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ON A SINGULAR VOLTERRA EQUATION

1. Introduction

The importance of equation

$$f(p) = 2\int_p^\infty \frac{F(r)}{\sqrt{1-(p/r)^2}}\,dr \tag{1.1}$$

in investigations of electromagnetic cascades was pointed out by B. Słowiński [5]. Here f is a given C^1 function on $[0,\infty[$ such that $\lim_{p\to\infty} pf(p)$ does exist. The unique solution F continuous on $]0,\infty[$ has the form

$$F(r) = \frac{1}{\pi r^2}\lim_{p\to\infty}[pf(p)] - \frac{1}{\pi r^2}\int_r^\infty \frac{d}{dp}[pf(p)]\frac{dp}{\sqrt{1-(r/p)^2}}. \tag{1.2}$$

Consult also [6]. This solution can be obtained in the following way. When one changes variables in (1.1) by introducing

$$y := \frac{1}{r}, \quad t := \frac{1}{p} \tag{1.3}$$

and by defining new functions

$$\eta(t) := f\left(\frac{1}{t}\right), \quad u(y) := \frac{2}{y^2}F\left(\frac{1}{y}\right), \tag{1.4}$$

then the equation (1.1) becomes

$$\eta(t) = \int_0^t \frac{u(y)dy}{\sqrt{1-(y/t)^2}}. \tag{1.5}$$

The equation (1.5) was derived by H. Wagner [9] in a connection with his studies on landing of seaplanes. The unique continuous on $]0,\infty[$ solution of

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equation (1.5) is (see W. Tollmien [8])

$$u(y) = \frac{2}{\pi} \lim_{t\to 0^+} \left[\frac{\eta(t)}{t}\right] + \frac{2}{\pi} y \int_0^y \frac{d}{dt}\left[\frac{\eta(t)}{t}\right] \frac{dt}{\sqrt{y^2 - t^2}}. \tag{1.6}$$

Then the formula (1.2) follows from (1.6) by means of (1.3) and (1.4).

More general Abel-type equations were investigated by M. Soupline [7]. We refer to W. Schmeidler [4] for further informations on Volterra equations and for useful references to original papers.

For some physical reasons (consult, e.g., G. A. Acopdjanov [1] and B. Słowiński papers), the most interesting solution of (1.1) is for the function

$$f(p) = \frac{1}{a} e^{-p/a}, \tag{1.7}$$

where $a > 0$ is constant. For this function we obtain, by (1.2),

$$F(r) = \frac{1}{\pi a^2} \int_0^1 \left(\frac{1}{s} - \frac{a}{r}\right) e^{-r/as} \frac{ds}{s^2\sqrt{1 - s^2}}. \tag{1.8}$$

The integrand in (1.8) is a C^∞ function of s on $[0,1[$ (if we let 0 for $s = 0$) and the singularity of this function at 1 is weak. Moreover, the function F defined by (1.8) is a very well defined analytic function on $]0,\infty[$.

Our aim is to study the behaviour of F at infinity and at zero. Some results in this topic were announced in [3].

2. On Kostritsa conjecture

A. A. Kostritsa [2] suggests that the solution of (1.1) for the function (1.7) has the form

$$F(r) = \text{const.}\, \mathrm{K}_0\left(\frac{r}{a}\right), \tag{2.1}$$

where K_0 is the MacDonals's function. If the function F were as in (2.1) it would satisfy the equation

$$r^2 F''(r) + rF'(r) - \frac{r^2}{a^2} F(r) = 0. \tag{2.2}$$

But the result of substitution of F described in (1.8) into the left-hand side of equation (2.2) is

$$\frac{1}{\pi a^2} \int_0^1 \left(\frac{r}{a} - \frac{a}{r} - \frac{1}{s} - \frac{r^2}{a^2 s} - \frac{2r}{as^2} + \frac{r^2}{a^2 s}\right) e^{-r/as} \frac{ds}{s^2\sqrt{1 - s^2}}.$$

It is clear that the above function does not identically vanish, and hence, for every real constants c_1 and c_2, there exists a positive r such that

$$F(r) \neq c_1 \mathrm{I}_0\left(\frac{r}{a}\right) + c_2 \mathrm{K}_0\left(\frac{r}{a}\right).$$

In particular, (2.1) does not hold.

