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ON THE HASEMAN PROBLEM

1. This article is devoted to the theory in a Lebesque space $L_p(\Gamma, \varrho)$ with a power weight ϱ of a singular integral operator (SIO) with a shift

$$N = WP_+ + GP_-, \tag{1}$$

where $P_+ + P_- = I$ is the identity operator, $P_+ - P_- = S$ is the singular integral operator with Cauchy kernel given by

$$(S\varphi)(t) = (\pi i)^{-1} \int_\Gamma (\tau - t)^{-1}\varphi(\tau)\, d\tau, \quad t \in \Gamma,$$

W is the shift operator defined by $(W\varphi)(t) = \varphi[\alpha(t)]$, α is an orientation-preserving diffeomorphism of a simple open oriented smooth curve Γ onto itself and at last G is a function or matrix-valued function (briefly MVF) on Γ.

The operator (1) is closely connected with the boundary value problem of Haseman: find a piecewise analytic function $\Phi(z)$ having a representation of it in the form of the Cauchy type integral with a density of class $L_p(\Gamma, \varrho)$ on the basis of the boundary condition

$$\Phi^+[\alpha(t)] = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{2}$$

where $\Phi^\pm(t)$ are angular limit values of the function $\Phi(z)$. D. A. Kveselava, N. P. Vekua, G. F. Mandzhavidze and B. V. Khvedelidze, I. B. Simonenko, L. I. Chibrikova, S. N. Antoncev and V. N. Monakhov studied the Haseman problem (2) under the various assumptions (see [1]). On the basis of the investigations of Banach algebras of SIO with noncarleman shifts and ruled coefficients (that is uniform limits of step functions) V.G. Kravchenko and

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the author [2], [3] obtained necessary and sufficient Noetherian conditions and a formula for computing the index of the operator (1).

THEOREM 1. *Let $\Gamma = [t_0, t_1]$ be a simple open oriented smooth curve, let G be a continuous function on Γ and*

$$\varrho(t) = |t - t_0|^{\beta_0}|t - t_1|^{\beta_1}, -p^{-1} < \beta_j < 1 - p^{-1}, 1 < p < \infty.$$

Then the operator (1) *is Noetherian in the space $L_p(\Gamma, \varrho)$ if and only if $G(t) \neq 0$ for all $t \in \Gamma$ and the numbers*

$$\varphi_j = (p^{-1} + \beta_j)(1 + \delta_j^2) + (2\pi)^{-1}[\delta_j \ln |G(t_j)| - (-1)^j \arg G(t_j)] \quad (j = 0, 1)$$

are not integers, where $\delta_j = (2\pi)^{-1} ln|\alpha'(t_j)|$. If these conditions are satisfied, the index of the operator (1) *is equal to*

$$\text{(3)} \quad \operatorname{ind} N = (2\pi)^{-1}\{\arg G(t)\}_{t\in\Gamma} + \sum_{j=0}^{1}[E(\varphi_j) + (-1)^j(2\pi)^{-1} \arg G(t_j)],$$

where $(2\pi)^{-1}\{\arg G(t)\}_{t\in\Gamma}$ is the Cauchy index of the invertible function G and $E(x)$ is the integer part of a real number x.

Later the Noether theory of the operator (1) was constructed under more generalized conditions: for piecewise continuous matrix-valued coefficients, composite contours and piecewise smooth shifts (see [4], [3]).

The next step was calculation of defect numbers of the operator (1). This result was obtained jointly by A. V. Ajzenshtat, G. S. Litvinchuk and the author [5], but under more strict conditions: Γ is a simple open Ljapunov curve and the derivative α' in addition satisfies a Hölder condition. It is based on the next sewing theorem.

Let $\mathfrak{R}$ be a set consisting from all curves γ, such that the operator S is bounded in all spaces $L_p(\gamma)$, $1 < p < \infty$. G. David [6] proved that

$$\mathfrak{R} = \{\gamma : \sup_{t\in\Gamma} \sup_{r>0} r^{-1} \operatorname{mes}(\gamma \cap \{z : |z - t| < r\}) < \infty\}.$$

DEFINITION 1. We call an open curve $\mathcal{L} \in \mathfrak{R}$ with endpoints $\tau_0, \tau_1 \in \mathbb{C} \setminus \{\infty\}$ a spiral of logarithmic type (Δ_0, Δ_1) if there exist limits

$$\lim_{\tau\to\tau_j, \tau\in\mathcal{L}} [\arg(\tau - \tau_j)/ \ln |\tau - \tau_j|^{-1}] = \Delta_j \quad (j = 0, 1).$$

THEOREM 2. *Let Γ be a simple open Ljapunov arc with endpoints t_0, t_1 and let α be an orientation-preserving H-smooth diffeomorphism of the contour Γ onto itself. Then the following propositions are valid:*

1) *there exists a conformal and one-sheeted mapping $\omega : \mathbb{C} \setminus \Gamma \to \mathbb{C} \setminus \mathcal{L}$, where $\mathcal{L} \in \mathfrak{R}$ is a spiral of logarithmic type (δ_0, δ_1), which is a Ljapunov curve outside of an arbitrary neighborhood of the endpoints $\tau_j = \omega(t_j)$, $\delta_j = (2\pi)^{-1} \ln |\alpha'(t_j)|$;*

2) *the function* ω *has Hölder limit values* $\omega^{\pm}(t)$ *on* Γ *satisfying the boundary condition* $\omega^{+}[\alpha(t)] = \omega^{-}(t),\ t \in \Gamma$;

