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GENERALIZED INITIAL VALUE PROBLEM FOR FOURTH ORDER EQUATION OF ELLIPTIC TYPE

Dedicated to the memory of Professor WITOLD POGORZELSKI

1. Introduction

Initial value problems for elliptic partial differential equations occur in many problems of mathematical physics for instance in geology, elasticity, aerodynamics. Problems of such a type were studied by many authors, among others A. K. Aziz and R. P. Gilbert [1], [2], D. L. Colton [3], [4] for the second order elliptic equations in two independent variables. In [5] D. L. Colton was studying the Cauchy problem for a class of fourth-order elliptic equations. J. Conlan and R. P. Gilbert were investigating the equation of the type [6]

$$\Delta\Delta u = f(x, y, u, u_x, u_y, u_{xx}, u_{yy}, \Delta u_x, \Delta u_y) \tag{1}$$

where f was analytic, and subject to the following initial conditions

$$a_1^{(k)}\Delta u_x + a_2^{(k)}\Delta u_y + a_3^{(k)}u_{xx} + a_4^{(k)}u_{yy} + a_5^{(k)}u_{xy} + a_6^{(k)}u_x + a_7^{(k)}u_y + a_8^{(k)}u + a_9^{(k)} = 0\,. \tag{2}$$

In conditions (2) $a_i^{(k)} = a_i^{(k)}(x)$ for $k = 1, 3$, $i = 1, \dots, 9$ on $y = f_k(x)$, and $a_i^{(k)} = a_i^{(k)}(y)$ for $k = 2, 4$, $i = 1, \dots, 9$ on $x = f_k(y)$, the data being analytic.

R. P. Gilbert and Wei Lin [7] were studying the equation

$$\Delta\Delta u + au_{xx} - 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \tag{3}$$

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known in the theory of elasticity, giving an interesting representation for the solution of the Goursat problem.

In [9] there was studied some initial value problem for equation (1) in a special form i.e. the equation

$$\nu\Delta\Delta U = \frac{\partial U}{\partial y}\frac{\partial \Delta U}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial \Delta U}{\partial y} \tag{4}$$

known in the theory of the plane motion of the viscous fluid. In the equation (4) the coefficient of viscosity ν is constant and the unknown function U, called the stream function, is defined by the equalities $v_x = \frac{\partial U}{\partial y}$, $v_y = -\frac{\partial U}{\partial x}$, where v_x and v_y denote the components of velocity vector. The conditions (2) were simplified, namely the coefficients $a_i^{(k)}$ were admitted to be equal to zero for $k = 1, \dots, 4$, $i = 1,2,3,4,5$, and for $k = 1, \dots, 4$, $i = 6,7,8,9$ to be constant. Besides the initial curves were defined by the functions being constants.

In this paper we intend to study the equation (4), subject to the following initial conditions

$$u_{xx}^{(k)} + u_{yy}^{(k)} + a_6^{(k)}u_x + a_7^{(k)}u_y + a_8^{(k)}u + a_9^{(k)} = 0 \tag{5}$$

where

$$a_i^{(k)} = a_i^{(k)}(x), \quad i = 6,7,8,9,\ k = 1,3, \quad a_i^{(k)} = a_i^{(k)}(y), \quad i = 6,7,8,9, \quad k = 2,4,\ f_k = \text{const}, \quad k = 1,2,3,4. \tag{6}$$

2. Existence and uniqueness of solution

Following the approach used in [8], [6], [5], [9] we introduce the two complex variables $z = x + iy$, $z^* = x - iy$, which admit the conjugate values if x and y are real. Thus equation (4) is replaced by a complex hyperbolic equation of fourth order, namely

$$\frac{\partial^4\varphi}{\partial z^2\partial z^{*2}} - \frac{i}{2\nu}\frac{\partial\varphi}{\partial z}\frac{\partial^3\varphi}{\partial z\partial z^{*2}} + \frac{i}{2\nu}\frac{\partial\varphi}{\partial z^*}\frac{\partial^3\varphi}{\partial z^2\partial z^*} = 0 \tag{7}$$

where

$$\varphi(z,z^*) = U\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right). \tag{8}$$

The differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

have the property

$$\frac{\partial^p}{\partial z^p}\left(\frac{\partial^q}{\partial z^{*q}}\right) = \frac{\partial^q}{\partial z^{*q}}\left(\frac{\partial^p}{\partial z^p}\right), \quad p,q = 0,1,2,\dots$$

After suitable transformations (cf. [9]) we can state that the function holomorphic φ satisfying the equation (7) must also satisfy the equation

$$(9)\quad \varphi(z,z^*) - \gamma \int_0^z (z-t)\left[\frac{\partial\varphi(t,z^*)}{\partial t}\right]^2 dt + \gamma \int_0^{z^*} (z^*-\tau)\left[\frac{\partial\varphi(z,\tau)}{\partial \tau}\right]^2 d\tau = \phi(z,z^*),$$

where $\gamma = \frac{i}{4\nu}$ and $\phi(z,z^*)$ is the holomorphic function being the solution of the equation $\frac{\partial^4\phi}{\partial z^2\partial z^{*2}} = 0$ i.e.

$$(10)\qquad \phi(z,z^*) = z\phi_1^*(z^*) + z^*\phi_1(z) + \phi_2^*(z^*) + \phi_2(z).$$

The arbitrary functions ϕ_1, ϕ_2, ϕ_1^*, ϕ_2^* are holomorphic in the cylindrical domain (D,D^*) for $z \in D$, $z^* \in D^*$ and $\phi_m^*(0) = \phi_m(0)$, $m = 1,2$.

According to I. N. Vekua [8], formula (32.9), we can represent the arbitraty holomorphic function ϕ in the form

$$(11)\quad \phi(z,z^*) = a_0 + a_1 zz^* + \int_0^{z^*} \psi_0^*(\tau)\,d\tau + \int_0^z \psi_0(t)\,dt + z\int_0^{z^*} (z^*-\tau)\psi_1^*(\tau)\,d\tau + z^*\int_0^z (z-t)\psi_1(t)\,dt,$$

where a_0, a_1 are constants and ψ_0, ψ_0^*, ψ_1, ψ_1^* are holomorphic functions in D and D^* respectively, arbitrarily chosen, satisfying the condition $\psi_m(0) = \psi_m^*(0)$, $(m = 0,1)$. The initial conditions (5) take the form:

$$(12)\quad \begin{aligned}
&\alpha_1(z)\frac{\partial\varphi(z,z^*)}{\partial z^*} + \beta_1(z)\varphi(z,z^*) = f_0(z) && \text{on } z^* = 0,\\
&\alpha_2(z^*)\frac{\partial\varphi(z,z^*)}{\partial z} + \beta_2(z^*)\varphi(z,z^*) = f_0^*(z^*) && \text{on } z = 0,\\
&\frac{\partial^2\varphi(z,z^*)}{\partial z\partial z^*} = f_1(z) && \text{on } z^* = 0,\\
&\frac{\partial^2\varphi(z,z^*)}{\partial z\partial z^*} = f_1^*(z^*) && \text{on } z = 0,
\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, f_0, f_0^*, f_1, f_1^*$ are given functions and $\beta_1(z) \neq 0$, $z \in D$, $\beta_2(z^*) \neq 0$, $z^* \in D^*$, $f_k^{(m)}(0) = f_m^{*(k)}(0)$, $k,m = 0,1$. Taking into account the equation (9) and the conditions (12) we come to the equalities