3. Behaviour of F at infinity

Let us define the function $h :]0,1[\times R_+ \to R$ by the formula

$$h(s,x) := \left(\frac{1}{s} - \frac{1}{x}\right)\frac{\exp(-x/s)}{s^2\sqrt{1-s^2}} \tag{3.1}$$

We see that for F defined by (1.8)

$$F(r) = \frac{1}{\pi a^2} \int_0^1 h\left(s, \frac{r}{a}\right) ds. \tag{3.2}$$

Hence, the behaviour of F for small $r > 0$ and range r, say $r \gg a$, is completely described by estimates of the function

$$H(x) := \int_0^1 h(s,x)\, ds \tag{3.3}$$

for small $x > 0$ and $x \gg 1$, respectively.

LEMMA 3.4. *There exists $C^+ > 0$ such that $H(x) < C^+ e^{-x}$ for every $x > 1$.*

P r o o f. We see from formulae (3.1) and (3.3) that

$$H(x) < \int_0^1 e^{-x/s} \frac{ds}{s^3\sqrt{1-s^2}}.$$

Since $\frac{x}{s} = \frac{1}{s} + \frac{x-1}{s}$ and $\frac{x-1}{s} > x - 1$ for $s \in]0,1[$, we infer that

$$H(x) < \int_0^1 e^{-1/s}\, e^{-(x-1)/s} \frac{ds}{s^3\sqrt{1-s^2}} < e^{1-x} \int_0^1 \frac{e^{-1/s} ds}{s^3\sqrt{1-s^2}}.$$

The result follows with

$$C^+ = e \int_0^1 \frac{e^{-1/s} ds}{s^3\sqrt{1-s^2}}. \quad \blacksquare \tag{3.4'}$$

LEMMA 3.5. *Let $c \in R$ and $c > 1$. Then there exists $C^- > 0$ such that*

$$H(x) > C^- e^{-x^2/2}$$

for every $x > c$.

Proof. By means of estimates $x \cdot \frac{1}{s} < \frac{1}{2}(x^2 + \frac{1}{s^2})$ and $\frac{1}{s^2} < \frac{1}{s^3}$ for $s \in]0,1[$, we derive inequality

$$H(x) > \int_0^1 \left(\frac{1}{s^3} - \frac{1}{xs^3}\right) \exp\left(-\frac{1}{2}x^2 - \frac{1}{2s^2}\right) \frac{ds}{\sqrt{1-s^2}}$$
$$= \left(1 - \frac{1}{x}\right) e^{-x^2/2} \int_0^1 e^{-1/s^2} \frac{ds}{s^3\sqrt{1-s^2}} > \left(1 - \frac{1}{c}\right) e^{-x^2/2} \int_0^1 \frac{e^{-1/2s^2} ds}{s^3\sqrt{1-s^2}}.$$

Thus, the result holds with

$$C^- = \frac{c-1}{c} \int_0^1 e^{-1/2s^2} \frac{ds}{s^3\sqrt{1-s^2}}. \qquad (3.5') \quad \blacksquare$$

Remark 3.6. It is easy to see that

$$\int_0^1 \frac{1}{s^3} e^{-1/2s^2} \frac{ds}{\sqrt{1-s^2}} > \int_0^1 \frac{1}{s^3} e^{-1/2s^2} ds = \frac{1}{\sqrt{e}}.$$

Thus, one can take the constant $C^- = \frac{c-1}{c\sqrt{e}}$ in Lemma 3.5. and $C^- = \frac{1}{2\sqrt{e}}$ if $c \geq 2$.

