,q)$ denote the number of matrices from $\mathcal{M}_{p,q}$ in which A appears. Then* $\lim_{q\to\infty} \frac{N(A,q)}{k^{pq}} = 1$.

Proof. Let $\phi(q) = \frac{N(A,q)}{k^{pq}}$. It is obvious that $\phi(q) \le 1$. If A appears in $B \in \mathcal{M}_{p,q}$ then A appears in BC for each $C \in \mathcal{M}_{p,1}$. $\mathrm{card}(\mathcal{M}_{p,1}) = k^p$ thus $N(A, q+1) \ge N(q)k^p$ and hence $\phi(q+1) \ge q$. Therefore

$$(1) \qquad \lim_{q\to\infty} \phi(q) = \alpha, \text{ where } 0 \le \alpha \le 1\,.$$

Now let $t \in N$ and let us define the sets

$$B_i = \{A_1 A_2 \dots A_t \in \mathcal{M}_{p,tp} : A_j \in \mathcal{M}_{p,p} \text{ for } 1 \le j \le t, A_j \ne A \text{ for } j < i,\ A_i = A\}$$

for $i = 1, 2, \dots, t$. Of course A appears in every member of B_i, and $B_i \cap B_j = \emptyset$ for $i \ne j$. Thus

$$N(A, tp) \ge \sum_{i=1}^{t} \mathrm{card}(B_i) = \sum_{i=1}^{t} (k^{p^2} - 1)^{i-1} (k^{p^2})^{t-1} =$$

$$= \frac{(k^{p^2}-1)^t - (k^{p^2})^t}{(k^{p^2}-1) - k^{p^2}} = k^{tp^2} - (k^{p^2}-1)^t$$

therefore

$$\phi(tp) \geq \frac{k^{tp^2} - (k^{p^2}-1)^t}{k^{tp^2}} = 1 - \left(1 - \frac{1}{k^{p^2}}\right)^t$$

and we have

$$\lim_{t\to\infty} \phi(tp) = 1 .$$

Because of (1) the proof is complete. ■

4.2. LEMMA. *Let p and q be the positive integers and let $A \in \mathcal{M}_{p,p}$. Assume that for each $r > q$ there is defined a partition*

$$\{D_1^{(r)}, D_2^{(r)}, \ldots, D_{k^{p(r-q)}}^{(r)}\}$$

of $\mathcal{M}_{p,r}$ into disjoint non-empty sets. Then there exists an integer $R > q$, such that some $D_i^{(R)}$ has the property that A appears in every its member.

Proof. Let us suppose that the lemma is false. Let $N(A,r)$ denote the number of elements of $\mathcal{M}_{p,r}$ in which A appears. Then for all $r > q$ we have

$$N(A,r) \leq \operatorname{card}(\mathcal{M}_{p,r}) - k^{p(r-q)} = k^{pr} - k^{p(r-q)}$$

hence $\frac{N(A,r)}{k^{pr}} \leq 1 - \frac{1}{k^{pq}}$ what contradicts Lemma 4.1. ■

4.1. DEFINITION. A configuration $\omega \in C_k^{(2)}$ is *transitive* if and only if for all $p \in N$, each $A \in \mathcal{M}_{p,p}$ appears in ω.

4.1. Remark. The existence of such configuration is obvious.

4.2. Remark. We infer that for all $p, q \in N$ each $A \in \mathcal{M}_{p,q}$ appears in a given transitive configuration.

4.3. LEMMA. *Let F be a one-dimensional transition rule on k symbols and let F be onto. Let $\mathbf{v} = (0,1)$ and let ω be a transitive two-dimensional configuration on k symbols. If $F_{\mathbf{v}}(\gamma) = \omega$ then γ is also transitive.*

Proof. Assume that $F_{\mathbf{v}}(\gamma) = \omega$. Let $A \in \mathcal{M}_{p,p}$ for certain $p \in N$. We will show that A appears in γ. Since A is arbitrary, the proof will be complete.