$$\begin{aligned}
&\alpha_1(z)\Big[2\gamma \int_0^z (z-t)u(t,0)w(t,0)\,dt + a_1 z + \int_0^z (z-t)\psi_1(t)\,dt \\
&\qquad + \beta_1(z)\Big[\gamma \int_0^z (z-t)[u(t,0)]^2\,dt + a_0 + \int_0^z \psi_0(t)\,dt = f_0(z), \\
&\alpha_2(z^*)\Big[-2\gamma \int_0^{z^*} (z^*-\tau)v(0,\tau)w(0,\tau)\,d\tau + a_1 z^* + \int_0^{z^*} (z^*-\tau)\psi_0^*(\tau)\,d\tau \\
(13)\quad &\quad + \beta_2(z^*)\Big[-\gamma \int_0^{z^*} (z^*-\tau)[v(0,\tau)]^2\,d\tau + a_0 + \int_0^{z^*} \psi_0^*(\tau)\,d\tau\Big] = f_0^*(z^*), \\
&\qquad 2\gamma \int_0^z u(t,0)w(t,0)\,dt + a_1 + \int_0^{z^*} \psi_1(t)\,dt = f_1(z) \\
&\qquad -2\gamma \int_0^{z^*} v(0,\tau)w(0,\tau)\,d\tau + a_1 + \int_0^{z^*} \psi_1^*(\tau)\,d\tau = f_1^*(z^*)
\end{aligned}$$

where we have denoted

$$(14)\qquad \begin{aligned}
&\frac{\partial\varphi(z,z^*)}{\partial z} = u(z,z^*), \quad \frac{\partial\varphi(z,z^*)}{\partial z^*} = v(z,z^*), \\
&\frac{\partial^2\varphi(z,z^*)}{\partial z\partial z^*} = \frac{\partial v(z,z^*)}{\partial z} = \frac{\partial u(z,z^*)}{\partial z^*} = w(z,z^*).
\end{aligned}$$

Thus the initial conditions are satisfied if the arbitrary holomorphic function (11) is represented in the form

$$(15)\qquad \begin{aligned}
\phi(z,z^*) = \varphi_0(z,z^*) &+ \gamma \int_0^{z^*} (z^*-\tau)[v(0,\tau)]^2\,d\tau \\
&- \gamma \int_0^z (z-t)[u(t,0)]^2\,dt \\
&+ 2\gamma z \int_0^{z^*} (z^*-\tau)v(0,\tau)w(0,\tau)\,d\tau \\
&- 2\gamma z \int_0^z (z-t)u(t,0)w(t,0)\,dt
\end{aligned}$$

where

$$(16)\qquad \begin{aligned}
\varphi_0(z,z^*) = &-f_0(0) - f_1(0)zz^* + \frac{1}{\beta_2(z^*)}f_0^*(z^*) + \frac{1}{\beta_1(z)}f_0(z) \\
&- \frac{\alpha_2(z^*)}{\beta_2(z^*)}\int_0^{z^*} f_1^*(\tau)\,d\tau - \frac{\alpha_1(z)}{\beta_1(z)}\int_0^z f_1(t)\,dt +
\end{aligned}$$

$$+ z^* \int_0^z f_1(t)\,dt + z \int_0^{z^*} f_1^*(\tau)\,d\tau\,.$$

Let us further denote

$$u_0(z,z^*) = \frac{\partial\varphi_0(z,z^*)}{\partial z}, \quad v_0(z,z^*) = \frac{\partial\varphi_0(z,z^*)}{\partial z^*}, \tag{17}$$
$$w_0(z,z^*) = \frac{\partial u_0(z,z^*)}{\partial z^*} = \frac{\partial v_0(z,z^*)}{\partial z}$$

and consider the auxiliary system for the fourtuple $F[\varphi, z, v, w]$

$$\varphi(z,z^*) = \varphi_0(z,z^*) + \gamma \int_0^z (z-t)[u(t,z^*)]^2\,dt - \gamma \int_0^z (z-t)[u(t,0)]^2\,dt$$
$$- \gamma \int_0^{z^*} (z^*-\tau)[v(z,\tau)]^2\,d\tau + \gamma \int_0^{z^*} (z^*-\tau)[v(0,\tau)]^2\,d\tau$$
$$+ 2\gamma z \int_0^{z^*} (z^*-\tau)v(0,\tau)w(0,\tau)\,d\tau$$
$$- 2\gamma z^* \int_0^z (z-t)u(t,0)w(t,0)\,dt \equiv T_1F(z,z^*)\,,$$

