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ON A COMPREHENSIVE STRUCTURE ON A DIFFERENTIABLE MANIFOLD

Introduction

In recent years several structures, notably almost contact structure [2], [6], [8], almost r-contact structure [4], [15], almost paracontact structure [9], almost r-paracontact structure [2], almost contact hyperbolic structure [14] and almost r-contact hyperbolic structure [5], have been defined and studied on a differentiable manifold by many geometers. Some generalized structures, including almost $(\varepsilon_1, \varepsilon_2)$-contact structure [10], almost $(\varepsilon_1, \varepsilon_2)-$ r-contact structure [11], [12] and unified structure [1], [13], have also been defined.

In this paper we define and study a comprehensive structure having all structures mentioned above as its special cases.

1. A comprehensive structure

We first define a comprehensive structure Σ on a differentiable manifold as follows.

DEFINITION 1.1. Let M be an m-dimensional differentiable manifold admitting a tensor field F of type (1, 1), linearly independent vector fields (T_x) and 1-forms (A^x), $x = 1, \ldots, r$, $r < n$, such that

(1.1) $$F(T_x) = 0,$$

(1.2) $$F^2 X = ea^2 x + cA^x(X)T_x,$$

where e, c take values ± 1 and a^2 is a (complex) constant. We define the structure $\Sigma \equiv (F, T_x, A^x)$ to be a *comprehensive structure* on M and the pair (M, Σ) or simply M to be a *comprehensive structure manifold*.

AGREEMENT 1.1. In the above and in what follows the indices $x, y, z, \ldots$ run over $(1, \ldots, r)$ and the equations containing $X, Y, Z, \ldots$ hold for arbitrary vector fields unless otherwise stated.

THEOREM 1.1. *If M is a comprehensive structure manifold, then*

$$A^x \cdot F = 0\,, \tag{1.3}$$

$$A^x(T_y) = -eca^2\delta^x_y\,, \tag{1.4}$$

$$\operatorname{rank}(F) = m - r\,, \tag{1.5}$$

δ^x_y being Kronecker's symbol.

Now introduce a metric on M.

DEFINITION 1.2. On a comprehensive structure manifold (M,Σ) let a metric g be introduced such that

$$g(FX,FY) = a^2 g(X,Y) + ec\sum_x A^x(X)A^x(Y)\,. \tag{1.6}$$

We define $(\Sigma,g) \equiv (F,T_x,A^x,g)$ to be a *comprehensive metric structure* and M equipped with such a metric structure to be a *comprehensive metric structure manifold*. The above metric g is said to be a metric associated to the comprehensive structure on M.

Setting $X = T_x$, an immediate consequence is that A^x is the covariant form of T_x, that is

$$A^x(Y) = g(T_x,Y)\,. \tag{1.7}$$

THEOREM 1.2. *On a comprehensive structure manifold (M,Σ) there always exists a metric g, given by* (1.6).

Proof. Let h' be any Riemannian metric on M and let h be defined by

$$a^2h(X,Y) \overset{\text{def}}{=} -ec\Big[h'(F^2X,F^2Y) + \sum_x A^x(X)A^x(Y)\Big]\,.$$

Then $h(T_x,Y) = A^x(Y)$ and it is easy to check that h is a metric. Now let us define g by

$$2a^2g(X,Y) \overset{\text{def}}{=} h(FX,FY) + a^2h(X,Y) - ec\sum_x A^x(X)A^x(Y)\,.$$

Again g is clearly a metric and the relation

$$\begin{aligned} 2a^2g(FX,FY) &= a^2h(FX,FY) + h(ea^2X + cA^x(X)T_x, ea^2Y + cA^x(Y)T_x) \\ &= a^2h(FX,FY) + a^4h(X,Y) + eca^2\sum_x A^x(X)A^x(Y) \\ &= 2a^4g(X,Y) + 2eca^2\sum_x A^x(X)A^x(Y) \end{aligned}$$

implying (1.6). However, the metric g is, of course, not unique.

