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ON PERIODIC SOLUTIONS OF EQUATIONS WITH RIGHT INVERTIBLE OPERATORS INDUCED BY FUNCTIONAL SHIFTS

Dedicated to the memory of Professor WITOLD POGORZELSKI

Real shifts induced by right invertible operators were studied by D. Przeworska-Rolewicz [1]–[4]. Complex and functional extensions of these shifts were considered by the author [5]–[10]. In the present paper periodic solutions of equations and initial value problems, induced by functional shifts are studied. Periodic solutions of an equation with an operator of complex differentation are considered.

0. Denote by $L(X)$ the set of all linear operators with domains and ranges in a linear space X over the field $\mathbb{C}$ of the complex numbers and by $L_o(X)$ the set of all operators $A \in L(X)$ with dom $A = X$. The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. If $D \in R(X)$ then we denote by $\mathcal{R}_D$ the set of all right inverses of D. In the sequel we shall assume that $\dim \ker D \neq 0$ and that right inverses belong to $L_o(X)$. An operator $F \in L_o(X)$ is said to be an initial operator for D corresponding to an $R \in \mathcal{R}_D$ if

$$F^2 = F, \quad FX = \ker D \quad \text{and} \quad FR = 0\,.$$

The set of all initial operators for a given $D \in R(X)$ is denoted by $\mathcal{F}_D$.

Here and in the sequel we admit that $0^0 := 1$. We also write $\mathbb{N}$ for the set of all positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

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For a given operator $D \in R(X)$ we shall write (cf. [2], [4]):

$$(0.1) \qquad S := \bigcup_{i=1}^{\infty} \ker D^i .$$

If $R \in \mathcal{R}_D$ then the set S is equal to the linear span $P(R)$ of all D-monomials, i.e.

$$(0.2) \qquad S = P(R) := \operatorname{lin}\{R^k z : z \in \ker D,\ k \in \mathbb{N}_0\} .$$

Evidently, the set $P(R)$ is independent of the choice of the right inverse R.

In the sequel K will stand either for the unit disk $K_1 := \{h \in \mathbb{C} : |h| < 1\}$ or for the complex plane $\mathbb{C}$. Denote by $H(\Omega)$ the class of all functions analytic on a set $\Omega \subseteq \mathbb{C}$.

DEFINITION 0.1. Suppose that a function $f \in H(K)$ has the following expansion

$$(0.3) \qquad f(h) = \sum_{n=0}^{\infty} a_n h^n \quad \text{for all } h \in K .$$

Suppose that $D \in R(X)$ and $\dim \ker D > 0$. A family $T_{f,K} = \{T_{f,h}\}_{h \in K} \subset L_o(X)$ is said to be a family of *functional shifts* for the operator D induced by the function f if

$$(0.4) \qquad T_{f,h}x = [f(hD)]x := \sum_{n=0}^{\infty} a_n h^n D^n x$$

for all $h \in K$; $x \in S$.

We should point out that, by definition of the set S, the last sum has only a finite number of components different than zero.

PROPOSITION 0.1 (cf. [9]) *Suppose that $D \in R(X)$, $\dim \ker D \neq 0$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $T_{f,K} = \{T_{f,h}\}_{h \in K}$ is a family of functional shifts for D induced by the function f. Then*

$$(0.5) \quad \text{(i)} \qquad T_{f,h}R^k F = \sum_{j=0}^{k} a_j h^j R^{k-j} F ,$$

$$FT_{f,h}R^k z = a_k ,$$

for all $h \in K$; $k \in \mathbb{N}_0$; $z \in \ker D$,

(ii) *If a family $W_{f,K} = \{W_{f,h}\}_{h \in K} \subset L(X)$ satisfies the condition: $W_{f,h}R_1^k z = \sum_{j=0}^{k} a_j h^j R_1^{k-j} z$ for all $h \in K$; $k \in \mathbb{N}_0$; $z \in \ker D$, where $R_1 \in \mathcal{R}_D$, then*

$$W_{f,h}|_S = T_{f,h}|_S \quad \textit{for all } h \in K .$$

PROPOSITION 0.2 (cf. [8]) *Suppose that all assumptions of Proposition* 0.1 *are satisfied. Then for all* $h \in K$ *the operators* $T_{f,h}$ *commute on the set* S *with the operator* D.