$$u(z,z^*) = u_0(z,z^*) + \gamma \int_0^z [u(t,z^*)]^2\,dt - \gamma \int_0^z [u(t,0)]^2\,dt$$
$$+ 2\gamma \int_0^{z^*} (z^*-\tau)v(0,\tau)w(0,\tau)\,d\tau \tag{18}$$
$$- 2\gamma \int_0^{z^*} (z^*-\tau)v(z,\tau)w(z,\tau)\,d\tau - 2\gamma \int_0^z z^*u(t,0)w(t,0)dt$$
$$\equiv T_2F(z,z^*)\,,$$

$$v(z,z^*) = v_0(z,z^*) + 2\gamma \int_0^z (z-t)u(t,z^*)w(t,z^*)\,dt$$
$$- 2\gamma \int_0^z (z-t)u(t,0)w(t,0)dt - \gamma \int_0^{z^*} [v(z,\tau)]^2\,d\tau$$
$$+ \gamma \int_0^{z^*} [v(0,\tau)]^2\,d\tau + 2\gamma \int_0^{z^*} zv(0,\tau)w(0,\tau)\,d\tau \equiv T_3F(z,z^*)\,,$$

$$w(z,z^*) = w_0(z,z^*) + 2\gamma \int_0^z u(t,z^*)w(t,z^*)\,dt - 2\gamma \int_0^{z^*} v(z,\tau)w(z,\tau)\,d\tau$$

$$+ 2\gamma \int_0^{z^*} v(0,\tau)w(0,\tau)\,d\tau - 2\gamma \int_0^z u(t,0)w(t,0)\,dt \equiv T_4F(t,t^*)\,.$$

Denoting

(19) $$TF(z,z^*) = [T_1F(z,z^*),\ T_2F(z,z^*),\ T_3F(z,z^*),\ T_4F(z,z^*)]$$

we represent the system (18) in the form

(20) $$F = TF\,.$$

In order to show that the system (20) has a unique solution in a linear space X of fourtuples of bounded holomorphic functions over a fixed neighbourhood of the origin we define the norm

(21) $$\|F\|_\lambda = \max\{\|\varphi\|_\lambda,\ \|u\|_\lambda,\ \|v\|_\lambda,\ \|w\|_\lambda\}, \qquad F \in X\,,$$

where

(22) $$\|\varphi\|_\lambda = \sup e^{-\lambda(|z|+|z^*|)}|\varphi(z,z^*)| : (z,z^*) \in (\widetilde{D},\widetilde{D}^*)\,, \quad \lambda > 0$$

and

$$\widetilde{D} = \{z : |z| < \varrho\}, \qquad D^* = \{z^* : |z^*| < \varrho\}\,.$$

The norms $\|u\|_\lambda$, $\|v\|_\lambda$, $\|w\|_\lambda$ are defined in the similar way.

We denote this Banach space by B.

Let $F_1, F_2 \in B$ and consider the difference $TF_1 - TF_2$.

The right hand sides of equations (18) involve integrals of the functions u, v, w.

As example let us study

(23) $$\left\|\gamma \int_0^z (z-t)[u_1(t,z^*)]^2\,dt - \gamma \int_0^z (z-t)[u_2(t,z^*)]^2\,dt\right\|_\lambda$$

$$= |\gamma| \sup e^{-\lambda(|z|+|z^*|)}\left|\int_0^z [u_1(t,z^*)-u_2(t,z^*)][u_1(t,z^*)+u_2(t,z^*)](z-t)dt\right|\Big\}$$

$$\le |\gamma| \sup e^{-\lambda(|z|+|z^*|)}\|u_1 - u_2\|_\lambda 2M \int_0^{|z|} (|z|-t)e^{2\lambda(t+|z^*|)}\,dt$$

$$\le \frac{|\gamma|M}{2\lambda^2}e^{\lambda(|z|+|z^*|)}\|u_1 - u_2\|_\lambda \le \frac{|\gamma|M}{2\lambda^2}e^{\lambda(|z|+|z^*|)}\|F_1 - F_2\|_\lambda\,.$$