THEOREM 1.3. *On a comprehensive metric structure manifold* (M, Σ, g) *the following relations hold good:*

(1.8) $$g(T_x, FX) = 0\,,$$

(1.9) $$g(FX, Y) = eg(X, FY)\,.$$

The proof is obvious.

Using (1.1) and (1.2), it is easy to verify the following result.

THEOREM 1.4. *Let* (F, T_x, A^x) *and* $(F, T_x, \overline{A}^x)$ *[resp.* $(F, \overline{T}_x, A^x)$*] be two comprehensive structure on a differentiable manifold* M*; then we have* $A^x = \overline{A}^x$ *[resp.* $T_x = \overline{T}_x$*].*

Thus we see that two comprehensive structures having same F and same (T_x) [resp. (A^x)] on a differentiable manifold are always identified. However, a comprehensive structure on a differentiable manifold M always induces another comprehensive structure on M. So we can prove the following theorem.

THEOREM 1.5. *A comprehensive structure on a differentiable manifold* M *is not unique.*

Proof. Let H be an arbitrary non-singular tensor field of type (1, 1) on M. Defining

(1.10) $$\overline{F} \overset{\text{def}}{=} H^{-1}FH, \quad \overline{A}^x \overset{\text{def}}{=} A^x \cdot H, \quad \overline{T}_x \overset{\text{def}}{=} H^{-1}(T_x)\,,$$

it can be easily seen that $(\overline{F}, \overline{T}_x, \overline{A}^x)$ is also a comprehensive structure on M. Moreover, if g is an associated metric to the structure (F, T_x, A^x) on M, then a metric $\overline{g}$ on M defined by

(1.11) $$\overline{g}(X, Y) \overset{\text{def}}{=} g(HX, HY)$$

provides an associated metric to the structure $(\overline{F}, \overline{T}_x, \overline{A}^x)$ on M.

We can state this fact as follows.

COROLLARY 1.1. *A comprehensive metric structure on a differentiable manifold is not unique.*

2. Existence of a comprehensive structure

Let λ be an eigenvalue of F corresponding to an eigenvector P. We now consider the following two possible cases.

CASE 1. P is linearly independent of (T_x). Then (1.2) implies $(\lambda^2 - ea^2)P = cA^x(P)T_x$. Hence $\lambda = \pm\sqrt{ea^2}$ and $A^x(P) = 0$.

CASE 2. P is a linear combination of (T_x). Then $F(P) = 0$ that is $\lambda = 0$. Therefore, there are r eivenvalues 0.

Since M is of dimension m and $\operatorname{rank}(F) = m - r$, there are, say, r eigenvalues 0, s eigenvalues $+\sqrt{ea^2}$ and $m - r - s$ eigenvalues $-\sqrt{ea^2}$. Let L, K and N denote the distributions corresponding to the eigenvalues 0, $+\sqrt{ea^2}$ and $-\sqrt{ea^2}$, respectively.

LEMMA 2.1. *The distributions L, K and N are complementary distributions generated by the complementary projection operators l, k and n, defined by*

$$l \overset{\text{def}}{=} (a^2 I - eF^2)a^{-2}\,, \tag{2.1}$$

$$2k \overset{\text{def}}{=} (eF^2 + dF)a^{-2}\,, \tag{2.2}$$

and

$$2n \overset{\text{def}}{=} (eF^2 - dF)a^{-2}\,, \tag{2.3}$$

respectively, where I is the identity tensor field and $d = e\sqrt{ea^2}$.

Proof. We see that $l + k + n = I$. We also have

$$l^2 = (a^4 I + F^4 - 2ea^2F^2)a^{-4} = (a^4 I + ea^2F^2 - 2ea^2F^2)a^{-4} = l\,,$$

$$k^2 = (F^4 + d^2F^2 + 2edF^3)\tfrac{1}{4}a^{-4} = (ea^2F^2 + ea^2F^2 + 2da^2F)\tfrac{1}{4}a^{-4} = k$$

and similarly $n^2 = n$. Again, we get

$$\begin{aligned} 2lk &= (ea^2F^2 + da^2F - F^4 - edF^3)a^{-4} \\ &= (ea^2F^2 + da^2F - ea^2F^2 - da^2F)a^{-4} = 0\,, \\ 2ln &= (ea^2F^2 - da^2F - F^4 + edF^3)a^{-4} \\ &= (ea^2F^2 - da^2F - ea^2F^2 + da^2F)a^{-4} = 0\,, \\ 4kn &= F^4 - d^2F^2 = ea^2F^2 - ea^2F^2 = 0\,. \end{aligned}$$