We may assume that the neighbourhood for F is the sequence $(-m, -m+1, \ldots, m)$ and that f is the generating function. Let $q = 2m$. For each $s \geq 1$ we may define function $f_s : N_k^{s+q} \to N_k^s$ by equality

$$f_s(x_1, x_2, \ldots, x_{s+q}) = (f(x_1, x_2, \ldots, x_{1+q}), \ldots, f(x_s, x_{s+1}, \ldots, x_{s+q})) .$$

F is onto hence (see [3]) we know that

$$\operatorname{card}(f_s^{-1}(x)) = k^q \text{ for every } x \in N_k^s . \quad (2)$$

Now for every $r > q$ define a partition of the set $\mathcal{M}_{p,r}$ in the following way. B and C from $\mathcal{M}_{p,r}$ belong to the same class if and only if

$$f_{r-q}(b_{i_1}, b_{i_2}, \ldots, b_{i_r}) = f_{r-q}(c_{i_1}, c_{i_2}, \ldots, c_{i_r}) \text{ for all } 1 \le i \le p\,.$$

Thus for each $r > q$ we have established a partition of $\mathcal{M}_{p,r}$ into disjoint non-empty sets. Moreover, from (2) we see that such a partition contains $\frac{k^{pr}}{k^{qr}} = k^{p(q-r)}$ sets. Applying Lemma 4.2 we infer that there exists $R > q$ and matrix $B \in \mathcal{M}_{p,R-q}$ such that for every $C \in \mathcal{M}_{p,R}$ satisfying the condition

$$f_{R-q}(b_{i_1}, b_{i_2}, \ldots, b_{i_R}) = f_{R-q}(c_{i_1}, c_{i_2}, \ldots, c_{i_R}) \text{ for all } 1 \le i \le p\,.$$

A appears in C. Since ω is transitive, every $B \in \mathcal{M}_{p,R-q}$ appears in ω. Thus because of Definition 3.1 A appears in γ. ∎

In order to prove Theorem 4.1 which is almost the same as Lemma 4.3 but allows $\mathbf{v}$ to be an arbitrary element from Z^2, we will introduce the following notation.

Notation. Let $\mathbf{v}, \mathbf{w} \in Z^2$. By $\mathcal{L}_{\mathbf{v},\mathbf{w}}$ we denote an operation $\mathcal{L}_{\mathbf{v},\mathbf{w}} : \mathcal{C}_k^{(2)} \to \mathcal{C}_k^{(2)}$ such that for all $\mathbf{x} = (x_1, x_2) \in Z^2, \omega \in \mathcal{C}_k^{(2)}$ we have $(\mathcal{L}_{\mathbf{v},\mathbf{w}}\omega)(\mathbf{x}) = \omega(x_1\mathbf{v} + x_2\mathbf{w})$.

Direct calculation shows that for every $F \in \Phi_k^{(1)}$ and every $\mathbf{v}, \mathbf{w} \in Z^2$ we have

$$\mathcal{L}_{\mathbf{v},\mathbf{w}} F_{\mathbf{v}} = F_{(1,0)} \mathcal{L}_{\mathbf{v},\mathbf{w}} \tag{3}$$

The next important property of $\mathcal{L}_{\mathbf{v},\mathbf{w}}$ demands detail proof.

4.4. Lemma. *Let* $\mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2) \in Z^2, v_1w_2 - v_2w_1 = 1$ *and* $\mathcal{L}_{\mathbf{v},\mathbf{w}}\omega = \gamma$ *for some* $\omega, \gamma \in \mathcal{C}_k^{(2)}$. ω *is transitive if and only if* γ *is transitive.*

Proof. First we will prove that if ω is transitive then γ also is. To do it, suppose ω is transitive, let $A = (a_{ij}) \in \mathcal{M}_{p,p}$ for some p and we will show that A appears in γ.

As ω is transitive, there exist $\mathbf{y} \in Z^2$ such that $\omega(\mathbf{y} + i\mathbf{v} + j\mathbf{w}) = a_{ij}$ for $1 \le i, j \le p$. Let $\overline{\mathbf{y}} = (\overline{y}_1, \overline{y}_2)$ where $\overline{y}_1 = y_1w_2 - y_2w_1$ and $\overline{y}_2 = -y_1v_2 + y_2v_1$. We have

$$\gamma((\overline{y}_1 + i, \overline{y}_2 + j)) = \omega((\overline{y}_1 + i)\mathbf{v} + (\overline{y}_2 + j)\mathbf{w})$$

and after some calculations we obtain

$$\gamma((\overline{y}_1 + i, \overline{y}_2 + j)) = \omega((y_1(v_1w_2 - v_2w_1) + iv_1 + jw_1, y_2(v_1w_2 - v_2w_1) + \\ + iv_2 + jw_1))\,.$$

But $v_1w_2 - v_2w_1 = 1$, so finally we have

$$\gamma((\overline{y}_1 + i, \overline{y}_2 + j)) = \omega(\mathbf{y} + i\mathbf{v} + j\mathbf{w}) = a_{ij} \qquad \text{for } 1 \le i, j \le p$$

hence A appears in γ. As A is an arbitrary matrix we infer that γ is transitive.