We denote by $X_{T_{f,h}}$ the space of $T_{f,h}$-periodic elements, i.e. the space

$$X_{T_{f,h}} := \{x \in X : T_{f,h}x = x\}, \quad h \in K. \tag{0.6}$$

1. In this section, K will stand either for the unit disk or for the complex plane $\mathbb{C}$. As before a function $f \in H(K)$ has the following expansion:

$$f(h) = \sum_{n=0}^{\infty} a_n h^n \quad \text{for all } h \in K.$$

Let $T_{f,K} = \{T_{f,h}\}_{h \in K}$ be a family of functional shifts for $D \in R(X)$ induced by the function f.

The general form of the solution of the equation

$$Dx = y \tag{1.1}$$

is given by the formula

$$x = z + Ry, \tag{1.2}$$

where $z \in \ker D$ is arbitrary and $R \in \mathcal{R}_D$ is arbitrarily fixed (cf. [2]).

PROPOSITION 1.1. *Suppose that* $f \in H(K)$, $f(0) = 1$, $D \in R(X)$ *and* $\dim \ker D \neq 0$. *Let* $R \in \mathcal{R}_D$ *be arbitrarily fixed. Then the equation* (1.1) *has a solution belonging to the space* $X_{T_{f,h}}(h \in K)$ *defined by Formula* (0.6) *if and only if* $Ry \in X_{T_{f,h}}$. *If this condition is satisfied then Formula* (1.2) *determines all solutions of Equation* (1.1) *which belong to the space* $X_{T_{f,h}}$.

Proof. Let $x \in X_{T_{f,h}}$ be a solution of Equation (1.1). Then there exists $z_1 \in \ker D$ such that $x = z_1 + Ry$. Since $T_{f,h}z_1 = z_1$, therefore

$$x = T_{f,h}x = T_{f,h}(z_1 + Ry) = z_1 + T_{f,h}Ry.$$

Hence,

$$z_1 + Ry = z_1 + T_{f,h}Ry,$$

i.e. $Ry = S_h Ry$.

Conversely, let $Ry \in X_{T_{f,h}}$, then $x = z + Ry$, where $z \in \ker D$, is a solution of Equation (1.1) and

$$T_{f,h}x = T_{f,h}(z + Ry) = T_{f,h}z + T_{f,h}Ry = z + Ry = x,$$

i.e. $x \in X_{T_{f,h}}$.

Note that if $f(0) = 1$ and $R_1 y \in X_{T_{f,h}}$ for an operator $R_1 \in \mathcal{R}_D$ then the set

$$R_D y := \{Ry : R \in \mathcal{R}_D\} \subset X_{T_{f,h}} \quad (h \in K).$$

Indeed, let $R \in \mathcal{R}_D$ be arbitrarily fixed. Then there exist $x \in \operatorname{dom} D$, $z, z_1 \in \ker D$ such that

$$Dx = y; \quad x = z + Ry \quad \text{and} \quad x = z_1 + R_1 y .$$

We have

$$T_{f,h}x = z_1 + R_1 y = x \quad \text{and} \quad T_{f,h}x = z + T_{f,h}Ry .$$

Hence

$$T_{f,h}Ry = Ry \quad \text{i.e. } Ry \in X_{T_{f,h}} .$$

Observe that, in general, Equation (1.1) may have not a solution in the space $X_{T_{f,h}}(h \in K)$ although the right side of this equation is a member of $X_{T_{f,h}}$. For example, let $0 \neq h \in K$ and $0 \neq z \in \ker D$ be arbitrarily fixed and let the families $T_{\exp,K}, T_{\cos,h}, T_{ch,h}$ be given (cf. [5]). Proposition 0.1 implies that

$$T_{\exp,h}Rz = Rz + hz \neq Rz \quad \text{for } R \in \mathcal{R}_D .$$

This and Proposition 1.1 together imply that the equation

$$Dx = z \tag{1.3}$$

has not a solution belonging to the space $X_{T_{\exp,h}}$, although $\ker D \subset X_{T_{\exp,h}}$. The following equalities (cf. Proposition 0.1)

$$T_{\cos,h}Rz = Rz, \qquad T_{ch,h}Rz = Rz, \qquad z \in \ker D$$

and Proposition 1.1 together imply that every solution of the Equation (1.3) belongs to $X_{T_{\cos,h}}, X_{T_{ch,h}}$, respectively.