Now, composing $(3.2 \div 5)$, we obtain the following result.

THEOREM 3.7. *Let $c > 1$. Then the function F defined by formula* (1.8) *has the estimates:*

$$\begin{cases} F(r) < \dfrac{C^+}{\pi a^2} e^{-r/a} & \text{for } r \geq a \\ F(r) > \dfrac{C^-}{\pi a^2} e^{-r^2/a^2} & \text{for every } r > ca, \end{cases}$$

where positive constants C^+ and C^- are defined by (3.4′) *and* (3.5′), *respectively.*

Remark 3.8. By the methods used above one can also prove that $F^{(n)}(r) = 0(e^{-r/a})$ as $r \to \infty$ for any derivative $F^{(n)}$ of the function F.

4. Estimates of F on $]0, a[$

We assume throughout this paragraph that $0 < x < b < 1$. It is not difficult to calculate

$$I_1(x,b) := \int_b^1 \left(\frac{1}{s^3} - \frac{1}{xs^2}\right) \frac{ds}{\sqrt{1-s^2}} \qquad (4.1)$$
$$= \frac{\sqrt{1-b^2}}{2b^2} + \frac{1}{2} \ln \frac{1+\sqrt{1-b^2}}{b} - \frac{1}{xb}\sqrt{1-b^2},$$

$$I_2(x,b) := \int_x^b e^{-x/s}\left(\frac{1}{s^3} - \frac{1}{xs^2}\right)ds = \frac{1}{xb}e^{-x/b} - \frac{1}{ex^2}, \tag{4.2}$$

$$I_3(x) := \int_0^x e^{-x/s}\left(\frac{1}{s^3} - \frac{1}{xs^2}\right)ds = \frac{1}{ex^2}. \tag{4.3}$$

Note that $I_1(x,b) < 0, I_2(x,b) < 0$ and $I_3(x) > 0$, since the functions under integrals in (4.1)–(4.3) have constant sign.

It is visible from formulae (3.1) and (4.3) that

$$\frac{1}{ex^2} < \int_0^x h(s,x)ds < \frac{1}{ex^2\sqrt{1-x^2}}$$

for $x \in]0,1[$. This implies that

$$\lim_{x\to 0^+} x^2 \int_0^x h(s,x)ds = \frac{1}{e}$$

which means that $h(\cdot,x)$ has a very high local maximum in the interval $]0,x[$ if $x > 0$ is a fixed small real number.

Dividing the interval $[0,1]$ into three ones: $[0,x],[x,b]$ and $[b,1]$, and taking into account (3.3), (4.1)–(4.3), we obtain

$$H(x) < \frac{1}{\sqrt{1-x^2}}I_3(x) + \frac{1}{\sqrt{1-x^2}}I_2(x,b) + e^{-x/b}I_1(x,b), \tag{4.4}$$

$$H(x) > I_3(x) + \frac{1}{\sqrt{1-b^2}}I_2(x,b) + e^{-x}I_1(x,b). \tag{4.5}$$

These estimates are valid for all x,b such that $0 < x < b < 1$.

LEMMA 4.6. *If* $0 < x < b < 1$ *then*

$$H(x) < \frac{1}{2}\ln 2 - \frac{1}{2}\ln b + \frac{1}{2b^2} + \frac{1}{xb}\left(\frac{1}{\sqrt{1-x^2}} - \sqrt{1-b^2}\right) =: Q^+(x,b). \tag{4.6.1}$$

Moreover, for $c \in]0,1[$ *and* $b = x^c$, *we have*

$$\begin{cases} \lim_{x\to 0^+} x^{2/3}Q^+(x,x^{1/3}) = 1, & \\ \lim_{x\to 0^+} x^{1-c}Q^+(x,x^c) = \frac{1}{2} & \textit{for } c \in]0,\tfrac{1}{3}[, \\ \lim_{x\to 0^+} x^{2c}Q^+(x,x^c) = \frac{1}{2} & \textit{for } c \in]\tfrac{1}{3},1[. \end{cases} \tag{4.6.2}$$

