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UNBOUNDED SOLUTIONS OF QUASILINEAR DIFFERENTIAL-FUNCTIONAL EQUATIONS

1. Introduction

We will denote by $C^0(X,Y)$ the class of all continuous functions defined on X and taking values in Y where X,Y are metric spaces. Let $h_0 \in R_+$, $h = (h_1,\dots,h_n) \in R^n_+$ where $R_+ = [0,+\infty)$ and let $D = [-h_0,0] \times [-h,h]$.

Suppose that $a > 0$ and $z : [-h_0,a]\times R^n \to R$. For $(x,y) = (x,y_1,\dots,y_n) \in [0,a] \times R^n$ we denote by $z_{(x,y)} : D \to R$ the function defined in the following way:

$$z_{(x,y)}(t,s) = z(x+t,y+s), \quad (t,s) \in D.$$

Let $a_0 > 0$ and $\Omega = [0,a_0] \times R^n \times C^0(D,R)$. Suppose that $\varrho = (\varrho_1,\dots,\varrho_n) : \Omega \to R^n$ and $f : \Omega \to R$ are the functions of variables (x,y,w), $(x,y,w) \in [0,a_0] \times R^n \times C^0(D,R)$. Let $\varphi : [-h_0,0] \times R^n \to R$ be given function.

We consider the differential-functional equation

$$D_x z(x,y) + \sum_{k=1}^{n} \varrho_k(x,y,z_{(x,y)}) D_{y_k} z(x,y) = f(x,y,z_{(x,y)}) \tag{1}$$

with initial condition

$$z(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in [-h_0,0] \times R^n. \tag{2}$$

In this paper we give sufficient conditions for the existence and uniqueness of unbounded classical solutions of (1), (2). The method used in this paper is based on results due to P. Bassanini and L. Cesari for solutions in the sense "almost everywhere" of systems which are not functional (see [1]–[4]). Bounded solutions of initial value problems for quasilinear hyperbolic

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systems of differential-functional equations have been discussed in [6]. The cases of bounded and unbounded classical solutions of nonlinear equations have been treated in [5].

We will introduce some notations and definitions. For $y = (y_1, \dots, y_n) \in R^n$ we denote $\|y\|_n = |y_1| + \dots + |y_n|$ and $\|y\| = \max\{|y_i| : i = 1, \dots, n\}$. For $w \in C^0(D, R)$ we define $\|w\|_0 = \max\{|w(t,s)| : (t,s) \in D\}$. Let $C^0_{(L)}(D,R)$ denotes the space of all continuous functions of variables $(t,s) = (t, s_1, \dots, s_n)$ which satisfy the Lipschitz condition with respect to s on D. If $w \in C^0_{(L)}(D,R)$ then we define $\|w\|_{(1+L)} = \sup\{|w(t,s) - w(t,\bar{s})| \|s - \bar{s}\|^{-1} : (t,s),(t,\bar{s}) \in D\}$. Denote by $C^{(1)}(D,R)$ the set of all continuous functions $w : D \to R$ which have continuous derivatives $D_{s_k} w, k = 1, \dots, n$ and for $w \in C^{(1)}(D,R)$ we define $\|w\|_{(1)} = \sum_{k=1}^n \|D_{s_k} w\|_0$. Let $C^{(1)}_{(L)}(D,R)$ denotes the space of functions $w \in C^{(1)}(D,R)$ such that their first order derivatives $D_{s_k} w, k = 1, \dots, n$ satisfy the Lipschitz condition with respect to s. For $w \in C^{(1)}_{(L)}(D,R)$ we define $\|w\|_{(1+L)} = \|w\|_{(1)} + \sum_{k=1}^n \|D_{s_k} w\|_{(L)}$. If X, Y are Banach spaces then $CL(X,Y)$ denotes the set of all linear continuous operators defined on X and taking values in Y, and $\|\cdot\|_X$ is the norm in $CL(X,Y)$. Let us denote by J_+ the set of all functions $\alpha : R_+ \to R_+$ which are nondecreasing on R_+. Let $\Omega^{(1)} = [0, a_0] \times R^n \times C^{(1)}(D,R)$.