PROPOSITION 1.2. *Suppose that $f \in H(K)$, $f(0) \neq 1$, $D \in R(X)$ and $\dim \ker D \neq 0$. Let an operator $R \in \mathcal{R}_D$ be arbitrarily fixed. Then Equation* (1.1) *has a solution in the space $X_{T_{f,h}}(h \in K)$ if and only if $(T_{f,h} - I)Ry \in \ker D$. If this condition is satisfied then the unique solution of Equation* (1.1), *which belongs to $X_{T_{f,h}}$, has the form*

$$x = z_1 + Ry ,$$

where $z_1 = z_2/[1 - f(0)]$, $z_2 = (T_{f,h} - I)Ry$; $h \in K$.

Proof. Let $x \in X_{T_{f,h}}$ be a solution of Equation (1.1). Then there exists $z_1 \in \ker D$ such that $x = z_1 +$ Ry. By the definition, we have

$$x = T_{f,h}x = T_{f,h}(z_1 + Ry) = f(0)z_1 + T_{f,h}Ry . \tag{1.4}$$

Hence,

$$(T_{f,h} - I)Ry = [1 - f(0)]z_1 \in \ker D .$$

Conversely, if $(T_{f,h} - I)Ry = z_2 \in \ker D$ and $x = z_1 + Ry$, where $z_1 = z_2/[1 - f(0)]$, then $Dx = y$ and

$$T_{f,h}x = f(0)z_1 + T_{f,h}Ry = f(0)z_1 + z_2 + Ry$$
$$= f(0)z_1 + (1 - f(0))z_1 + Ry = z_1 + Ry = x\,.$$

In order to prove the uniqueness of solutions to Equation (1.1) in the space $X_{T_{f,h}}$, suppose that there exist $x_1, x_2 \in X_{T_{f,h}}$ such that $Dx_1 = Dx_2 = y$. Formula (1.4) implies that there exists $z \in \ker D$ such that $x_2 = x_1 + z$. We have

$$x_2 = T_{f,h}x_2 = T_{f,h}(x_1 + z) = T_{f,h}x_1 + T_{f,h}z = x_1 + f(0)z\,.$$

Hence $z = f(0)z$. Our assumptions imply that $z = 0$, i.e. that $x_2 = x_1$.

Note that if $f(0) \neq 1, h \in K$ is arbitrarily fixed and there exists an $R_1 \in \mathcal{R}_D$ such that $(T_{f,h} - I)R_1y \in \ker D$, then $(T_{f,h} - I)Ry \in \ker D$ for all $R \in \mathcal{R}_D$.

Suppose that $D \in R(X)$, $\dim\ker D \neq 0$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then the initial value problem

(1.1) $$Dx = y\,, \qquad y \in X\,,$$

(1.5) $$Fx = z_0, \qquad z_0 \in \ker D\,,$$

has the unique solution of the form

(1.6) $$x = z_0 + Ry,$$

(cf. [2]).

Proposition 1.1 implies the following

THEOREM 1.1. *Suppose that $D \in R(X)$, $\dim\ker D \neq 0$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$, $f \in H(K)$ and $f(0) = 1$. Then a necessary and sufficient condition for the initial value problem* (1.1), (1.5) *to have solutions in the space $X_{T_{f,h}}$ is that $Ry \in X_{T_{f,h}}$ $(h \in K)$. If this condition is satisfied then a unique solution of the problem exists and is of the form* (1.6).

Proposition 1.2 implies the following

THEOREM 1.2. *Suppose that $D \in R(X)$, $\dim\ker D \neq 0$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$, $f \in H(K)$ and $f(0) \neq 1$. Then a necessary and sufficient condition for the initial value problem* (1.1), (1.5) *to have solutions in the space $X_{T_{f,h}} (h \in K)$ is that*

$$z_0 = (T_{f,h} - I)Ry\,.$$

If this condition is satisfied then a unique solution of the problem exists and is of the form

$$x = (1 - f(0))^{-1}(T_{f,h} - f(0)I)Ry\,.$$

2. Let X be the set of all polynomials defined on K and with coefficients in $\mathbb{C}$. The set X is a linear space over $\mathbb{C}$, if the multiplication by a scalar and the addition are defined in usual way. Let X_m will stand for the set of all polynomials of order $m \in \mathbb{N}_0$ belonging to the space X.