Consequently l, k, n are complementary projection operators. Moreover

$$\begin{aligned} Fl &= (a^2F - eF^3)a^{-2} = (a^2F - a^2F)a^{-2} = 0\,, \\ Fk &= (eF^3 + dF^2)\tfrac{1}{2}a^{-2} = (a^2F + e\sqrt{ea^2}F^2)\tfrac{1}{2}a^{-2} \\ &= \sqrt{ea^2}(eF^2 + dF)\tfrac{1}{2}a^{-2} = k\sqrt{ea^2} \end{aligned}$$

and similarly $Fn = -n\sqrt{ea^2}$. We also get $k + n = a^{-2}eF^2$.

Now it remains to show that L, K and N are the complementary distributions generated by the complementary projection operators l, k and n, that is $L = \{lX; X \in \mathcal{X}(M)\}$, $K = \{kX; X \in \mathcal{X}(M)\}$, and $N = \{nX; X \in \mathcal{X}(M)\}$. Let $Z \in L$. Then, since 0 is the eigenvalue for L, we have $FZ = 0$. Also, since $Z = lZ + kZ + nZ$, we get $0 = FZ = FlZ + DkZ + FnZ = 0 + \sqrt{ea^2}kZ - \sqrt{ea^2}nZ$ or $kZ - nZ = 0$. But $k + n = a^{-2}eF^2$; therefore $kZ + nZ = 0$. Hence, $kZ = 0$ and $nZ = 0$ and thus $Z = lZ$, that is $L \subset \{lX; X \in \mathcal{X}(M)\}$.

Conversely, let $Z = lX$, then $FZ = FlX = 0$ which shows that $Z \in L$, that is $\{lX; X \in \mathcal{X}(M)\} \subset L$. Thus $L = \{lX; X \in \mathcal{X}(M)\}$.

Again, if $Z \in K$, then since $\sqrt{ea^2}$ is the eigenvalue for K we have $\sqrt{ea^2}Z = FZ = FlZ + FkZ + FnZ = 0 + \sqrt{ea^2}kZ - \sqrt{ea^2}nZ$ or $Z = kZ - nZ$. Also $kZ + nZ = ea^{-2}F^2Z = Z$. Thus $Z = kZ$, that is $K \subset \{kX; X \in \mathcal{X}(M)\}$. On the other hand, let $Z = kX$, then $FZ = FkX = \sqrt{ea^2}kX = \sqrt{ea^2}Z$. Thus $Z \in K$; that is, $\{kX; X \in \mathcal{X}(M)\} \subset K$. Hence $K = \{kX; X \in \mathcal{X}(M)\}$. Similarly, we can prove that $N = \{nX; X \in \mathcal{X}(M)\}$.

AGREEMENT 2.1. In what follows the indices i, j [resp. i', j'] run over $\{1, \ldots, s\}$ [resp. $\{1, \ldots, m - r - s\}$].

Now we are in a position to prove the main theorem of this section.

THEOREM 2.1. *A necessary and sufficient condition for M to admit a comprehensive structure is that there exists complementary projection operators l, k and n which bring together the complementary distributions L, K and N of dimensions r, s and $m - r - s$, respectively, which together span the manifold.*

