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ON SOME FIXED POINT THEOREMS IN HILBERT SPACES

Introduction. There are several papers on fixed point theorems for nonexpansive maps in Hilbert space, dealing with different boundary conditions.

The famous Ky Fan [4] states below as Theorem (F) yields as a consequence fixed point theorems under different boundary conditions. The aim of this paper is to give results of Ky Fan type in Hilbert space and then derive fixed point theorems for nonexpansive mappings.

For different types of mappings with fixed point theorems one should refer to a well-known paper of Rhoades [15].

In the end weakly nonexpansive multivalued mappings are considered.

1. THEOREM (F). *Let C be a compact, convex subset of a Banach space X and $f : C \to X$ a continuous function. Then there is a $y \in C$ such that $\|y - fy\| = d(fy, C)$.*

In case $fy \in C$ then f has a fixed point. Several fixed point theorems are derived as corollaries from Theorem (F).

We need the following definitions.

Let C be a nonempty subset of a Banach space X. Then C is called proximinal if each $x \in X$ has a best approximation in C, i.e. if the set

$$P_C(x) = \{y \in C : \|x - y\| = \inf \|x - x\| : z \in C\}$$

is nonempty for every $x \in X$. The map P_C is called the metric projection onto C. In case $P_C(x)$ is a singleton for each $x \in X$ then C is said to be

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Chebyshev. If C is a closed convex subset of a Hilbert space H then P is a proximity map [3] *and C is a Chebyshev set.*

For a map $f : C \to X$, where C is a nonempty subset of X, one tries to find an $x \in C$ that is closest to $f(x)$, in other words, finds a solution of the problem

$$(*) \qquad x \in C \text{ and } \|x - fx\| = d(fx, C).$$

It is clear that a fixed point of $P \circ f$ is a solution of $(*)$.

The measure of noncompactness of a bounded set A in a Banach space X denoted by $\alpha(A)$, is defined as

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by a finite number of sets each of diameter} \leq \varepsilon\}.$$

Let $f : C \to X$ be a continuous mapping. Then f is said to be densifying if $\alpha(f(A)) < \alpha(A)$, for each bounded set A with $\alpha(A) > 0$.

In case $\alpha(f(A)) \leq \alpha(A)$, then f is called a 1-set contraction.

A map $f : X \to X$ satisfying the condition

$\|fx - fy\| \leq \|x - y\|$ for all $x, y \in X$ is called a nonexpansive map.

A contraction map is densifying and a nonexpansive map is a 1-set contraction, (see for details [13]).

We state the following

THEOREM 1. *Let C be a weakly compact subset of a Hilbert space H and $f : C \to H$ a nonexpansive map. Then there is a $y \in \overline{\text{co}}(C) = D$, such that*

$$\|y - Fy\| = d(Fy, D)$$

where F is a nonexpansive extension of f to D.

The following extension result is very useful [9], [19].

Let $f : C \to H$ be a nonexpansive map. Then there exists a nonexpansive map $F : \overline{\text{co}}(C) \to H$ such that $F|_C = f$.

Proof of Theorem 1. Let $P : H \to D$ be the metric projection. Then P is a nonexpansive map and therefore $P \circ F$ is a nonexpansive map. Let $T = P \circ F : D \to D$. Then T has a fixed point in D (Browder [1]) say $u = Tu = P \circ Fu$. Then we get

$$\|u - Fu\| = d(Fu, D).$$

Remark 1. If C is a closed, bounded, convex subset of H and $f : C \to H$ is a nonexpansive map, then there is a $u \in C$ such that $\|u - fu\| = d(fu, C)$.

In this case we take $D = C$.

Results given in [2], [5], [11], [12], [16], and [18] can be derived easily.

We give the following to illustrate application of Theorem 1.

EXAMPLE 1. Let B_r be a ball of radius r and center at 0 in a Hilbert space H. If $f : B_r \to H$ is a nonexpansive map satisfying

$$(**) \qquad \|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2 \text{ for } x \in \partial B_r,$$

then f has a fixed point.

In fact, by Remark 1 there is a $u \in B_r$ such that

$$\|u - fu\| = d(fu, B_r).$$

If $fu \in B_r$ then f has a fixed point. We assume that $fu \notin B_r$ and then seek a contradiction.

If $fu \notin B_r$ then $u = Pfu \in \partial B_r$ and $\|u\| = r$, $\|fu\| > r$. Since $\|fu - u\|^2 \geq \|fu\|^2 - \|u\|^2$ gives

$$\|fu\|^2 \leq \|fu - u\|^2 + \|u\|^2 < (\|fu - u\| + \|u\|)^2 = \|fu\|^2,$$

a contradiction; so $fu \in B_r$ and f has a fixed point.

Note. If in place of $(**)$ f satisfies $f(\partial B_r) \subset B_r$, (Rothe type condition) then f was a fixed point.

Again, Remark 1 implies that there is a $u \in B_r$ with $\|u - fu\| = d(fu, B_r)$. The condition $f(\partial B_r) \subset B_r$ guarantees that $fu \in B_r$ and f has a fixed point. Now we give a theorem where C or $f(C)$ need not be bounded.