3) *the derivative* ω' *is continuously extended onto* $\Gamma \setminus \{t_0, t_1\}$ *from left and right sides and has the next asymptotics in neighborhoods of endpoints*

$$|\omega'(z) \asymp |z - t_j|^{-\delta_j^2/(1+\delta_j^2)} \quad \text{for } z \to t_j,\ z \overline{\in} \Gamma\ (j = 0, 1).$$

With the help of Theorem 2 the Haseman problem (2) is reduced to the equivalent Riemann problem on the logarithmic type spiral $\mathcal{L} = \omega(\Gamma)$:

$$F^{+}(\tau) = \widehat{G}(\tau)F^{-}(\tau) + \widehat{g}(\tau), \quad \tau \in \mathcal{L}, \tag{4}$$

where $\widehat{G}(\tau) = G[(\omega^{-})^{-1}(\tau)]$, $\widehat{g}(\tau) = g[(\omega^{-})^{-1}(\tau)] \prod_{j=0}^{1} (\tau - \tau_j)^{m_j}$,

$$F(z) = \Psi[\omega^{-1}(z)] \prod_{j=0}^{1} (z - \tau_j)^{m_j} \quad \text{and} \quad m_j = E[(p^{-1} + \beta_j)(1 + \delta_j^2)].$$

With problem (4) we associate a SIO

$$\widehat{N} = P_{+} + \widehat{G}P_{-} \tag{5}$$

acting in the space $L_p(\mathcal{L}, \widehat{\varrho})$, where

$$\widehat{\varrho}(\tau) = \prod_{j=0}^{1} |\tau - \tau_j|^{\widehat{\beta}_j}, \quad -p^{-1} < \widehat{\beta}_j = (p^{-1} + \beta_j)(1 + \delta_j^2) - p^{-1} - m_j < 1 - p^{-1}.$$

From the connection between problems (2) and (4) we can receive the following.

THEOREM 3. *The operators* (1) *and* (5) *are Noetherian only simultaneously. If they are Noetherian, then*

$$\dim \operatorname{Ker} N = \max\{0, \operatorname{ind} \widehat{N} + m\}, \quad \dim \operatorname{Coker} N = \max\{0, -\operatorname{ind} \widehat{N} - m\},$$

where $m = m_0 + m_1$.

COROLLARY 1. *If the operator* (1) *is Noetherian, then*

$$\dim \operatorname{Ker} N = \max\{0, \operatorname{ind} N\},$$
$$\dim \operatorname{Coker} N = \max\{0, -\operatorname{ind} N\},$$

where ind N *is calculated by the formula* (3).

2. Let us consider the operator (1) with a coefficient G having points of discontinuity of semi-almost periodic type on a piecewise smooth contour Γ. At first in the space $L_p^n(\mathbb{R})$ we shall study SIO's.

$$T_G = P_{+} + GP_{-} \tag{6}$$

with semi-almost periodic MVF's G and corresponding convolution type operators

$$W_G = \chi_- I + \mathcal{F}^{-1} G \mathcal{F} \chi_+ I, \tag{7}$$

where $\chi_\pm$ are characteristic functions of semi-axes $\mathbb{R}_\pm$ and $\mathcal{F}$ is the Fourier transformation:

$$(\mathcal{F}\varphi)(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ixy} \varphi(y)\, dy, \quad x \in \mathbb{R}.$$

In [7] D. Sarason defined the class SAP of semi-almost periodic functions as the subalgebra of L_∞ generated by the class AP of uniform almost periodic functions and the class $C(\overline{\mathbb{R}})$ of functions continuous on the two-point compactification of $\mathbb{R}$. In parallels we consider the algebra $\mathcal{M}_p$ of Fourier multipliers on $L_p(\mathbb{R})$ and the subalgebra $SAA_p \subset \mathcal{M}_p$ generated by AP_p and $C_p(\overline{\mathbb{R}})$, where AP_p is the closure in $\mathcal{M}_p$ of the class AP_W of absolutely convergent trigonometrical series $\sum f_\lambda e^{i\lambda x}$ ($\lambda \in \mathbb{R}$) and $C_p(\overline{\mathbb{R}})$ is the closure of the set of continuous functions on $\overline{\mathbb{R}}$ of bounded variation.

Any MVF $G \in SAP$ can be represented in the form

$$G = G_+ u_+ + G_- u_- + G_0, \tag{8}$$

where MVF's $G_\pm \in AP$, MVF $G_0 \in C(\overline{\mathbb{R}})$ and $G_0(\pm\infty) = 0$, the functions $u_\pm = (1 \pm \tanh \pi x)/2$ and the mappings $G \to G_\pm$ are homomorphisms of the Banach algebra SAP onto AP. This fact remains correct if SAP, AP and $C(\overline{\mathbb{R}})$ are replaced by SAP_p, AP_p and $C_p(\overline{\mathbb{R}})$, respectively. According to [8] the MVF's $G_\pm$ in (8) will be called the almost periodic components (or local representatives) of G at $\pm\infty$.

With the use of limit operators technique (see, for example, [9]) we can prove the following.

THEOREM 4. *If a MVF $G \in SAP$ (respectively, $G \in SAP_p$) and the operator T_G (W_G) is Noetherian in the space $L_p^n(\mathbb{R})$ then the operators $T_{G_\pm}$ ($W_{G_\pm}$) with almost periodic components $G_\pm$ of the MVF G are invertible in $L_p^n(\mathbb{R})$.*

In the case $n = 1$ Theorem 4 was obtained earlier in [10], [11]. The proof of Theorem 4 in the case $n > 1$ is based on the following statements.

