DEMONSTRATIO MATHEMATICA
Vol. XXVI No 3–4 1993

B. G. Pachpatte

SOME INEQUALITIES SIMILAR TO OPIAL'S INEQUALITY

1. Introduction

An inequality due to Opial [7] can be stated as follows

$$\int_a^b |y(t)y'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |y'(t)|^2\,dt, \tag{1}$$

where $y(t)$ is absolutely continuous function on $[a,b]$ such that $y(a)=0$ (see [5], Theorem 2′, p. 154). A summary of different proof and various generalizations of (1) can be found in [5] (pp. 154–162), and also in [1], [4], [6], [8]–[12]. In particular, Godunova and Levin [2] found interesting generalizations of (1), (see [5], Theorem 12 and 13, p. 159). The main purpose of this note is to establish some new integral inequalities which in the special cases yield (1) and also the new inequalities of the Godunova and Levin type.

2. Main results

For the sake of brevity we write f_i for $f_i(|u_i(t)|)$, f_i' for $f_i'|u_i(t))$, u_i' for $u_i'(t)$ with $t\in[a,b]$ and we use the notation

$$\begin{aligned}L[f_1,\dots,f_n,f_1',\dots,f_n',u_1',\dots,u_n']\\ &= f_1,\dots,f_{n-1}f_n'|u_n'| + f_1\dots f_{n-2}f_{n-1}'|u_{n-1}'|f_n +\\ &\quad \dots + f_1'|u_1'|f_2\dots f_n, \qquad n\ge 2.\end{aligned}$$

The main result of this paper is established as follows.

THEOREM 1. *Let $u_i(t)$, $i=1,\dots,n$, be real-valued absolutely continuous functions on $[a,b]$ with $u_i(a)=0$. Let $f_i(r)$, $i=1,\dots,n$, be real-valued nonnegative continuous nondecreasing functions for $r\ge 0$ and $f_i(0)=0$ such that $f_i'(r)$ exist nonnegative continuous and nondecreasing for $r\ge 0$.*

Then

$$(2)\qquad \int_a^b L[f_1,\dots,f_n,f_1',\dots,f_n',u_1',\dots,u_n']\,dt \le \prod_{i=1}^n f_i\Big(\int_a^b |u_i'(t)|dt\Big).$$

The inequality (2) *also holds if we replace the condition* $u_i(a) = 0$ *by* $u_i(b) = 0$.

As an immediate consequence of Theorem 1 we have the following one.

THEOREM 2. *Assume that in the hypotheses of Theorem* 1 *we have* $u_i = u$ *and* $f_i = f$. *Then*

$$(3)\qquad \int_a^b \{f(|u(t)|\}^{n-1}f'(|u(t)|)|u'(t)|\,dt \le \frac{1}{n}\Big\{f\Big(\int_a^b |u'(t)|\,dt\Big)\Big\}^n.$$

The inequality (3) *also holds if we replace the condition* $u(a) = 0$ *by* $u(b) = 0$.

Remark 1. If we take $n = 2$ in (3), then we get the following inequality

$$(4)\qquad \int_a^b f(|u(t)|)f'(|u(t)|)|u'(t)|\,dt \le \frac{1}{2}\Big\{f\Big(\int_a^b |u'(t)|\,dt\Big)\Big\}^2$$

which is analogous to that given in [5] (Theorem 13, p. 159). Further, by taking $f(r) = r^{m+1}$ in (4), where $m \ge 0$ is a constant, and using the Hölder inequality with indices $2(m+1)$ and $\frac{2(m+1)}{2m+1}$ to the resulting integral on the right, we see that (4) reduces to the following inequality

$$(5)\qquad \int_a^b |u(t)|^{2m+1}|u'(t)|\,dt \le \frac{(b-a)^{2m+1}}{2(m+1)}\int_a^b |u'(t)|^{2(m+1)}\,dt$$

which reduces to (1), when $m = 0$.

A slight variant of Theorem 1 is as follows.

THEOREM 3. *Let* u_i, f_i, f_i' *be as in Theorem* 1. *Let* $p_i(t) > 0$ *be defined on* $[a,b]$ *and* $\int_a^b p_i(t)\,dt = 1$, $i = 1,\dots,n$. *If* $h(r)$ *is a positive convex and increasing function for* $r > 0$, *then*

$$(6)\qquad \int_a^b L[f_1,\dots,f_n,f_1',\dots,f_n',u_1',\dots,u_n']\,dt$$

$$\le \prod_{i=1}^n f_i\Big(h^{-1}\Big(\int_a^b p_i(t)h\Big(\frac{|u_i'(t)|}{p_i(t)}\Big)\,dt\Big)\Big).$$

The inequality (6) *also holds if we replace the condition* $u_i(a) = 0$ *by* $u_i(b) = 0$.