THEOREM 2. *Let C be a closed, convex subset of a Hilbert space H and $f : C \to H$ a nonexpansive map. Let P be the metric projection on C. If there is an $x_0 \in C$ such that $\{(P \circ f)^n x_0\}$ is bounded then $P \circ f$ has a fixed point say u, i.e., $\|u - fu\| = d(fu, C)$.*

Proof. Let $T = P \circ f : C \to C$. Then T is a nonexpansive map with $\{T^n x_0\}$ bounded and therefore has a fixed point say u [6]. This gives that

$$\|u - fu\| = d(fu, C|.$$

We give an example to show that $(P \circ f)^n x_0$ may be bounded when C or $f(C)$ is not.

EXAMPLE 2. Let $C = [0, \infty) \subset R$, and $f : C \to R$ given by $f(x) = -x$. Then C and $f(C)$ both are unbounded. For $x_0 \in C$, $\{(P \circ f)^n x_0\}$ is bounded. As an application of Theorem 2 we give the following:

Let all the hypotheses of Theorem 2 be satisfied and assume further, that, for each $x \in C$ with $fx \neq x$ the line segment $[x, fx]$ contains at least two points of C. Then f has a fixed point.

In fact, by Theorem 2 there is an $x \in C$ such that

$$\|x - fx\| = d(fx, C).$$

If $x \neq fx$ then $[x, fx]$ has at least two elements of C so let $z = \lambda fx + (1-\lambda)x \in C$ for $0 \leq \lambda < 1$. Now,

$$\begin{aligned}\|x - fx\| &\leq \|x - z\| = \|x - \lambda x - (1-\lambda)fx\| \\ &= |\lambda|\|x - fx\| < \|x - fx\| \quad \text{since } \lambda < 1, \text{ a contradiction.}\end{aligned}$$

So $x = fx$.

Ray and Cramer [14] proved the following results giving important remarks on Leray-Schauder boundary conditions.

THEOREM A. *Let H be a Hilbert space and $f : D \to H$ a nonexpansive map, where D is a closed, bounded, convex subset of H. Let $g : D \to [0,1]$ be an arbitrary function.*

Suppose for each $x \in D$,

$$\text{(R)} \qquad \lim_{h \to 0+} \inf \frac{d((1-h)x + hf(x), D)}{h} \leq g(x)\|x - fx\|.$$

Then f has a fixed point.

THEOREM B. *Let D be a closed, bounded, convex subset of a Hilbert space H, $f : D \to H$ a densifying map and $g : D \to [0,1]$ an arbitrary function. If* (R) *holds then f has a fixed point.*

2. In this section we study the above theorems for 1-set contraction maps using a well-known theorem due to Lin and Yen [12] stated below. Recall that a nonexpansive map is a 1-set contraction and so is a densifying map. This approach unifies several previous results in fixed point theory.

We state the following due to Lin and Yen [12].

THEOREM 3. *Let C be a closed, convex subset of a Hilbert space H and $f : C \to H$ be a continuous 1-set contraction mapping with $f(C)$ bounded. Let $P : H \to C$ be a proximity map with $(1 - P \circ f)(\overline{\text{co}}((P \circ f)(C))$ closed. Then there is a $y \in C$ such that*

$$\|y - fy\| = d(fy, C).$$

We include the proof for the sake of completeness.

Proof. Let $D = \overline{\text{co}}[(P \circ f)(C)]$. Then $P \circ f : D \to D$ is 1-set contraction and $(1 - P \circ f)(D)$ is closed so $P \circ f$ has a fixed point, say $y = (P \circ f)y$ [13]. This gives that

$$\|y - fy\| = d(fy, C).$$

In case $fy \in C$ then f has a fixed point.

We have the following as corollaries.

Let C be a closed bounded convex subset of H and $f : C \to H$ a nonexpansive map. Then there is a $y \in C$ such that

$$\|y - fy\| = d(fy, C).$$

If further, $f(\partial C) \subset C$ then f has a fixed point.

If $C = B_r$, a ball of radius r and center at 0, then f has a fixed point if the following condition is satisfied

(1) if $f(x) = \alpha x$ for $x \in \partial B_r$ then $\alpha \leq 1$ (Leray–Schauder condition).

Williamson [20] has shown that the boundary condition (R) considered by Ray and Cramer [14] is equivalent to the Leray-Schauder condition. It, therefore, follows that f has a fixed point if the boundary condition (R) is added in the hypotheses of Theorem 3.

3. In this section we would like to give results for multivalued mappings and fixed points. Several interesting results for multivalued nonexpansive mappings have been given recently. Recall that a mappings $F : C \to 2^C$ is nonexpansive if

$$H(Fx, Fy) \leq d(x, y) \quad \text{for all } x, y \in C,$$

where H stands for the Hausdorff metric.

Husain and Tarafdar [7] considered a mapping of a different nature defined below.