Let $\mathfrak{C}$ and $\mathfrak{G}$ be Banach algebras of SIO in $L_p^n(\mathbb{R})$ with matrix-valued coefficients of class SAP and AP, respectively. Let $\mathfrak{B}$ and $\mathfrak{D}$ be Banach algebras in $L_p^n(\mathbb{R})$ generated by operators $\operatorname{sgn}(\cdot)I$ and $\mathcal{F}^{-1}G\mathcal{F}$, where MVF's G belong to SAP_p and AP_p, respectively. It is clear that algebras $\mathfrak{C}$ and $\mathfrak{B}$ contain the ideal $\mathcal{L}o$ of compact operators in $L_p^n(\mathbb{R})$.

LEMMA 1. *The mappings $\nu_\pm : T \to T_\pm$ defined on generators GI (MVF's $G \in SAP$) and $S = P_+ - P_-$ of the algebra $\mathfrak{C}$ by the equalities*

$$\nu_\pm(GI) = G_\pm I, \quad \nu_\pm(S) = S$$

are continued to homomorphisms of the algebra $\mathfrak{C}$ onto the algebra $\mathfrak{S}$; moreover

$$(\forall T \in \mathfrak{C})\ \|T_\pm\| \leq |T| = \inf\{\|T + K\| : K \in \mathcal{L}o\}.$$

LEMMA 2. *The mappings $\mu_\pm : W \to W_\pm$ defined on generators* $\mathrm{sgn}(\cdot)I$ *and $\mathcal{F}^{-1}G\mathcal{F}$ (MVF's $G \in SAP_p$) of the algebra $\mathfrak{B}$ by the equalities*

$$\mu_\pm(\mathcal{F}^{-1}G\mathcal{F}) = \mathcal{F}^{-1}G_\pm\mathcal{F}, \quad \mu_\pm(\mathrm{sgn}(\cdot)I) = \mathrm{sgn}(\cdot)I$$

are continued to homomorphisms of the algebra $\mathfrak{B}$ onto the algebra $\mathfrak{D}$; moreover

$$(\forall W \in \mathfrak{B})\ \|W_\pm\| \leq |W|.$$

According to [9], the limits of any strongly convergent subsequences $B'_{h_n} T B'^{-1}_{h_n}$ and $e^{-ixh'_n} W e^{ixh'_n} I$, respectively, of sequences $B_{h_n} T B^{-1}_{h_n}$ and $e^{-ixh_n} W e^{ixh_n} I$, where $(B_h\varphi)(x) = \varphi(x+h)$ and $h_n \to \pm\infty$ as $n \to \pm\infty$, will be called limit operators for T and W.

THEOREM 5. *If the operator $T \in \mathfrak{C}$ ($W \in \mathfrak{B}$) is Noetherian, then limit operators $T_\pm = \nu_\pm(T) \in \mathfrak{S}$ ($W_\pm = \mu_\pm(W) \in \mathfrak{D}$) are invertible.*

Theorem 4 follows from Theorem 5 for $T = T_G$ and $W = W_G$.

COROLLARY 2. *For a MVF $G \in AP$ (respectively, $G \in AP_p$) the Noether property and the invertibility of the operator $T_G(W_G)$ in $L_p^n(\mathbb{R})$ are equivalent.*

It is clear that Lemma 1 and Theorems 4–5 are extended on algebras SIO's in $L_p^n(\Gamma)$ with coefficients having semi-almost periodic type discontinuities.

In the case $p = 2$ Theorem 4 admits the strengthening.

THEOREM 6. *If a MVF $G \in SAP$ and the operator T_G or W_G is n-normal (d-normal) in the space $L_2^n(\mathbb{R})$, then the operators $T_{G_\pm}$ and $W_{G_\pm}$ are left (right) invertible in $L_2^n(\mathbb{R})$.*

If $n = 1$, then Theorem 6 is valid for all $p \in (1, \infty)$ (see [10], [11]).

The investigation of the Noether property of the operators (6)–(7) is closely connected with the study of a special form of factorization of almost periodic MVF's $G_\pm$, which I. M. Spitkovskii factorization of almost periodic MVF's $G_\pm$, which I.M. Spitkovskii and the author have called P-factorization (see [8], [12]).

For a MVF $G \in AP$ let $\mathbb{M}(G)$ and $\Omega(G)$ denote its mean value and spectrum, respectively:

$$\mathbb{M}(G) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(x)\, dx, \quad \Omega(G) = \{\lambda \in \mathbb{R} : \mathbb{M}(e^{-i\lambda x} G(X)) \neq 0\}.$$

Let $AP^{\pm} = \{G \in AP : \Omega(G) \subset \mathbb{R}_{\pm}\}$ and $AP^{\pm}_W = AP^{\pm} \cap AP_W$.

DEFINITION 2. A P-factorization (correspondingly a P_W-factorization) of an $n \times n$ MVF G defined on $\mathbb{R}$ is defined to be a representation of it in the form

$$G = G^{+} \wedge G^{-}, \tag{9}$$

where $\wedge(X) = \operatorname{diag}[e^{i\lambda_1 x}, \ldots, e^{i\lambda_n x}]$, the numbers λ_j are real and MVF's $G^{\pm}$ $(G^{\pm})^{-1} \in AP^{\pm}$ (respectively, $AP^{\pm}_W$).