Suppose that we are given families $T_{\exp,K}$, $T_{\cos,K}$, $T_{ch,K}$, $T_{\sin,K}$, $T_{sh,K}$. Similarly as in [5] we write:

$$S_K := T_{\exp,K}\,, \quad c_K := T_{\cos,K}\,,$$
$$ch_K := T_{ch,K}\,, \quad s_K := T_{\sin,K}\,,$$
$$sh_K := T_{sh,K}\,.$$

Let $\{a_k\}$ be a sequence, where $a_k \in \mathbb{C}$ and $a_k \neq 0$ for every $k \in \mathbb{N}_0$. Define linear operators D, R as follows:

$$D\mathbf{1} = 0, \quad Dw^{k+1} = a_k/a_{k+1}w^k,$$
$$Rw^k = a_{k+1}/a_k w^{k+1}, \quad k \in \mathbb{N}_0,\ w \in K\,,$$

where $\mathbf{1} \equiv 1$ on K.

Cleary, the operators D, R are uniquely determined on the whole space X, i.e. $D, R \in L_0(X)$. It is easy to show that $D \in R(X)$ and $R \in R_D$. An initial operator F for the operator D corresponding to R has the form (cf. [7])

$$(Fx)(w) = x(0), \quad x \in X\,.$$

Observe that

$$\ker D = X_0\,,$$

$$DX_n \subset X_{n-1}\,, \quad n \in \mathbb{N}\,, \tag{2.1}$$

$$RX_m \subset X_{m+1}\,, \quad m \in \mathbb{N}_0\,. \tag{2.2}$$

Let $x \in X_m$, $m \in \mathbb{N}_0$. Consider elements S_hx, c_hx, ch_hx, s_hx, sh_hx for $h \in K$. We have (cf. [5])

$$S_hx = \sum_{j=0}^{m}(j!)^{-1}h^jD^jx\,, \tag{2.3}$$

$$c_hx = \sum_{j=0}^{m}(j!)^{-1}h^j\cos(j\pi/2)D^jx\,, \tag{2.4}$$

$$ch_hx = \sum_{j=0}^{m}(j!)^{-1}h^j|\cos(j\pi/2)|D^jx\,,$$

$$s_hx = \sum_{j=0}^{m}(j!)^{-1}h^j\sin(j\pi/2)D^jx\,, \tag{2.5}$$

$$sh_h x = \sum_{j=0}^{m} (j!)^{-1} h^j |\sin(j\pi/2)| D^j x\,.$$

PROPOSITION 2.1. *For all $0 \neq h \in K$ the space of S_h-periodic elements X_{S_h} coincides with $X_0 = \ker D$.*

Proof. Let $n \in \mathbb{N}$ and $x \in X_n$ are arbitrarily fixed. Consider $S_h x$ for an arbitrary $0 \neq h \in K$. By Formula (2.3) we obtain

$$S_h x = x + \sum_{j=1}^{n} \frac{h^j}{j!} D^j x\,.$$

This and Inclusion (2.1) together imply that $S_h x = x + x_1$, where $x_1 \in X_{n-1}$.

In the case $n = 1$ we have $x_1 \neq 0$. Indeed, let $x = t_1 w + t_0 \in X_1$, where $t_0, t_1 \in \mathbb{C}$ ($t_1 \neq 0$). We have

$$x_1 = S_h x - x = hDx = hD(t_0 + t_1 w) = (ht_1 a_0/a_1)1 \neq 0\,.$$

Thus the equation $S_h x = x$ is not solvable for $x \notin \ker D$. Since $\ker D = X_0$, we conclude that the space of S_h-periodic elements $X_{S_h} = X_0$.

Propositon 1.1., Inclusion (2.2) and Proposition 2.1. together imply

THEOREM 2.1. *The equation*

$$Dx = y, \quad y \in X \ (0 \neq h \in K)\,, \tag{2.6}$$

has not S_h-periodic solutions.