2. Bicharacteristics

The following assumptions will be needed through the paper.

ASSUMPTION H_1. Suppose that $\varphi : [-h_0, 0] \times R^n \to R$ is of class C^1 and there are $\lambda_1, \lambda_2 \in R_+$ such that

$$\|D_y\varphi(x,y)\| \le \lambda_1, \|D_y\varphi(x,y) - D_y\varphi(x,\bar{y})\| \le \lambda_2 \|y - \bar{y}\|_n,$$
$$(x,y),(x,\bar{y}) \in [-h_0, 0] \times R^n.$$

ASSUMPTION H_2. Suppose that

1° $\varrho \in C^0(\Omega, R)$, there exist on $\Omega^{(1)}$ the derivatives $D_y\varrho, D_w\varrho$ and they are continuous on $\Omega^{(1)}$;

2° there exists $\alpha_1 \in J_+$ such that for $(x,y,w) \in \Omega^{(1)}$ we have

$$\|D_y\varrho(x,y,w)\| \le \alpha_1(\|w\|_{(1)}), \|D_w\varrho(x,y,w)\|_{C^0(D,R)} \le \alpha_1(\|w\|_{(1)});$$

3° there exists $\beta_1 \in J_+$ such that for $(x,y),(x,\bar{y}) \in [0,a] \times R^n$, $w \in C^{(1)}_{(L)}(D,R), h \in C^{(1)}(D,R)$ we have

$$\|D_y\varrho(x,\bar{y}, w + h) - D_y\varrho(x,y,w)\| \le \beta_1(\|w\|_{(1+L)})(\|y - \bar{y}\|_n + \|h\|_{(1)})$$

$$\|D_w\varrho(x,\overline{y},w+h)-D_w\varrho(x,y,w)\|_{C^0_{(L)}(D,R)}\leq$$
$$\leq\beta_1(\|w\|_{(1+L)})(\|y-\overline{y}\|_n+\|h\|_{(1)}).$$

Suppose that $a\in(0,a_0],Q_1,Q_2\in R_+$ and $Q_1\geq\lambda_1,Q_2\geq\lambda_2$. We say that the function $z:[-h_0,a]\times R^n\to R$ belongs to $K_a(Q_1,Q_2)$ if z is continuous and

(i) $z(x,y)=\varphi(x,y)$ for $(x,y)\in[-h_0,0]\times R^n$

(ii) there exist continuous derivatives with respect to y and

$$\|D_yz(x,y)\|\leq Q_1,\quad \|D_yz(x,y)-D_yz(x,\overline{y})\|\leq Q_2\|y-\overline{y}\|_n,$$
$$(x,y),(x,\overline{y})\in[-h_0,a]\times R^n.$$

For $z\in K_a(Q_1,Q_2)$ we define $\|z\|_*=\sup\{|z(x,y)|(1+\|y\|)^{-1}:(x,y)\in[-h_0,a]\times R^n\}$.

Suppose that $a\in(0,a_0],(x,y)\in[0,a]\times R^n,z\in K_a(Q_1,Q_2)$ be fixed.

Consider the following problem

$$\eta'(t)=\varrho(t,\eta(t),z_{(t,\eta(t))}),\eta(x)=y \tag{3}$$

LEMMA 1. *If assumptions* H_1, H_2 *are satisfied and* $a\in(0,a_0],z\in K_a(Q_1,Q_2)$ *then for each* $(x,y)\in[0,a]\times R^n$ *there exists an unique solution* $g(\cdot,x,y)=g[z](\cdot,x,y)$ *of the problem* (3). *This solution is defined on* $[0,a)$ *and the derivatives* $D_xg,D_{y_k}g$, $k=1,\ldots,n$ *exist on* $[0,a]\times[0,a]\times R^n$ *and*