The set of the values λ_j is invariant for the P-factorable MVF G. We call these numbers the partial P-indices of the MVF G. If all partial P-indices are equal, then the matrix $\mathbb{M}(G^{+})\mathbb{M}(G^{-})$ is uniquely determined, and we denote it below by $d(G)$.

3. Let a MVF $G \in AP_W$.

THEOREM 7. *If a MVF $G \in AP_W$, then the following affirmations are equivalent:*

1) *the operator T_G is Noetherian in the space $L^n_p(\mathbb{R})$,*
2) *the operator T_G is invertible in $L^n_p(\mathbb{R})$,*
3) *the operator W_G is Noetherian in $L^n_p(\mathbb{R})$,*
4) *the operator W_G is invertible in $L^n_p(\mathbb{R})$,*
5) *the MVF G is P-factorable with zero partial P-indices,*
6) *the MVF G is P_W-factorable with zero partial P-indices.*

The implications 6)⇒2)⇒1) and 6)⇒4)⇒3) are obvious. The equivalence 5)⇔6) was established in [13] with the use of the results from [14]. The implications 1)⇒2) and 3)⇒4) were proved in Theorem 4. It remains to prove the implications 2)⇒6) and 4)⇒6), moreover only for $n > 1$, since in the case $n = 1$ they were established in [15], [16].

At first let $p = 2$. Then $W_G = \mathcal{F}^{-1} T_G \mathcal{F}$, where $\mathcal{F}$ is the isometry of $L^n_2(\mathbb{R})$ onto itself. Hence in the case $p = 2$ it is sufficient to consider only the operator T_G.

We denote by B_2 the Hilbert space of Besicovitch almost periodic functions, i.e. the supplement of the set of almost periodic polynomials $f(x) =$

$\sum f_\lambda e^{i\lambda x}$ by the norm

$$\|f\|_{B_2} = \left(\sum |f_\lambda|^2\right)^{1/2} = [\mathbb{M}(|f|^2)]^{1/2}.$$

As is known, the space B_2 is identified with the Hilbert space $L_2(\mathbb{R}_B)$ with respect to the Haar measure $d\mu$ on the Bohr compact $\mathbb{R}_B$. Parallel with B_2 we consider also Besicovitch spaces B_p $(1 \le p \le \infty)$ identified with Banach spaces $L_p(\mathbb{R}_B)$ with the Haar measure $d\mu$. In the Besicovitch space B_2 we introduce the orthoprojectors $\mathcal{P}_\pm$ and $\widehat{\mathcal{P}}_\pm$ defined by equalities

$$\mathcal{P}_- = I - \mathcal{P}_+, \quad \widehat{\mathcal{P}}_+ = I - \widehat{\mathcal{P}}_-, \quad \mathcal{P}_+(e^{i\lambda x}) = \chi(\lambda)e^{i\lambda x}, \quad \widehat{\mathcal{P}}_-(e^{i\lambda x}) = \chi(-\lambda)e^{i\lambda x},$$

where $\chi(\lambda) = 1$ for $\lambda \ge 0$ and $\chi(\lambda) = 0$ for $\lambda < 0$.

Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators acting in a Banach space X. In parallels with the C^*-algebra $\mathfrak{S} \subset \mathcal{L}(L_2^n(\mathbb{R}))$ generated by the operators $P_\pm$ and operators of multiplication by MVF's $G \in AP$ we consider the C^*-algebras $\mathcal{C}, \widehat{\mathcal{C}} \subset \mathcal{L}(B_2^n)$ generated, respectively, by operators $\mathcal{P}_\pm, GI$ and $\widehat{\mathcal{P}}_\pm, GI$, where MVF's $G \in AP$.

With the help of the theorem on an isomorphism of C^*-algebras of operators with shifts established by the author in [17] we can prove the following.

THEOREM 8. *The C^*-algebras $\mathfrak{S}, \mathcal{C}$ and $\widehat{\mathcal{C}}$ are isometrically isomorphic.*

COROLLARY 3. *If a MVF $G \in AP$, then the operators*

$$T_G = P_+ + GP_- \in \mathcal{L}(L_2^n(\mathbb{R})), \quad \mathcal{T}_G = \mathcal{P}_+ + G\mathcal{P}_- \in \mathcal{L}(B_2^n),$$
$$\widehat{\mathcal{T}}_G = \widehat{\mathcal{P}}_+ + G\widehat{\mathcal{P}}_- \in \mathcal{L}(B_2^n)$$

are invertible only simultaneously.

The following gives the implication 2)⇒6) in Theorem 7 in the case $p = 2$.