Let X be a normed linear space and C a nonempty subset of X. A multivalued mapping $F : C \to 2^C$ is called weakly nonexpansive map if given $x \in C$ and $u_x \in Fx$, for each $y \in C$ there is a $u_y \in Fy$ such that

$$\|u_x - u_y\| \leq \|x - y\|.$$

If $x \in Fx$ then x is called a fixed point of F.

If f_α $(\alpha \in I)$ is a family of single valued nonexpansive self maps of C, then $Fx = \bigcup_{\alpha \in I} f_\alpha x$, $(x \in C)$ is a weakly nonexpansive multivalued map.

We need the following definitions.

A subset C in a normed linear space X is said to be starshaped if there exists a point p such that all line segments joining p to other points of C lie in C, i.e. if

$$x \in C \text{ then } \alpha p + (1 - \alpha)x \in C, \; 0 \leq \alpha \leq 1.$$

The point p is called a star centre.

Every convex set is star shaped but not conversely.

Let X be a Banach space. X is said to satisfy Opial's condition if for each $x_0 \in X$ and each sequence $\{x_n\}$ converging weakly to x the inequality

$$\lim_{n \to \infty} \inf \|x_n - x\| > \lim_{n \to \infty} \inf \|x_n - x_0\| \text{ holds for all } x \neq x_0.$$

Every Hilbert space and the spaces $\ell^p (1 \leq p < \infty)$ satisfy Opial's conditions (see for details [10]).

We state our theorem:

THEOREM 4. *Let C be a nonempty weakly compact starshaped subset of a Banach space satisfying Opial's condition. Let $F : C \to 2^C$ be a compact valued weakly nonexpansive mapping satisfying the following condition:*

(c) *For a fixed $p \in C, 0 < r_n < 1, r_n \to 0$, there is a $u_x \in Fx$ for all $x \in C$ such that each single valued map $f_n(x) = r_n u_x + (1 - r_n)p$ of C has a fixed point $x_n \in C$.*

Then F has a fixed point.

Proof. Since C is a weakly compact set and $x_n \in C$, so x_n has a convergent subsequence $x_{n_i} \rightharpoonup x_0 \in C$, say, ($\rightharpoonup$ stands for weak convergence).

Now (write u_n for $u_{x_{n_i}}$, for short):

$$x_n = f_n x_n = r_n u_n + (1 - r_n)p,$$

so we get

$$\|u_n - x_n\| = \|u_n - r_n u_n - (1 - r_n)p\| = (1 - r_n)\|u_n - p\|.$$

Since C is bounded and $u_n \in Fx_n \subset C$, therefore $\|u_n - p\|$ is bounded and we get that $\|u_n - x_n\| \to 0$ as $n \to \infty$.

Since F is weakly nonexpansive, for each $u_n \in Fx_n$, there is a $v_n \in Fx_0$ such that

$$\|u_n - v_n\| \leq \|x_n - x_0\|.$$

Since Fx_0 is compact v_n has a subsequence $v_{n_i} \to v_0 \in Fx_0$. We write $\{v_n\}$ for $\{v_{n_i}\}$. So $\liminf \|u_n - v_n\| \leq \liminf \|x_n - x_0\| < \infty$. Since $\|u_n - x_n\| \to 0$ and $v_n \to v_0$ we get

$$\liminf \|x_n - v_0\| \leq \liminf \|x_n - x_0\|.$$

Now Opial's condition implies that $x_0 = v_0 \in Fx_0$.

We derive the following as corollaries:

1. *If C is a weakly compact starshaped subset of a Hilbert space H and $F : C \to 2^C$ is weakly nonexpansive multivalued map with compact values satisfying* (c), *then F has a fixed point.*

Note. In a Hilbert space H Opial's condition is satisfied.

2. *If C is a closed, bounded, convex subset of a reflexive Banach space X with Opial's condition and $F : C \to 2^C$ is a weakly nonexpansive map with compact values satisfying condition* (c), *then F has a fixed point.*

We get results due to Husain and Latig [8] as corollaries to our theorem since a convex set is always starshaped.

THEOREM 5. *Let C be a compact starshaped subset of a Banach space X and $F : C \to 2^C$ a weakly nonexpansive multivalued mapping satisfying condition* (c). *Then F has a fixed point.*

Proof. Since C is compact $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to x_0. Also, we get $\|u_n - x_n\| \to 0$ as in Theorem 4. Since F is weakly nonexpansive, for each $u_n \in Fx_n$, there is a $v_n \in Fx_0$ such that

$$\|u_n - v_n\| \leq \|x_n - x_0\|.$$

Since Fx_0 is compact again let $\{v_n\}$ for $\{v_{n_i}\}$ converge to $v_0 \in Fx_0$. Therefore $\|u_n - v_n\| \leq \|u_n - x_n\| + \|x_n - v_n\|$ gives that $x_0 = v_0 \in Fx_0$.

Note. In case C is a compact, convex subset of a Banach space X and $F : C \to 2^C$ a weakly nonexpansive map satisfying condition (c). Then F has a fixed point. It follows since a convex set is starshaped.

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