$$\|D_{y_k}g(t,x,y)\|\leq C_1,\quad \|D_{y_k}g(t,x,y)-D_{y_k}g(t,x,\overline{y})\|\leq C_2\|y-\overline{y}\|_n,$$
$$k=1,\ldots,n,\ (t,x,y),(t,x,\overline{y})\in[0,a]\times[0,a]\times R^n,$$

where $C_1=\exp(a\alpha_1(Q_1)(1+Q_1))$, $C_2=a(\beta_1(Q_1+Q_2)C_1^2(1+Q_1)^2+$ $+\alpha_1(Q_1)Q_2C_1^2)C_1$. *Moreover, if* $z,\overline{z}\in K_a(Q_1,Q_2)$ *then*

$$\|g[z](t,x,y)-g[\overline{z}](t,x,y)\|(1+\|y\|)^{-1}\leq\alpha_1(Q_1)C_1\|z-\overline{z}\|_*,$$
$$(t,x,y)\in[0,a]\times[0,a]\times R^n.$$

The proof of this lemma is based on the Gronwall inequality and we omit the details.

3. Transformation T

ASSUMPTION H_3. Suppose that:

1° $f\in C^0(\Omega,R)$, there exist on $\Omega^{(1)}$ the derivatives D_yf,D_wf and they are continuous on $\Omega^{(1)}$;

2° there exists $\alpha_2\in J_+$ such that for $(x,y,w)\in\Omega^{(1)}$ we have

$$\|D_yf(x,y,w)\|\leq\alpha_2(\|w\|_{(1)}),\quad \|D_wf(x,y,w)\|_{C^0(D,R)}\leq\alpha_2(\|w\|_{(1)});$$

3° there exists $\beta_2 \in J_+$ such that for $(x,y),(x,\bar{y}) \in [0,a] \times R^n$, $w \in C^{(1)}_{(L)}(D,R), h \in C^{(1)}(D,R)$ we have

$$\|D_y f(x,\bar{y},x+h) - D_y f(x,y,w)\| \le \beta_2(\|w\|_{(1+L)})(\|y-\bar{y}\|_n + \|h\|_{(1)}),$$
$$\|D_w f(x,\bar{y},w+h) - D_w f(x,y,w)\|_{C^0_{(L)}(D,R)} \le \le \beta_2(\|w\|_{(1+L)})(\|y-\bar{y}\|_n + \|h\|_{(1)}).$$

ASSUMPTION H_4. If $H_0 > 0$ then suppose that the following consistency condition

$$D_x\varphi(0,y) + \sum_{k=1}^{n} \varrho_k(0,y,\varphi_{(0,y)}) D_{y_k}\varphi(0,y) = f(0,y,\varphi_{(0,y)})$$

holds for $y \in R^n$.

Suppose that $a \in (0,a_0]$. For $u \in K_a(Q_1,Q_2)$ we define $T : u \mapsto Tu$ in the following way:

$$(4)\quad \begin{aligned} &(Tu)(x,y) = \varphi(0,g[u](0,x,y)) + \int_0^x f(t,g[u](t,x,y),u_{(t,g[u](t,x,y))})\,dt \\ &\qquad\qquad \text{for } (x,y) \in [0,a]\times R^n \\ &(Tu)(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in [-h_0,0]\times R^n. \end{aligned}$$

LEMMA 2. *Let Assumptions* H_1–H_4 *hold. Then there are* $Q_1,Q_2 \in R_+$ *and* $a \in (0,a_0]$ *such that the transformation* T *maps the set* $K_a(Q_1,Q_2)$ *into itself.*

Proof. Let choose $Q_1,Q_2 \in R_+, a \in (0,a_0]$ such that

$$(5)\quad \begin{aligned} &Q_1 \ge \lambda_1 C_1 + a\alpha_2(Q_1)C_1(1+Q_1), \\ &Q_2 \ge \lambda_2 C_1^2 + \lambda_1 C_2 + a(\alpha_2(Q_1)(C_2(1+Q_1) + Q_2C_1^2) + \\ &\qquad + \beta_2(Q_1+Q_2)C_1^2(1+Q_1)^2). \end{aligned}$$