THEOREM 9. *If a MVF $G \in AP_W$ and the operator T_G is invertible in the space $L_2^n(\mathbb{R})$, then $G = G^+G^-$, where MVF's $G^\pm$, $(G^\pm)^{-1} \in AP_W^\pm$.*

Really, let I_n be the identity matrix of order n, let G' be the matrix transposed to G and $(\mathbb{C}\varphi)(X) = \overline{\varphi(X)}, X \in \mathbb{R}$. According to Corollary 3, the operators

$$\mathcal{T}_G = \mathcal{P}_+ + G\mathcal{P}_-, \quad \mathbb{C}\mathcal{T}_G^*\mathbb{C} = \widehat{\mathcal{P}}_- + \widehat{\mathcal{P}}_+ G'I$$

are invertible parallel with T_G. Hence, the operators

$$\mathcal{T}_{G'} = G^{-1}\mathcal{P}_+ + \mathcal{P}_-, \quad \mathcal{T}_{G''} = (G')^{-1}\mathcal{P}_- + \mathcal{P}_+$$

are invertible in the space B_2 of Besicovitch almost periodic $n \times n$ MVF's. Then the MVF's

$$\Phi^\pm = \mathcal{P}_\pm(\mathcal{T}_{G'})^{-1}I_n, \quad \Psi^\pm = \widehat{\mathcal{P}}_\pm(\mathcal{T}_{G''})^{-1}I_n,$$

respectively, belong to $B_2^{\pm} = \{A \in B_2 : \Omega(A) \subset \mathbb{R}_{\pm}\}$ and satisfy the relations

$$(10) \quad \Phi^+ = G(I_n - \Phi^-), \quad (\Psi^-)' = (I_n - \Psi^+)'G, \quad \mathbf{M}(\Phi^-) = \mathbf{M}(\Psi^+) = 0.$$

Since $G \in AP_W$, according to [14], the MVF's $\Phi^{\pm}, \Psi^{\pm} \in AP_W^{\pm}$. It follows from (10) that

$$(11) \quad C \stackrel{\text{def}}{=} (I_n - \Psi^+)'G(I_n - \Phi^-) = (I_n - \Psi^+)'\Phi^+ = (\Psi^-)'(I_n - \Phi^-).$$

Then $C \in AP_W^+ \cap AP_W^-$ and, consequently, C is a constant matrix. By analogy with Theorem 3.4 [18] we can prove that $\det C \neq 0$. It remains to set $G^+ = \Phi^+$, $G^- = C^{-1}(\Psi^-)'$ and to use (11).

In the case $p \neq 2$ the proof of the implications 2)⇒6) and 4)⇒6) is based on the following.

THEOREM 10. *If a MVF $G \in AP_W$, then the operator T_G (respectively, W_G) is invertible in all spaces $L_p^n(\mathbb{R})$, $1 < p < \infty$, only simultaneously.*

This theorem for the operator W_G follows from Corollary 2.2.12 in V. G. Kurbatov [19]. With a view to prove Theorem 10 for T_G we introduce the algebra $\mathfrak{S}_p$ $(1 < p < \infty)$ of operators $T = \sum A_\lambda e^{i\lambda x} I$ acting in the Banach space $L_p^n(\mathbb{R})$, where $A_\lambda = \mathcal{F}^{-1} a_\lambda \mathcal{F}$, $\mathcal{F}$ is the Fourier transformation, elements of MVF's $a_\lambda (\lambda \in \mathbb{R})$ belong to the algebra $\mathcal{M}_p$ of Fourier multipliers on $L_p(\mathbb{R})$ and

$$\|T\|_W = \sum \|A_\lambda\|_{\mathcal{L}(L_p^n(\mathbb{R}))} < \infty.$$

The algebra $\mathfrak{S}_p$ is Banach under this norm.

Let G be a discrete abelian group and let $K = K(G)$ be its character group (all characters are continuous). Let $\mathcal{L}$ denote a Banach algebra with identity and let $W(K, \mathcal{L})$ denote the subalgebra of the algebra $C(K, \mathcal{L})$ consisting of functions of the form

$$T(\kappa) = \sum \langle \kappa, \lambda \rangle b_\lambda, \quad \kappa \in K,$$

where $b_\lambda \in L$, $\langle \kappa, \lambda \rangle$ is the value of the character κ at the element $\lambda \in G$ and $\sum \|b_\lambda\| < \infty$.

In [20] S. Bochner and R. Phillips proved that the algebra $W(K, \mathcal{L})$ is full. With a help of this fact we can prove, according to [21], [19], the following.

THEOREM 11. *The algebra $\mathfrak{S}_p$ is full for every $p \in (1, \infty)$.*

COROLLARY 4. *If the operator $T \in \mathfrak{S}_p$ is invertible, then it is invertible in any space $L_r^n(\mathbb{R})$, where r belongs to the segment with endpoints p and $q = p/(p-1)$.*

Theorem 10 follows from Corollary 4 and Theorem 9. As a result, Theorem 7 is proved completely.

Remark 1. A P-factorization of a MVF $G \in AP_W$ is not a necessary condition of one-sided invertibility of the operator T_G in contrast to two-sided invertibility. In fact, it is sufficient to consider the left (right) invertible in the space $L_p^2(\mathbb{R})$ operator T_G with the MVF $G = e^{-i\lambda x}A$ ($G = e^{-\lambda x}A$), where the MVF

$$A(x) = \begin{bmatrix} e^{i(1+\alpha)x} & 0 \\ e^{-ix} - 1 + e^{i\alpha x} & e^{-i(1+\alpha)x} \end{bmatrix}$$

with $\alpha = (5^{1/2} - 1)/2$ does not have a P-factorization (see [8]).

For any MVF $G \in AP$ we denote

$$\|G\|_\infty = \|s(G(.))\|_{L_\infty(\mathbb{R})},$$

where $s(G(x))$ is the maximal singular number of $G(x)$.

Theorem 7 implies also the following.