Proof. By means of the estimate (4.4) and equalities (4.1)–(4.3), we obtain

$$H(x) < \frac{1}{ex^2\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\left(\frac{1}{xb}e^{-x/b} - \frac{1}{ex^2}\right) + e^{-x/b}I_1(x,b) <$$

$$< \frac{1}{xb\sqrt{1-x^2}} + I_1(x,b)$$
$$< \frac{1}{xb\sqrt{1-x^2}} + \frac{\sqrt{1-b^2}}{2b^2} + \frac{1}{2}\ln(1+\sqrt{1-b^2}) - \frac{1}{2}\ln b - \frac{\sqrt{1-b^2}}{xb}$$

and, since $\sqrt{1-b^2} < 1$ and $\ln(1+\sqrt{1-b^2}) < \ln 2$, we see that (4.6.1) holds. To prove (4.6.2) let us notice that $x^c > x$ for any $c \in]0,1[$ and $x \in]0,1[$. Hence, by (4.6.1) $H(x) < Q^+(x,x^c)$, where

$$Q^+(x,x^c) = \frac{1}{2}\ln 2 - \frac{c}{2}\ln x + \frac{1}{2}x^{-2x} + x^{c-1}\frac{x^{2-2c} - x^2 + 1}{\sqrt{1-x^2}(1+\sqrt{(1-x^2)(1-x^2c)})}.$$

This formula for $Q^+(x,x^c)$ makes all limits in (4.6.2) evident. ■

LEMMA 4.7. *If* $0 < x < b < 1$ *then*

$$H(x) > \frac{-b^2}{x^2e\sqrt{1-b^2}} - \frac{1}{b^2\sqrt{1-b^2}} + \frac{e^{-x}\sqrt{1-b^2}}{2b^2} =: Q^-(x,b) \tag{4.7.1}$$

and, for $c \in]0,1[$ *and* $b = x^c$, *we have*

$$\begin{cases} \lim_{x\to 0+} x^{2-2c}Q^-(x,x^c) = -\frac{1}{e} & \text{if } c \in]0,\frac{1}{2}[, \\ \lim_{x\to 0+} xQ^-(x,\sqrt{x}) = -\frac{1}{e} - \frac{1}{2}, & \\ \lim_{x\to 0+} x^{2c}Q^-(x,x^c) = -\frac{1}{2} & \text{if } c \in]\frac{1}{2},1[. \end{cases} \tag{4.7.2}$$

Proof. By (4.5) and (4.1)–(4.3), we infer

$$H(x) > \frac{1}{ex^2} + \frac{e^{-x/b}}{xv\sqrt{1-b^2}} - \frac{1}{ex^2\sqrt{1-b^2}} + \frac{\sqrt{1-b^2}}{2b^2}e^{-x} - \frac{\sqrt{1-b^2}}{xb}e^{-x}. \tag{4.7.3}$$

We dropped the term with $\ln(\ldots)$, since it is positive for all $b \in]0,1[$ and, when multiplied by x^c, it tends to zero as $x \to 0$ for $b = x^c$. Combining the first term with third, and the second one with last in (4.7.3),

$$H(x) > \frac{-b^2}{x^2e\sqrt{1-b^2}(1+\sqrt{1-b^2})} + \frac{1}{xb\sqrt{1-b^2}}(e^{-x/b} - e^{-x} + b^2e^{-x}) + \frac{\sqrt{1-b^2}}{2b^2}e^{-x}$$
$$> \frac{-b^2}{x^2e\sqrt{1-b^2}} + \frac{1}{xb\sqrt{1-b^2}}(e^{-x/b} - e^{-x}) + \frac{b}{x}e^{-x} + \frac{\sqrt{1-b^2}}{2b^2}e^{-x}.$$