Proof. The necessary part follows from Lemma 2.1. For sufficient part, let $(T_x, U_i, U_{i'})$ be a set such that $(T_x), (U_i)$ and $(U_{i'})$ are the basis vectors in L, K and N, respectively, and let $(-eca^{-2}A^x, V^i, V^{i'})$ be the inverse set. Therefore, we get

$$\tag{2.4} \begin{cases} -eca^{-2}A^x(T_y) = \delta^x_y, & A^x(U_i) = 0, \quad A^x(U_{i'}) = 0\,, \\ V^i(T_x) = 0, & V^i(U_j) = \delta^i_j, \quad V^i(U_{j'}) = 0\,, \\ V^{i'}(T_x) = 0, & V^{i'}(U_j) = 0, \quad V^{i'}(U_{j'}) = \delta^{i'}_{j'}\,, \end{cases}$$

and

$$V^i(X)U_i + V^{i'}(X)U_{i'} - eca^{-2}A^x(X)T_x = X\,,$$

or

$$\tag{2.5} ea^2V^i(X)U_i + ea^2V^{i'}(X)U_{i'} - cA^x(X)T_x = ea^2X\,.$$

Putting

$$\tag{2.6} FX = \sqrt{ea^2}V^i(X)U_i + \sqrt{ea^2}V^{i'}(X)U_{i'}$$

we have $F(T_x) = 0$ and

$$\begin{aligned} F^2X &= \sqrt{ea^2}V^i(FX)U_i + \sqrt{ea^2}V^{i'}(X)U_i \\ &= \sqrt{ea^2}V^j(\sqrt{ea^2}V'(X)U_i + \sqrt{ea^2}V^{i'}(X)U_{i'})U_j \\ &\quad + \sqrt{ea^2}V^{j'}(\sqrt{ea^2}V^i(X)U_i + \sqrt{ea^2}V^{i'}(X)U_{i'})U_{j'} \\ &= ea^2V^i(X)U_i + ea^2V^{i'}(X)U_{i'} = ea^2X + cA^x(X)T_x\,. \end{aligned}$$

Thus (F, T_x, A^x) defines a comprehensive structure on M.

3. Integrability conditions

Let us recall some relations of the previous section as follows:

$$lk = kl = ln = nl = kn = nk = 0\,, \tag{3.1}$$

$$l^2 = l, \quad k^2 = k, \quad n^2 = n\,, \tag{3.2}$$

$$Fl = lF = 0\,, \tag{3.3}$$

$$Fk = kF = k\sqrt{ea^2}\,, \tag{3.4}$$

$$Fn = nF = -n\sqrt{ea^2}\,, \tag{3.5}$$

$$F^2 l = 0, \quad F^2 k = ea^2 k, \quad F^2 n = ea^2 n\,. \tag{3.6}$$

LEMMA 3.1. *If $[F,F]$ is the Nijenhuis tensor of F, then*

$$l[F,F](lX,lY) = 0\,, \tag{3.7}$$

$$k[F,F](kX,kY) = 0\,, \tag{3.8}$$

$$n[F,F](nX,nY) = 0\,, \tag{3.9}$$

$$l[F,F](kX,kY) = ea^2 l[kX,kY]\,, \tag{3.10}$$

$$l[F,F](nX,nY) = ea^2 l[nX,nY]\,, \tag{3.11}$$

$$k[F,F](lX,lY) = ea^2 k[lX,lY]\,, \tag{3.12}$$

$$k[F,F](nX,nY) = 4ea^2 k[nX,nY]\,, \tag{3.13}$$

$$n[F,F](lX,lY) = ea^2 n[lX,lY]\,, \tag{3.14}$$

$$n[F,F](kX,kY) = 4ea^2 n[kX,kY]\,. \tag{3.15}$$

Proof. The Nijenhuis tensor $[F,F]$ of F is defined by

$$[F,F](X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]\,. \tag{3.16}$$

But $k + n = e^{-1}F^2a^{-2}$, therefore we get

$$\begin{aligned}[F,F](X,Y) = {}& [FX,FY] - F[FX,Y] - F[X,FY] \\ & + ea^2 k[X,Y] + ea^2 n[X,Y]\,.\end{aligned} \tag{3.17}$$

On putting in (3.17) nX and nY in place of X and Y, respectively, operating the whole equation by k and using (3.1), (3.6), we get (3.13). Similarly we get (3.7)–(3.12), (3.14), (3.15).

Finally, we prove main theorem of this section.