Corollary 5. *If a MVF $G \in AP_W$ is P-factorable with coincident partial P-indices, then all MVF's of class AP_W from a sufficiently small by the norm $\|\cdot\|_\infty$ neighborhood of the MVF G are P_W-factorable with the same partial P-indices.*

4. Now we consider the operators T_G and W_G with semi-almost periodic MVF G having almost periodic components $G_\pm \in AP_W$ in the space $L_p^n(\mathbb{R}), 1 < p < \infty$.

If the operator T_G is Noetherian in $L_p^n(\mathbb{R})$, then according to Theorem 4, the operators $T_{G_\pm}$ are invertible in $L_p^n(\mathbb{R})$. Then, in view of conditions $G_\pm \in AP_W$ and Theorem 7, the Noether property of T_G implies P_W-factorability of MVF's $G_\pm$ with zero partial P-indices. A similar fact is correct for W_G too. Hence from Theorem 3.2 in [8] and Theorem 2 in [12] we can obtain the following.

Theorem 12. *If G is an $n \times n$ MVF of class SAP (respectively, SAP_p) and its local representatives $G_\pm \in AP_W$, then the operator $T_G(W_G)$ is Noetherian in the space $L_p^n(\mathbb{R})$ if and only if the following conditions hold:*

1) $\det G(x) \neq 0$ *for all* $X \in \mathbb{R}$,

2) *MVF's $G_\pm$ are P-factorable (equivalently, P_W-factorable) with zero partial P-indices,*

3) *the eigenvalues ξ_j $(j = 1, \ldots, n)$ of the matrix $d(G_-)^{-1}d(G_+)$ satisfy the inequalities*

(12) $\gamma_j \overset{\text{def}}{=} \{p^{-1} - (2\pi)^{-1} \arg \xi_j) \neq 0\}$ $\quad(\widehat{\gamma}_j \overset{\text{def}}{=} \{q^{-1} - (2\pi)^{-1} \arg \xi_j\} \neq 0)$,

where $\{x\}$ *denotes the fractional part of a number* $X \in \mathbb{R}$ *and* $q = p/(p-1)$. *If conditions* 1) – 3) *are satisfied, then*

$$\text{ind}\, T_G = \text{Ind}\det G - n/p + \sum_1^n \gamma_j \tag{13}$$

$$\Big(\text{ind}\, W_G = \text{Ind}\det G - n/q + \sum_1^n \widehat{\gamma}_j\Big),$$

where

$$\text{Ind}\det G = \frac{1}{2\pi}\lim_{t\to+\infty}\frac{1}{t}\int_{-t}^{t} \arg\det G(x)\,\text{sgn}\, x\, dx.$$

Remark 2. Theorem 12 remains true for T_G acting in the weight space

$$L^n_{p,\beta} = \{f : (1+|x|^\beta) f \in L^n_p(\mathbb{R})\}, \quad 1 < p < \infty,\ -p^{-1} < \beta < 1 - p^{-1},$$

if in relations (12)–(13) p^{-1} is replaced by $p^{-1} + \beta$.

5. Now we get criteria for T_G with any $G \in SAP$ to be Noetherian or $n(d)$-normal in the space $L^n_2(\mathbb{R})$. As a result we get rid of the condition $G_\pm \in AP_W$. For this it is necessary to generalize a concept of P-factorability for MVF's $G \in AP$.

We consider the Banach spaces B_p $(p = 1, 2, \infty)$ of $n\times n$ MVF's A with elements belonging to $L_p(\mathbb{R}_B)$ and norms

$$\|A\|_1 = \int_{\mathbb{R}_B} \text{tr}(A(t)A^*(t))^{1/2}\, d\mu,$$

$$\|A\|_2 = \Big(\int_{\mathbb{R}_B} \text{tr}(A(t)A^*(t))\, d\mu\Big)^{1/2},$$

$$\|A\|_\infty = \|s(A(.))\|_{L_\infty(\mathbb{R}_B)},$$

respectively, where tr is a matrix trace, $s(\cdot)$ is a maximal singular number and $d\mu$ is the Haar measure on $\mathbb{R}_B$. Let $B^\pm_p = \{A \in B_p : \Omega(A) \subset \mathbb{R}_\pm\}$.

Definition 3. We call a representation (9) a generalized P-factorization of an $n\times n$ MVF $G \in AP$, if Λ has a previous form, MVF's $G^\pm, (G^\pm)^{-1} \in B^\pm_2$ and the operators $G^+\mathcal{P}_+(G^+)^{-1}I$ and $(G^{-1})^{-1}\Lambda^{-1}\mathcal{P}_-(G^+)^{-1}I$ are bounded in the space B^n_2.

Similarly to Theorem 2.1 in [8] we obtain the following

Theorem 13. *Let a MVF* $G \in AP$ *parallel with the* P*-factorization* (9), *in which* $\lambda_1 \geq \ldots \geq \lambda_n$, *admits a representation* $G = F^+MF^-$, *where* $M(x) = \text{diag}[e^{i\mu_j x}]^n_{j=1}$, $\mu_j \in \mathbb{R}$, $\mu_1 \geq \ldots \geq \mu_n$ *and MVF's* $F^\pm$, $(F^\pm)^{-1} \in$

$B_2^{\pm}$. *Then* $\mu_j = \lambda_j$ $(j = 1, \ldots, n)$ *and the factors* $F^{\pm}$ *are determined by the formulas*

$$F^{+} = G^{+}z, \quad F^{-} = \Lambda^{-1}z^{-1}\Lambda G^{-1},$$

where a MVF $z = (z_{kj})_{k,j=1}^{n} \in B_1^{+}$ *together with* $z^{-1}, z_{kj} = 0$ *if* $\lambda_k < \lambda_j$ *and* $\Omega(z_{kj}) \subset [0, \lambda_k - \lambda_j]$ *if* $\lambda_k \geq \lambda_j$.