In similar way we can obtain

PROPOSITION 2.2. *The space of c_h-periodic (ch_h-periodic) elements is of the form*

$$X_{c_h} = X_0 \cup X_1 \quad (X_{ch_h} = X_0 \cup X_1)\,,$$

where $0 \neq h \in K$.

THEOREM 2.2. *Let $0 \neq h \in K$. Equation (2.6) has a c_h-periodic (ch_h-periodic) solution if and only if $y \in \ker D = X_0$.*

PROPOSITION 2.3. *The space of s_h- periodic (sh_h-periodic) elements contains only the zero element.*

Proof. Let $h \in K$ be arbitrarily fixed. Formula (2.5) implies

$$s_h x = 0 \quad \text{for } x \in X_0$$

and

$$s_h x = \sum_{j=1}^{n} \frac{h^j}{j!} \sin(j\pi/2) D^j x \quad \text{for } x \in X_n,\ n \in \mathbb{N}\,.$$

This and Inclusion (2.1) together imply

$$s_h X_0 = \{0\}\,, \qquad s_h X_n \subset X_{n-1}\,, \quad n \in \mathbb{N}\,. \tag{2.7}$$

Since for all $k, m \in \mathbb{N}_0$, $k \neq m$ $X_k \cap X_m = \emptyset$, we conclude that

$$X_{s_h} = \{0\}\,.$$

Similarly, we can obtain that $X_{sh_h} = \{0\}$.

Observe that Formula (2.7) and Inclusion (2.2) together imply

$$(s_h - I)RX_m \subset X_{m+1}, \qquad m \in \mathbb{N}_0\,. \tag{2.8}$$

A similar inclusion holds for sh_h.

Proposition 1.2 and Formula (2.8) together imply

THEOREM 2.3. *Equation* (2.6) *has only a trivial s_h-periodic* (sh_h-*periodic*) *solution.*

An immediate consequence of Theorems 2.1, 2.2, 2.3 and Propositions 1.3, 1.4 is the following

COROLLARY 2.1. *Let $0 \neq h \in K$. The following initial boundary value problem*

$$Dx = y, \qquad 0 \neq y \in X\,,$$
$$Fx = z_0, \qquad z_0 \in \ker D$$

(i) *has not S_h-periodic, s_h-periodic, sh_h-periodic solutions,*

(ii) *has a c_h-periodic, ch_h-periodic solution if and only if $y \in \ker D$.*

References

[1] D. Przeworska-Rolewicz, *Shift and periodicity for right invertible operators*, Research Notes in Mathematics 43, Pitman Advanced Publish Program, Boston—London—Melbourne, 1980.

[2] D. Przeworska-Rolewicz, *Algebraic Analisis*, PWN-Polish Scientific Publishers and D. Reidel, Warszawa—Dordrecht, 1988.

[3] D. Przeworska-Rolewicz, *True Shifts*, J. Math. Anal. Appl., 170 (1) 1992, 27–48.

[4] D. Przeworska-Rolewicz, *Spaces of D-paraanalitic elements*, Dissertationes Math., 302, Warszawa, 1990.

[5] Z. Binderman, *Complex R-shifts for right invertible operators*, Demonstratio Math., 23 (1990) 1043–1053.

[6] Z. Binderman, *On some properties of complex R-shifts*, Demonstratio Math., (25) 1-2, 1992, 207–217.

[7] Z. Binderman, *Some properties of operators of complex differentation and shifts*, Zeszyty Nauk. Politech. Łódź, Matematyka, 24 (1993), 5–18.

[8] Z. Binderman, *Cauchy integral formula induced by right invertible operators*, Demonstratio Math., 25, 3 (1992) 671–690.

[9] Z. Binderman, *Functional shifts induced by right invertible operators*, Math. Nachrichten, 157 (1992) 211–224.

[10] Z. Binderman, *A unified approach to shifts induced by right invertible operators*, Math. Nachrichten, 161 (1993), 239–252.

[11] Z. Binderman, *On singular boundary value problems for generalized analitic functions*, Zeszyty Nauk. Politech. Łódź, Matematyka, 23 (1993), 11–16.

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