Relation (4) implies for $U = Tu$

$$\begin{aligned} D_{y_k}U(x,y) = &\sum_{j=1}^{n} D_{y_j}\varphi(0,f(0,x,y))D_{y_k}g_j(0,x,y) + \\ &+ \int_0^x \Big(\sum_{j=1}^{n} D_{y_j}f(P(t,x,y))D_{y_k}g_j(t,x,y) + \\ &\qquad + D_w f(P(t,x,y)) \sum_{j=1}^{n} (D_{y_j}u)_{(t,g(t,x,y))} D_{y_k}g_j(t,x,y)\Big)\,dt, \end{aligned}$$

where $(x,y) \in [0,a]\times R^n$ and $P(t,x,y) = (t,g(t,x,y),u_{(t,g(t,x,y))})$.

By Assumptions H_2, H_3 and by inequalities (5) we have

$$(6)\quad \|D_yU(x,y)\| \le \lambda_1 C_1 + a\alpha_2(Q_1)C_1(1+Q_1) \le Q_1, \qquad (x,y)\in[0,a]\times R^n,$$

$$(7)\quad \|D_yU(x,y)-D_yU(x,\overline{y})\| \le (\lambda_1 C_2 + \lambda_2 C_1^2)\|x-\overline{y}\|_n + \\ + a(\alpha_2(Q_1)(C_2(1+Q_1)+Q_2C_1^2)+\beta_2(Q_1+Q_2)C_1^2(1+Q_1)^2)\|y-\overline{y}\|_n \le \\ \le Q_2\|y-\overline{y}\|_n, \quad (x,y),(x,\overline{y})\in[0,a]\times R^n.$$

From (6) and (7) we obtain Lemma 2.

ASSUMPTION H_5. Suppose that $Q_1, Q_2 \in R_+$ and $a \in (0, a_0]$ satisfy (5) and $a(\lambda_1\alpha_1(Q_1)C_1 + a\alpha_2(Q_1)\alpha_1(Q_1)C_1(1+Q_1) + \alpha_2(Q_1)) \le q$ where $0 \le q < 1$.

LEMMA 3. *Suppose that Assumptions* H_1–H_5 *are satisfied. Then transformation* $T : K_a(Q_1,Q_2) \to K_a(Q_1,Q_2)$ *has an unique fixed point.*

Proof. For $u, v \in K_a(Q_1,Q_2)$ we have

$$\|Tu - Tv\|_* \le a(\lambda_1\alpha_2(Q_1)C_1 \\ + a\alpha_2(Q_1)\alpha_1(Q_1)C_1(1+Q_1) + \alpha_2(Q_1))\|u-v\|_* \le q\|u-v\|_*$$

where $0 \le q < 1$. Thus T is contraction and Lemma 3 we obtain from the Banach fixed-point theorem.

4. The main theorem

We are now in position to show a theorem on existence and uniqueness of solutions of problem (1), (2).

THEOREM. *Suppose that Assumptions* H_1–H_5 *are satisfied. Then there is a function* $z \in K_a(Q_1,Q_2)$ *which is an unique solution of problem* (1), (2) *in class* $K_a(Q_1,Q_2)$.

Proof. For z^* which is an unique fixed point of $T : K_a(Q_1,Q_2) \to K_a(Q_1,Q_2)$ we have

$$(8)\quad z^*(x,y) = \varphi(0, g(0,x,y)) \\ + \int_0^N f(t, g(t,x,y), z^*_{(t,g(t,x,y))})\,dt, \quad (x,y)\in[0,a]\times R^n$$

and $z^*(x,y) = \varphi(x,y)$ for $(x,y) \in [-h_0, 0]\times R^n$.