The following result is an easy consequence of Theorem 3.

THEOREM 4. *Assume that in the hypotheses of Theorem* 3 *we have* $u_i = u$, $f_i = f$ *and* $p_i = p$. *Then*

$$(7)\qquad \int_a^b \{f(|u(t)|)\}^{n-1} f'(|u(t)|)|u'(t)|\,dt \le \frac{1}{n}\left\{f\left(h^{-1}\left(\int_a^b p(t)h\left(\frac{|u'(t)|}{p(t)}\right)dt\right)\right)\right\}^n .$$

The inequality (7) *also holds if we replace the condition* $u(a) = 0$ *by* $u(b) = 0$.

Remark 2. If we take $n = 2$ in (7), then we get the following inequality analogous to that given in [5] (Theorem 12, p. 159), i.e.

$$(8)\qquad \int_a^b f(|u(t)|)f'(|u(t)|)u'(t)|\,dt \le \frac{1}{2}\left\{f\left(h^{-1}\left(\int_a^b p(t)h\left(\frac{|u'(t))}{p(t)}\right)dt\right)\right)\right\}^2 .$$

We also note that in the special cases the inequality (8) yields the various inequalities as discussed in Remark 1.

Proof of Theorem 1. Let $t \in [a,b]$ and define

$$(9)\qquad z_i(t) = \int_a^t |u_i'(s)|\,ds, \qquad i = 1,\ldots,n,$$

implying

$$(10)\qquad z_i'(t) = |u_i'(t)|, \qquad t \in [a,b], \quad i = 1,\ldots,n.$$

For $t \in [a,b]$ we have the following identities

$$(11)\qquad u_i(t) = \int_a^t u_i'(s)\,ds, \qquad i = 1,\ldots,n.$$

From (11) and (9) we observe that

$$(12)\qquad |u_i(t)| \le z_i(t), \qquad i = 1,\ldots,n.$$

Using (12), (10) and (9), we get

$$\int_a^b L[f_1,\ldots,f_n,f_1',\ldots,f_n',u_1',\ldots,u_n']\,dt$$

$$\leq \int_a^b [f_1(z_1(t))\ldots f_{n-1}(z_{n-1}(t))f_n'(z_n(t))z_n'(t)$$

$$+ f_1(z_1(t))\ldots f_{n-2}(z_{n-2}(t))f_{n-1}'(z_{n-1}(t))z_{n-1}'(t)f_n(z_n(t)) + \ldots$$

$$\ldots + f_1'(z_1(t))z_1'(t)f_2(z_2(t))\ldots f_n(z_n(t))]\,dt$$

$$= \int_a^b \frac{d}{dt}\Big[\prod_{i=1}^n f_i(z_i(t))\Big]\,dt = \prod_{i=1}^n f_i(z_i(b)) = \prod_{i=1}^n f_i\Big(\int_a^b |u_i'(t)|\,dt\Big),$$

being the required inequality (2). Defining $z_i(t) = \int_t^b |u_i'(s)|\,ds$ and hence $z_i'(t) = -|u_i'(t)|$, and representing $u_i(t) = -\int_t^b u_i'(s)\,ds$ in case of $u_i(b) = 0$, then observing that $|u_i(t)| \leq z_i(t)$, similarly as above, we get (2). This completes the proof of Theorem 1.

Proof of Theorem 3. Observe that, by hypotheses,

$$\int_a^b |u_i'(t)|\,dt = \int_a^b p_i(t)\frac{|u_i'(t)|}{p_i(t)}\,dt\Big(\int_a^b p_i(t)\,dt\Big)^{-1}, \quad i = 1,\ldots,n. \tag{13}$$

Since h is convex, from (13) and using Jensen's inequality (see [3], p. 113) we obtain

$$h\Big(\int_a^b |u_i'(t)|\,dt\Big) \leq \int_a^b p_i(t)h\Big(\frac{|u_i'(t)|}{p_i(t)}\Big)\,dt \tag{14}$$

which implies

$$\int_a^b |u_i'(t)|\,dt \leq h^{-1}\Big(\int_a^b p_i(t)h\Big(\frac{|u_i'(t)|}{p_i(t)}\Big)\,dt\Big). \tag{15}$$

All the hypotheses of Theorem 1 being satisfied we get (2). Using (15) in (2), we obtain the required inequality (6). This completes the proof of Theorem 3.

We omit the proofs of Theorems 2 and 4, being immediate from those of Theorems 1 and 3.

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DEPARTMENT OF MATHEMATIC AND STATISTICS
MARATHWADA UNIVERSITY
AURANGABAD 431 004
(MAHARASHTRA), INDIA

Received January 7, 1991.