The numbers λ_j uniquely to within transpositions determined by a generalized P-factorable MVF $G \in AP$ are also called the partial P-indices of G.

COROLLARY 6. *If all the partial P-indices of a generalized P-factorable MVF* $G \in AP$ *are equal, then a generalized P-factorization is determined to within the transformation* $G^{+} \to G^{+}z, G^{-} \to z^{-1}G^{-}$, *where* z *is a constant nonsingular matrix. In this case the nonsingular matrix* $d(G) = \mathbb{M}(G^{+})\mathbb{M}(G^{-})$ *is uniquely determined.*

LEMMA 3. *If* G *is a generalized P-factorable MVF of class AP and all* $\lambda_j = 0$ $(\leq 0, \geq 0)$, *then the operator* T_G *is invertible (letf-invertible, right-invertible) in the space* $L_2^n(\mathbb{R})$.

Let $\mathcal{P}$ denote the open with respect to the norm $\|\cdot\|_\infty$ set of all MVF's $G \in AP$ such that the operators T_G are invertible in $L_2^n(\mathbb{R})$. According to Corollary 3, the operators $\mathcal{T}_G$ and $\widehat{\mathcal{T}}_G$ are invertible in B_2^n for $G \in \mathcal{P}$. Let I_n be the identity $n \times n$ matrix in contrast to the identity in B_2^n operator I.

LEMMA 4. *The mappings* $G \to \mathbb{M}(\mathcal{T}_G^{-1}G)$ *and* $G \to \mathbb{M}(\widehat{\mathcal{T}}_G^{-1}I_n)$ *are continuous at* $\mathcal{P}$.

LEMMA 5. *Matrices* $\mathbb{M}(\mathcal{T}_G^{-1}G)$ *and* $\mathbb{M}(\widehat{\mathcal{T}}_G^{-1}I_n)$ *are nonsingular for all* $G \in \mathcal{P}$.

THEOREM 14. *If a MVF* $G \in AP$, *then the operator* T_G *is invertible in the space* $L_2^n(\mathbb{R})$ *if and only if* G *is the generalized P-factorable MVF with zero partial P-indices.*

The sufficiency was proved in Lemma 3. The necessity is proved similarly to Theorem 9 and ([15], pp. 273–274) with the use of Lemmas 4–5 and the relations

$$C = d(G) = \mathbb{M}(\mathcal{T}_G^{-1}G), \quad C^{-1} = [d(G)]^{-1} = \mathbb{M}(\widehat{\mathcal{T}}_G^{-1}I_n),$$

where C is defined by (11).

With a help of Theorems 13–14 we can prove the following.

THEOREM 15. *If* G *is a Hermitian-positive MVF of class AP and* $G^{-1} \in AP$, *then* $G = G^{+}(G^{+})^{*}$, *where* $(G^{+})^{\pm 1} \in B_\infty^{+}$.

Theorem 13 implies also the following.

COROLLARY 7. *If a MVF $G \in AP$ is generalized P-factorable with zero partial P-indices, then all MVF's of class AP in a sufficiently small neighborhood of G are also generalized P-factorable with zero partial P-indices.*

From Lemmas 4–5 we can deduce also the following.

COROLLARY 8. *The function $d : G \to d(G)$ defined by the formulas*

$$d(G) = \mathbf{M}(G^+)\mathbf{M}(G^-) = \mathbf{M}(\mathcal{T}_G^{-1}G)$$

continuously maps the open set $\mathcal{P}$ onto the set of nonsingular numerical $n \times n$ matrices.

Remark 3. The invertibility of a MVF G in AP doesn't guarantee both the P-factorability and the generalized P-factorability of G.

THEOREM 16. *If an $n \times n$ MVF $G \in SAP$, then the operator T_G is Noetherian in the space $L^n_{2,\beta}$ ($|\beta| < 2^{-1}$) if and only if the following conditions hold:*

1) $\det G(x) \neq 0$ *for all* $x \in \mathbb{R}$,
2) *the MVF's* $G_\pm$ *are generalized P-factorable with zero partial P-indices,*
3) *the eigenvalues* ξ_j *of the matrix* $d(G_-)^{-1}d(G_+)$ *satisfy the inequalities*

$$\gamma \stackrel{\text{def}}{=} \{2^{-1} + \beta - (2\pi)^{-1}\arg\xi_j\} \neq 0 \quad (j = 1, \ldots, n).$$

If conditions 1)–3) *are fulfilled, then the index of T_G can be computed from the formula* (13) *with p^{-1} replaced by $2^{-1} + \beta$.*

LEMMA 6. *Let a MVF $G \in L_\infty(\mathbb{R})$. Then the operator T_G is n-normal (d-normal) in the space $L^n_2(\mathbb{R})$ if and only if the operator T_{G_n} (respectively, T_{G_d}) is Noetherian in the space $L^{2n}_2(\mathbb{R})$, where*

$$G_n = \begin{bmatrix} G^* & 0 \\ I_n + GG^* & G \end{bmatrix}, \quad G_d = \begin{bmatrix} G & 0 \\ I_n + G^*G & G^* \end{bmatrix}.$$

From Lemma 6 and Theorem 16 we deduce the following.