THEOREM 3.1. *The comprehensive structure manifold M is completely integrable if and only if*

$$\begin{aligned}[F,F](X,Y) = {}& [F,F](lX,kY) + [F,F](lX,nY) \\ & + [F,F](kX,lY) + [F,F](kX,nY) + [F,F](nX,lY) + [F,F](nX,kY)\,.\end{aligned} \tag{3.18}$$

Proof. It is well known that any distribution D is integrable if and only if $[X,Y] \in D$ for all $X,Y \in D$. Thus, the distribution L is integrable if and only if

$$k[lX, lY] = 0\,, \tag{3.19}$$

$$n[lX, lY] = 0\,. \tag{3.20}$$

Equivalently, from (3.12) and (3.14) we have

$$k[F,F](lX, lY) = 0\,, \tag{3.21}$$

$$n[F,F](lX, lY) = 0\,. \tag{3.22}$$

The distribution K is integrable if and only if

$$l[kX, kY] = 0\,, \tag{3.23}$$

$$n[kX, kY] = 0\,, \tag{3.24}$$

and, equivalently,

$$l[F,F](kX, kY) = 0\,, \tag{3.25}$$

$$n[F,F](kX, kY) = 0\,. \tag{3.26}$$

Similarly, the distribution N is integrable if and only if

$$l[nX, nY] = 0\,, \tag{3.27}$$

$$k[nX, nY] = 0\,, \tag{3.28}$$

and equivalently,

$$l[F,F](nX, nY) = 0\,, \tag{3.29}$$

$$k[F,F](nX, nY) = 0\,. \tag{3.30}$$

The Nijenhuis tensor $[F,F]$ of F can be written in the form

$$[F,F](X,Y) = (l+k+n)[F,F]((l+k+n)X, (l+k+n)Y)\,.$$

Expanding right-hand side and using (3.7), (3.9), (3.21), (3.22), (3.25), (3.26), (3.29), (3.30), we get (3.18).

4. Special cases

The structure of this paper generalizes many known structures which may be obtained by taking particular values of a^2, e, c, r. We list these particular cases by giving different values to a^2, e, c, r, writing structural equations corresponding to (1.2), (1.4), (1.6), (1.9) and discussing the details.

CASE 1. ($a^2 = 1$, $e \equiv \varepsilon_1 = \pm 1$, $c \equiv \varepsilon_2 = \pm 1$). Almost $(\varepsilon_1, \varepsilon_2)-r$-contact Riemannian structure [10], [11]:

$$F^2X = \varepsilon_1 X + \varepsilon_2 A^x(X)T_x, \qquad A^x(T_y) = -\varepsilon_1\varepsilon_2\delta^x_y\,,$$

$$g(FX, FY) = g(X,Y) + \varepsilon_1\varepsilon_2 \sum_x A^x(X)A^x(Y)\,,$$

$$g(FX, Y) = \varepsilon_1 g(X, FY)\,.$$

CASE 2. ($a^2 = 1,\ e \equiv \varepsilon_1 = \pm 1,\ c \equiv \varepsilon_2 = \pm 1, r = 1$) Almost $(\varepsilon_1, \varepsilon_2)$-contact Riemannian structure [9]:

$$F^2 X = e_1 X + \varepsilon_2 A(X)T, \quad A(T) = -\varepsilon_1 \varepsilon_2\,,$$
$$g(FX, FY) = g(X,Y) + \varepsilon_1 \varepsilon_2 A(X)A(Y)\,,$$
$$g(FX, Y) = \varepsilon_1 g(X, FY)\,.$$

The existence theorem already has been discussed for cases 1, 2. Now, integrability conditions can be deduced from this paper.

AGREEMENT 4.1. In the above and in what follows, when $r = 1$, (A^i, T_1) will be identified by (A, T).