Now suppose γ is transitive and let us prove that ω is transitive also. Let $\mathbf{v}' = (v_1', v_2')$ and $\mathbf{w}' = (w_1', w_2')$ be given by equalities

$$v_1' = w_2, \quad v_2' = -v_2, \quad w_1' = -w_1, \quad w_2' = v_1 .$$

Then for all $\mathbf{x} = (x_1, x_2)$ we have

$$(\mathcal{L}_{\mathbf{v}',\mathbf{w}'}\gamma)(\mathbf{x}) = \gamma(x_1\mathbf{v}' + x_2\mathbf{w}')$$

and after simplifications we obtain

$$(\mathcal{L}_{\mathbf{v}',\mathbf{w}'}\gamma)(\mathbf{x}) = \omega(\mathbf{x})$$

therefore $\mathcal{L}_{\mathbf{v}',\mathbf{w}'}\gamma = \omega$ and we also have $v_1'w_2' - v_2'w_1' = 1$ so we can prove that ω is transitive in the way shown in the first part of the proof (exchanging γ and ω, and replacing $\mathbf{v}$, v_1, v_2, $\mathbf{w}$, w_1, w_2 by $\mathbf{v}'$, v_1', v_2', $\mathbf{w}'$, w_1', w_2', respectively). ∎

4.1. THEOREM. *Let F be a one-dimensional transition rule on k symbols and let F be onto. Let $\mathbf{v} \in Z^2$, $\omega, \gamma \in \mathcal{C}_k^{(2)}$ and let $F_{\mathbf{v}}(\gamma) = \omega$. If ω is transitive then γ also is.*

Proof. Let $\mathbf{v} = (v_1, v_2)$. Choose $\mathbf{w} = (w_1, w_2) \in Z^2$ so that $v_1w_2 - v_2w_1 = 1$ (such $\mathbf{w}$ exists, see [9]). $F_{\mathbf{v}}\gamma = \omega$ thus $\mathcal{L}_{\mathbf{v},\mathbf{w}}F_{\mathbf{v}}\gamma = \mathcal{L}_{\mathbf{v},\mathbf{w}}\omega$ and due to (3) we have

$$F_{(1,0)}\mathcal{L}_{\mathbf{v},\mathbf{w}}\gamma = \mathcal{L}_{\mathbf{v},\mathbf{w}}\omega . \tag{4}$$

ω is transitive, hence (Lemma 4.4) $\mathcal{L}_{\mathbf{v},\mathbf{w}}\omega$ is transitive, hence (Lemma 4.3 and (4)) $\mathcal{L}_{\mathbf{v},\mathbf{w}}\gamma$ is transitive, and finally (Lemma 4.4) γ is transitive. ∎

4.5. LEMMA. *Let G be a one-dimensional transition rule on k symbols and let $\mathbf{v} \in Z^2$. Then G is onto if and only if $G_{\mathbf{v}}$ is onto.*

Proof. If $G_{\mathbf{v}}$ is onto then let $b \in \mathcal{C}_k^{(1)}$. Define $\omega \in \mathcal{C}_k^{(2)}$ by equality $\omega(i\mathbf{v}) = b(i)$ and $\omega(\mathbf{x}) = 0$ for $\mathbf{x} \neq i\mathbf{v}$, $(i \in Z)$. Let $\gamma \in G_{\mathbf{v}}^{-1}(\omega)$ then $c \in \mathcal{C}_k^{(1)}$ given by $c(i) = \gamma(i\mathbf{v})$ belongs to $G^{-1}(b)$. Thus G is onto.