Evaluating the other integrals in the same manner we get

$$\|TF_1 - TF_2\|_\lambda \le 2M|\gamma|e^{2\lambda\varrho}\max\left\{\frac{4}{\lambda},\ \frac{1}{\lambda^2}+\frac{1}{\lambda}+\frac{\varrho}{\lambda}\right\}\|F_1 - F_2\|_\lambda . \tag{24}$$

Now for a fixed $\lambda > 0$, we can find $\varrho > 0$ such that if $|z| < \varrho$, $|z^*| < \varrho$ then

$$2M|\gamma|e^{2\lambda\varrho}\max\left\{\frac{4}{\lambda},\ \frac{1}{\lambda^2}+\frac{1}{\lambda}+\frac{\varrho}{\lambda}\right\} < 1 . \tag{25}$$

Thus the mapping T has a fixed point — the functions of the fourtuple F being holomorphic and bounded in $(\widetilde{D}, \widetilde{D}^*)$.

Denote the fixed point by $\widehat{F} = [\widehat{\varphi}, \widehat{u}, \widehat{v}, \widehat{w}]$. The equations (18) are satisfied by $\widehat{F}$, i.e.

$$\widehat{\varphi} = T_1\widehat{F}, \quad \widehat{u} = T_2\widehat{F}, \quad \widehat{v} = T_3\widehat{F}, \quad \widehat{w} = T_4\widehat{F} . \tag{26}$$

Through a simple computation we conclude that

$$\widehat{u} = \frac{\partial\widehat{\varphi}}{\partial z}, \quad \widehat{v} = \frac{\partial\widehat{\varphi}}{\partial z}, \quad \widehat{w} = \frac{\partial\widehat{u}}{\partial z^*} = \frac{\partial\widehat{v}}{\partial z} . \tag{27}$$

and that

$$\begin{cases} \alpha_1(z)\widehat{v}(z,z^*) + \beta_1(z)\widehat{\varphi}(z,z^*) = f_0(z) & \text{on } z^* = 0 , \\ \alpha_2(z^*)\widehat{u}(z,z^*) + \beta_2(z^*)\widehat{\varphi}(z,z^*) = f_0^*(z^*) & \text{on } z = 0 , \\ \widehat{w}(z,z^*) = f_1(z) & \text{on } z^* = 0 , \\ \widehat{w}(z,z^*) = f_1^*(z^*) & \text{on } z = 0 , \end{cases} \tag{28}$$

the functions φ_0, u_0, v_0, w_0 in (18) being given by (13), (14). The unique solution of the mapping T (20) is satisfying the initial conditions (12) and the equation (9), which is the complex form of the equation (4). Thus we can state that there exists the unique solution of the problem (4), (5).

References

[1] A. K. Aziz and R. P. Gilbert, *A generalized Goursat problem for elliptic equations*, J. Reine Angewandte Math. 222 (1966), 1–13.

[2] A. K. Aziz, R. P. Gilbert and H. C. Howard, *A second order nonlinear elliptic boundary value problem with generalized Goursat data*, Ann. Math. Pura Appl. IV 72 (1966), 325–341.

[3] D. L. Colton, *Cauchy problem for almost linear elliptic equations in two independent variables*, J. Approx. Theory 3 (1970), 66–71.

[4] D. L. Colton, *Cauchy's problem for almost linear elliptic equations in two independent variables*, J. Approx. Theory 4 (1971), 288–294.

[5] D. L. Colton, *Cauchy's problem for a class of fourth order elliptic equations in two independent variables*, Applicable Anal. 1 (1971), 13–22.

[6] J. Conlan and R. P. Gilbert, *Initial value problems for fourth order semi-linear elliptic equations*, Applicable Anal. 3 (1974), 321–331.

[7] R. P. Gilbert and Wei Lin, *Constructive methods for IV-th order elliptic equations*, J. Approx. Theory 44 (1985), 55–68.
[8] I. N. Vekua, *New Methods for solving Elliptic Equations*, North Holland, Amsterdam, 1967.
[9] J. Wolska-Bochenek, *Initial value problem for fourth order equation in the theory of viscous fluid*, Complex Variables Theory Appl. 13 (1990), no 3–4, 231–237.

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