CASE 3. ($a^2 = 1,\ e = -1,\ c = 1$) Almost r-contact Riemannian structure [4], [7], [15]:

$$F^2 X = -X + A^x(X)T_x, \quad A^x(T_y) = \delta^x_y\,,$$
$$g(FX, FY) = g(X,Y) - \sum_x A^x(X)A^x(Y)\,,$$
$$g(FX, Y) = -g(X, FY)\,.$$

CASE 4. ($a^2 = 1,\ e = -1,\ c = 1,\ r = 1$). Almost contact Riemannian structure [2], [6], [8]:

$$F^2 X = -X + A(X)T, \quad A(T) = 1\,,$$
$$g(FX, FY) = g(X,Y) - A(X)A(Y)\,,$$
$$G(FX, Y) = -g(X, FY)\,.$$

In cases 3, 4 the dimension of K becomes equal to the dimension of N and hence, in case of almost contact manifold, the manifold becomes odd dimensional.

CASE 5. ($a^2 = 1,\ e = 1,\ c = -1$) Almost r-paracontact Riemannian structure [3]:

$$F^2 X = X - A^x(X)T_x, \quad A^x(T_y) = \delta^x_y\,,$$
$$g(FX, FY) = g(X,Y) - \sum_x A^x(X)A^x(Y)\,,$$
$$g(FX, Y) = g(X, FY)\,.$$

CASE 6. ($a^2 = 1,\ e = 1,\ c = -1,\ r = 1$) Almost paracontact Riemannian structure [9]:

$$F^2 X = X - A(X)T, \quad A(T) = 1\,,$$
$$g(FX, FY) = g(X,Y) - A(X)A(Y)\,,$$
$$g(FX, Y) = g(X, FY)\,.$$

All the results can be deduced for cases 5, 6 by putting appropriate values for a^2, e, c, r.

CASE 7. ($a^2 = -1$, $e = -1$, $c = 1$) Almost r-contact hyperbolic Riemannian structure [5]:

$$F^2X = X + A^x(X)T_x, \quad A^x(T_y) = -\delta_y^x,$$
$$g(FX, FY) = -g(X,Y) - \sum_x A^x(X)A^x(Y),$$
$$g(FX,Y) = -g(X, FY).$$

CASE 8. ($a^2 = -1$, $e = -1$, $c = 1$, $r = 1$) Almost contact hyperbolic Riemannian structure [14]:

$$F^2X = X + A(X)T, \quad A(T) = -1,$$
$$g(FX, FY) = -g(X,Y) - A(X)A(Y),$$
$$g(FX,Y) = -g(X, FY).$$

To the best of my knowledge, existence and integrability in cases 7, 8 have not been studied so far.

CASE 9. (a^2 replaced by $-a^2$, $e = -1$, $c = 1$, $r = 1$) Unified metric structure [1], [13]:

$$F^2X = a^2X + A(X)T, \quad A(T) = -a^2,$$
$$g(FX, FY) = -a^2g(X,Y) - A(X)A(Y),$$
$$g(FX,Y) = -g(X, FY).$$

Putting $(\varepsilon_1, \varepsilon_2) = (-1,1)$, $(\varepsilon_1, \varepsilon_2) = (1,-1)$ and $(\varepsilon_1, \varepsilon_2) = (1,1)$ in case 2 we get almost contact Riemannian structure, almost paracontact Riemannian structure and almost contact hyperbolic structure (but not almost contact hyperbolic Riemannian structure), respectively. In fact, when $(\varepsilon_1, \varepsilon_2) = (1,1)$ we have

$$g(FX, FY) = g(X,Y) + A(X)A(Y), \quad g(FX,Y) = g(X, FY)$$

which does not coincide with the metric of case 8. However, if we take a particular case of the comprehensive metric structure by setting $a^2 = -1$, $e = -1$, $c = 1$, $r = 1$, it would be possible to find an almost contact hyperbolic Riemannian structure [14].

The unified metric structure [1], [13] only unifies an almost contact Riemannian structure [2], [6], [8] and an almost contact hyperbolic Riemannian structure [14]. However, if we take a particular case of comprehensive metric structure by setting $e = -1$, $c = 1$ and a^2 replaced by $-a^2$, it would be possible to find a metric structure which unifies an almost contact Riemannian structure [2], [6], [8], an almost r-contact Riemannian structure [4], [7],

[15], an almost contact hyperbolic Riemannian structure [14] and an almost r-contact hyperbolic Riemannian structure [5].

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