Since

$$e^{-x/b} - e^{-x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\left(\frac{x}{b}\right)^k + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} x^k$$

and, clearly,

$$0 < \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} x^k < x \quad \text{for } x \in]0,1[,$$

the following inequality holds

$$\text{(4.7.4)} \qquad H(x) > \frac{-b^2}{x^2 e\sqrt{1-b^2}} + \frac{1}{b^2\sqrt{1-b^2}} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\left(\frac{x}{b}\right)^{k-1} + \frac{\sqrt{1-b^2}}{2b^2} e^{-x},$$

where we dropped $\frac{b}{x}e^{-x}$, having $\lim_{x\to 0^+} x(\frac{x^c}{x}e^{-x}) = 0$ for every $c > 0$. Since $0 < \frac{x}{b} < 1$ and $\frac{1}{k!}(\frac{x}{b})^{k-1}$ tends monotonically to zero as $k \to \infty$, we have the estimate

$$-1 < \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\left(\frac{x}{b}\right)^{k-1} < 0$$

which, used in (4.7.4), gives (4.7.1).

For $b = x^c$ we have

$$Q^-(x, x^c) = \frac{-1}{e\sqrt{1-x^{2c}}} x^{-2+2c} + \left(\frac{1}{2}\sqrt{1-x^2}ce^{-x} - \frac{1}{\sqrt{1-x^{2c}}}\right) x^{-2c}$$

and thus all limits in (4.7.2) are evident. ∎

R e m a r k s. Taking the right-hand side of (4.4) as the upper bound $Q^+(x,b)$ of H in (4.6.1), we obtain the same limits for such $Q^+(x,b)$ as written down in (4.6.2). Similarly, if one replaces $Q^-(x,x^c)$ by the lower bound written in (4.5) with $b = x^c$, then the limits (4.7.2) remain still valid. From (4.6.2) it is easy to see that the best upper bound near zero (with $b = x^c$) is for $c = \frac{1}{3}$. The best lower bound that may be obtained by this method is for $c = \frac{1}{2}$.

Lemmas 4.6 and 4.7 imply the result as follows.

COROLLARY 4.8. *For every* $\varepsilon > 0, \lim_{x\to 0^+} x^{1+\varepsilon} H(x) = 0$.

P r o o f. For $x \in]0,1[$ and $\varepsilon > 0$, we obtain the followinf estimates, by virtue of (4.6.1) and (4.7.1),

$$x^\varepsilon x Q^-(x, \sqrt{x}) < x^{1+\varepsilon} H(x) < x^{1/3+\varepsilon} x^{2/3} Q^+(x, \sqrt[3]{x}).$$

Applying (4.6.2) and (4.7.2) to the above lower and upper bounds, we obtain the result. ∎

Let us define for $r \in]0,a[$ the following functions

$$F^+(r) := \frac{1}{\pi a^2} Q^+\left(\frac{r}{a}, \sqrt[3]{\frac{r}{a}}\right); \quad F^-(r) := \frac{1}{\pi a^2} Q^-\left(\frac{r}{a}, \sqrt{\frac{r}{a}}\right). \tag{4.9}$$

We now formulate results concerning asymptotics of F near zero.

THEOREM 4.10. *For every* $r \in]0,a[$

$$F^-(r) < F(r) < F^+(r) \tag{4.10.1}$$

and $F(r) = 0(\frac{1}{r})$ *as* $r \to 0$. *In particular,* $\lim_{r\to 0^+} r^{1+\varepsilon}F(r) = 0$ *for every* $\varepsilon > 0$. *Moreover,*

$$\begin{cases} \lim\limits_{r\to 0^+} r^{2/3}F^+(r) = \dfrac{1}{\pi a^{4/3}}, \\ \lim\limits_{r\to 0^+} rF^-(r) = -\dfrac{1}{\pi a}\left(\dfrac{1}{2} + \dfrac{1}{e}\right). \end{cases} \tag{4.10.2}$$

Proof. From (3.2) and (3.3) it follows that

$$F(r) = \frac{1}{\pi a^2} H\left(\frac{r}{a}\right). \tag{4.10.3}$$

Hence, viewing (4.6.1), (4.7.1) and (4.9), we obtain (4.10.1).