By uniqueness of solutions for (3) the following relation holds

$$(9)\qquad y = g(x,0,\eta) \Leftrightarrow \eta = g(0,x,y).$$

For fixed $x \in [0,a]$ this relation represents a 1-1 transformation of the space R^n into itself. By taking $y = g(x,0,\eta)$ and making use of (9), relation (8) is transformed into

$$(10) \qquad z^*(x, g(x,0,\eta)) = \varphi(0,\eta) + \int_0^x f(t, g(t,0,\eta), z^*_{t,g(t,0,\eta))})\,dt.$$

By differentiating (10) with respect to x and taking $\eta = g(0,x,y)$ we obtain that z^* is a solution of (1), (2).

Remark. The above result can be extended to weakly coupled quasi-linear systems

$$D_x z_i(x,y) + \sum_{k=1}^n \widetilde{\varrho}_{ik}(x,y,z_{(x,y)}) D_{y_k} z_i(x,y) = \widetilde{f}_i(x,y,z_{(x,y)}), \quad i = 1,\dots,m$$

where $z = (z_1,\dots,z_m)$ and $\widetilde{\varrho}_{ik}, \widetilde{f}_i : [0,a_0] \times R^n \times C^0(D,R^m) \to R$, $i = 1,\dots,m$, $k = 1,\dots,n$.

5. Examples

Let $\overline{\varrho}_k, \overline{f} : [0,a_0] \times R^n \times R \to R$, $k = 1,\dots,n$.

As a particular case of (1), (2) we obtain the initial problem for partial differential equations with a retarded argument

$$(11) \qquad D_x z(x,y) + \sum_{k=1}^n \overline{\varrho}_k(x,y,z(\gamma_1(x),\delta_1(x,y))) D_{y_k} z(x,y) = \overline{f}(x,y,z(\gamma_2(x)\delta_2(x,y)))$$

where $\gamma_1,\gamma_2 : [0,a_0] \to R$, $\delta_1,\delta_2 : [0,a_0] \times R^n \to R^n$.

We define $\varrho_k(x,y,w) = \overline{\varrho}_k(x,y,w(\gamma_1(x)-x,\delta_1(x,y)-y))$,

$$f(x,y,w) = \overline{f}(x,y,w(\gamma_2(x)-x,\delta_2(x,y)-y))$$

and it is easy to formulate a theorem on the existence and uniqueness of solutions of Cauchy problem for (11).

The differential-integral equation

$$D_x z(x,y) + \sum_{k=1}^n \overline{\varrho}\Big(x,y,\int_D z(x+t,y+s)\,dt\,ds\Big) D_{y_k} z(x,y) = \overline{f}\Big(x,y,\int_D z(x+t,y+s)\,dt\,ds\Big)$$

is a particular case of (1) with $\varrho_k(x,y,w) = \overline{\varrho}_k(x,y,\int_D z(t,s)\,dt\,ds)$ and $f(x,y,w) = \overline{f}(x,y,\int_D z(t,s)\,dt\,ds)$.

References

[1] P. Bassanini, *On a result proof concerning a boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form*, Boll. Un. Mat. Ital. (5) 14-A (1977), 325–332.
[2] P. Bassanini, *Iterative methods for quasilinear hyperbolic systems*, Boll. Un. Mat. Ital (6) 1-B (1982), 225–250.
[3] L. Cesari, *A boundary value problem for quasilinear hyperbolic systems*, Riv. Mat. Univ. Parma 3 (1974), 107–131.
[4] L. Cesari, *A boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form*, Ann. Sc. Norm. Sup. Pisa (4) 1 (1974), 311–358.
[5] D. Jaruszewska-Walczak, *Existence of solutions of first order partial differential-functional equations*, Boll. Un. Mat. Ital. (7) 4-B (1990), 57–82.
[6] D. Jaruszewska-Walczak, *Initial value problems for quasilinear hyperbolic systems of partial differential-functional equations*, Atti Sem. Mat. Fis. Univ. Modena 40 (1992), 229–246.

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