Now assume that G is onto. Let $\omega \in \mathcal{C}_k^{(2)}$. We will find γ, such that $G_{\mathbf{v}}(\gamma) = \omega$. Let $X \subset Z^2$ satisfy two conditions

$$Z^2 \subset \{\mathbf{x} + t\mathbf{v} : \mathbf{x} \in X, t \in R,\} \qquad \text{if } \mathbf{x}, \mathbf{y} \in X \text{ then } \mathbf{y} - \mathbf{x} \neq t\mathbf{v} \text{ for } t \in R .$$

It is obvious that such set exists. Let l be the greatest common divisor of p and q, where $\mathbf{v} = (p, q)$. For $x \in X$ we define configurations $b_{\mathbf{x}}^{(0)}, b_{\mathbf{x}}^{(1)}, \ldots$ $\ldots, b_{\mathbf{x}}^{(l-1)} \in \mathcal{C}_k^{(1)}$ by means of the following formula

$$b_{\mathbf{x}}^{(i)}(j) = \omega\left(\mathbf{x} + \left(\frac{i}{l} + j\right)\mathbf{v}\right) \text{ for } 1 \le i \le l - 1 .$$

We choose configuration $c_{\mathbf{x}}^{(i)}$ so that $G(c_{\mathbf{x}}^{(i)}) = b_{\mathbf{x}}^{(i)}$ (such configuration exists because G is onto). The configuration γ, such that $\gamma(\mathbf{x} + (\frac{i}{l} + j)\mathbf{v}) = c_{\mathbf{x}}^{(i)}(j)$ for all $\mathbf{x} \in X$, $i \in N_l$ and $j \in Z$ is well defined and satisfies equality $G_{\mathbf{v}}(\gamma) = \omega$. ■

4.2. THEOREM. *Let $F \in \Phi_k^{(2)}$, $G^{(1)}, G^{(2)}, \ldots, G^{(p)} \in \Phi_k^{(1)}$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in Z^2$ and $F = G_{\mathbf{v}_1}^{(1)} G_{\mathbf{v}_2}^{(2)} \ldots G_{\mathbf{v}_p}^{(p)}$. F is onto if and only if each function $G^{(i)}$ is onto.*

Proof. Of course if $G^{(1)}, G^{(2)}, \ldots, G^{(p)}$ are onto then (applying Lemma 4.5) $G_{\mathbf{v}_1}^{(1)}, G_{\mathbf{v}_2}^{(2)}, \ldots, G_{\mathbf{v}_p}^{(p)}$ are onto and hence F is onto.

Now assume that F is onto. Denote $H = G_{\mathbf{v}_2}^{(2)} G_{\mathbf{v}_3}^{(3)} \ldots G_{\mathbf{v}_p}^{(p)}$. Thus $F = G_{\mathbf{v}_1}^{(1)} H$. Obviously $G_{\mathbf{v}_1}^{(1)}$ is onto. We will show that H is onto.

Let $\omega \in \mathcal{C}_k^{(2)}$ be transitive. F is onto, therefore we can find γ and α so that $H(\alpha) = \gamma$ and $G_{\mathbf{v}_1}^{(1)}(\gamma) = \omega$. Since $G^{(1)}$ is onto, γ is also transitive (see Theorem 4.1).

Let $\beta \in \mathcal{C}_k^{(2)}$ be an arbitrary chosen configuration. One can easily note that for any $\epsilon > 0$, there exists $\mathbf{v} \in Z^2$ so that $d_2(\beta, \sigma_{\mathbf{v}}(\gamma)) < \epsilon$. This is implied by the fact that γ is transitive. Since $\sigma_{\mathbf{v}}(\gamma) = H(\sigma_{\mathbf{v}}(\alpha))$ (H commutes with $\sigma_{\mathbf{v}}$), we infer that $H(\mathcal{C}_k^{(2)})$ is dense in $\mathcal{C}_k^{(2)}$. For $H(\mathcal{C}_k^{(2)})$ is closed (H is continuous, $\mathcal{C}_k^{(2)}$ is compact so $H(\mathcal{C}_k^{(2)})$ is compact) we see that $H(\mathcal{C}_k^{(2)}) = \mathcal{C}_k^{(2)}$, what means H is onto.

Applying the same considerations for $F' = G_{\mathbf{v}_2}^{(2)} G_{\mathbf{v}_3}^{(3)} \ldots G_{\mathbf{v}_p}^{(p)}$, we see that $G^{(2)}$ is onto and $H' = G_{\mathbf{v}_3}^{(3)} \ldots G_{\mathbf{v}_p}^{(p)}$ is onto. After $p - 1$ such steps we obtain the thesis. ■

Final remarks. Results of this paper may be generalized by dealing with decomposition of n-dimensional transition rules. It would be interesting to find convenient method for determining the possibility of a decomposition. In my opinion it is important to find more properties of two-dimensional transition rules which are equivalent to some properties of one-dimensional components of the decomposition.

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