Now, by virtue of (4.6.2) and (4.9),

$$\lim_{r\to 0^+} r^{2/3}F^+(r) = a^{2/3} \lim_{r\to 0^+} \left(\frac{r}{a}\right)^{2/3} \frac{1}{\pi a^2} Q^+\left(\frac{r}{a}, \sqrt[3]{\frac{r}{a}}\right) = \frac{1}{\pi a^{4/3}}.$$

Similarly, by virtue of (4.7.2) and (4.9),

$$\begin{aligned} \lim_{r\to 0^+}[rF^-(r)] &= a \lim_{r\to 0^+}\left[\frac{r}{a}\cdot\frac{1}{\pi a^2} Q^-\left(\frac{r}{a}, \sqrt{\frac{r}{a}}\right)\right] \\ &= \frac{1}{\pi a}\lim_{x\to 0^+}[xQ^-(x,\sqrt{x})] = -\frac{1}{\pi a}\left(\frac{1}{2} + \frac{1}{e}\right). \end{aligned}$$

Thus, equalities (4.10.2) are valid. The remaining statements easily follow from (4.10.2) and from the fact that $r^c \ln r \to 0$ as $r \to 0^+$ for any $c > 0$. ∎

Theorems 3.7 and 4.10 imply the following result.

COROLLARY 4.11. *For every real* $m \geq 1$ *the integral* $\int_0^\infty r^m F(r)dr$ *is convergent.*

One important question remains still open. Is the function F positive everywhere on $\mathbb{R}_+$? From Theorem 3.7 we see that it is the case for $r \in [a,\infty[$. It is also not difficult to deduce from (3.1)÷(3.3) that $F'(r) < 0$ for all $r \geq a$. To prove that $F(r) > 0$ for $r \in]0,a[$, it is sufficient to show that $H(x) > 0$ for $x \in]0,1[$ (see (3.2) and (3.3)). Unfortunately, for a fixed small $x > 0$ the function $]0,1[\ni s \mapsto h(s,x) \in \mathbb{R}$ has a high positive local maximum in $]0,x[$

and a low negative local minimum for $s > x$, near zero. This can be seen from equality (4.3) and Lemma 4.6 or from the estimate

$$\frac{1}{\sqrt{1-x^{2/3}}} I_2(x, \sqrt[3]{x}) < \int_x^{\sqrt[3]{x}} h(s,x)ds < \frac{1}{\sqrt{1-x^2}} I_2(x, \sqrt[3]{x}),$$

where I_2 is defined by (4.2). Since

$$I_2(x, \sqrt[3]{x}) = x^{-4/3} \exp(-x^{2/3}) - \frac{1}{e} x^{-2},$$

we see that

$$\lim_{x \to 0+} x^2 I_2(x, \sqrt[3]{x}) = -\frac{1}{e}.$$

The same type of troubles appears for derivative $H'(x)$ in interval $]0,1[$. Is $H'(x) < 0$ for all $x \in]0,1[$? Computer calculations show that $H(x) > 0$ and H decreases for $x > 10^{-6}$ (Słowiński's private communication). If $F(r) > 0$ or, equivalently, $H(x) > 0$ for all positive r and x, then one can use F, after a suitable normalization, as a probability density μ on the plane, namely:

$$C \cdot F(\sqrt{x^2+y^2}) = \mu(x,y), \quad (x,y) \in \mathbb{R}^2,$$

where

$$C = \frac{1}{2\pi} \Big[\int_0^\infty rF(r)\, dr \Big]^{-1}.$$

The integral above is convergent, by virtue of Corollary 4.11, which would also guarantiee the existence of all moments for μ.

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