THEOREM 17. *If a MVF $G \in SAP$, then the operator T_G is n-normal (d-normal) in the space $L^n_{2,\beta}$ if and only if:*

1) $\det G(x) \neq 0$ *for all* $x \in \mathbb{R}$,

2) *the MVF's* $(G_n)_\pm$ *(respectively,* $(G_d)_\pm$*) are generalized P-factorable with zero partial P-indices,*

3) *the eigenvalues* ξ_j *of the matrix* $d[(G_n)_-]^{-1}Bd[(G_n)_+]B$ *(respectively,* $d[(G_d)_-]^{-1}B^{-1}d[(G_d)_+]B^{-1}$*) satisfy the relations*

$$2^{-1} - (2\pi)^{-1}\arg\xi_j \,\overline{\in}\, \mathbb{Z} \quad (j = 1, \ldots, 2n),$$

where $B = \operatorname{diag}[e^{\pi i\beta}I_n, e^{-\pi i\beta}I_n]$.

6. Let a contour Γ consist of simple open oriented smooth arcs Γ_k $(k = 1, \dots, k_0)$ intersecting only at their endpoints and not forming zero angles, $\mathcal{T}$ is the set of nodes of the contour Γ, α is an orientation-preserving diffeomorphism of each arc Γ_k onto itself.

In the space $L_p^n(\Gamma, \varrho)$, $\varrho(t) = \prod_{\tau \in \mathcal{T}} |t - \tau|^{\beta_\tau}$, $-p^{-1} < \beta_\tau < 1 - p^{-1}$, $1 < p < \infty$, we consider the operator (1), where G is a continuous on $\Gamma \setminus \tau$ MVF of order n having at nodes discontinuities of semi-almost periodic type, i.e. there exist such uniform almost periodic MVF's $G_{k,\pm}$ and such orientation-preserving diffeomorphisms γ_k of real axis $\mathbb{R}$ onto $\Gamma_k \setminus \mathcal{T}$ with finite non-zero limits $\lim_{x \to \pm\infty} x^2 \gamma_k'(x)$ that

$$\lim_{x \to \pm\infty} [G(\gamma_k(x)) - G_{k,\pm}(x)] = 0.$$

We assume that MVF's $G_{k,\pm} \in AP_W$ if $p \neq 2$.

LEMMA 7. *If the operator* (1) *is Noetherian in* $L_p^n(\Gamma, \varrho)$, *then for* $k = 1, \dots, k_0$ *the MVF's* $G_{k,\pm}$ *are* P_W*-factorable* (*generalized* P*-factorable if* $p = 2$) *with zero partial* P*-indices.*

To each node τ of the contour Γ (being for example a common end of arcs $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ numbered as a result of a circuit around τ in a counter-clockwise direction) we can associate the eigenvalues $\lambda_j(\tau)$ $(j = 1, \dots, n)$ of the matrix $[d(G_m)]^{\vartheta_m}[d(G_{m-1})]^{\vartheta_{m-1}} \dots [d(G_1)]^{\vartheta_1}$ and the number

$$\xi(\tau) = (2\pi)^{-1} \sum_{k=1}^{m} \vartheta_k \ln\left(\lim_{t \to \tau, t \in \Gamma_k} |\alpha'(t)|\right),$$

where $G_k = G_{k,-}$ and $\vartheta_k = 1$ (respectively, $G_k = G_{k,+}$ and $\vartheta_k = -1$) if τ is the origin (end) of the arc Γ_k.

THEOREM 18. *If an* $n \times n$ *MVF* G *is continuous on* $\Gamma \setminus \mathcal{T}$ *and admits discontinuities of semi-almost periodic type at nodes of the contour* Γ *and also the MVF's* $G_{k,\pm} \in AP_W$ *for all* $p \in (1, \infty) \setminus \{2\}$, *then the operator* (*1*) *is Noetherian in the space* $L_p^n(\Gamma, \varrho)$ *if and only if the following conditions hold*:

1) $\det G(t) \neq 0$ *for all* $t \in \Gamma \setminus \mathcal{T}$,

2) *for* $k = 1, \dots, k_0$ *the MVF's* $G_{k,\pm}$ *are generalized* P*-factorable with zero partial* P*-indices if* $p = 2$ *and* P_W*-factorable with zero partial* P*-indices for other* $p \in (1, \infty)$,

3) *for all* $\tau \in \mathcal{T}$ *and* $j = 1, \dots, n$ *the numbers*

$$\varphi_j(\tau) = (p^{-1} + \beta_\tau)[\xi^2(\tau) + 1] + (2\pi)^{-1}[\xi(\tau) \ln |\lambda_j(\tau)| - \arg \lambda_j(\tau)]$$

are not integers.

If these conditions are satisfied, then

$$\operatorname{ind} N = (2\pi)^{-1} \sum_{k=1}^{k_0} \lim_{t\to+\infty} t^{-1} \int_{-t}^{t} \arg\det G(\gamma_k(x)) \operatorname{sgn} x \, dx$$

$$+ \sum_{\tau\in T} \sum_{j=1}^{n} [E(\varphi_j(\tau)) + (2\pi)^{-1} \arg \lambda_j(\tau)],$$

where $\arg f$ *is an arbitrary continuous branch of the argument of* f *on* $\mathbb{R}$.

Remark 4. The results of sections 2-6 were partly announced in [22], [23], their systematic presentation is contained in [3].

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