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CHERN CLASSES OF BUNDLES OF RANK TWO ON P^5

1. Introduction

In [1] Schwarzenberger gives well known conditions (S_n^r) on $c_i \in Z$ for existence of a given rank r bundle E on P^n with Chern classes $c_i(E) = c_i$.The conditions (S_n^r) are :

If $c_t(E) = 1 + c_1(E)t + \cdots + c_r(E)t^r \in Z[t]$ is the Chern polynomial of E and $c_t(E) = \prod_{i=1}^r (1 + x_i t) \in C[t]$ then

$$\sum_{i=1}^{r} \binom{n + x_i + s}{n} \in Z$$

for any $s \in Z$.

We test a general method for calculation of the complete list of Schwarzenberger numbers when E is of rank 2 and $n = 5$. Let $\chi(E)$ be the Euler characteristic, $ch_t(E)$ be the Chern character of E.

2. Chern polynomial of a 2-bundle on P^5

Let E be a bundle on P^5. The existence of Beilinson's spectral sequence attached to the bundle E implies that, there exist $a, b, c, d, e \in Z$ such that the Chern polynomial of E is of the form:

$$\begin{aligned} c_t(E) &= c_t(\Omega^1(1))^a c_t(\Omega^2(2))^b c_t(\Omega^3(3))^c c_t(\Omega^4(4))^d c_t(\Omega^5(5))^e \\ &= (1 - t + t^2 - t^3 + t^4 - t^5)^a \cdot (1 - 4t + 9t^2 - 14t^3 + 14t^4)^b (1 - 6t \\ &\quad + 18t^2 - 34t^3 + 42t^4 - 42t^5)^c (1 - 4t + 7t^2 - 6t^3 + 3t^4)^d (1 - t)^e. \end{aligned}$$

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Let us compute five coefficients of the Taylor expansion of $\ln c_t(E)$:

$$\begin{aligned}\ln c_t(E) = -t(a+4b+6c+4d+e) &+ t^2(a+2b-2d-e)/2 \\ - t^3(a-2b-6c-2d+e)/3 &+ t^4(a-10b+10d-e)/4 \\ - t^5(a-26b+66c-26d+e)/5. &\end{aligned}$$

Substitute new integers:

$$\begin{aligned} v &:= -a-4b-6c-4d-e \\ w &:= a+2b-2d-e \\ x &:= -2(a-2b-6c-2d+e) \\ y &:= 6(a-10b+10d-e) \\ z &:= -24(a-26b+66c-26d+e). \end{aligned}$$

Then

$$\ln c_t(E) = vt + wt^2/2 + xt^3/6 + yt^4/24 + zt^5/120$$

$$c_t(E) = 1 + tv + t^2(v^2+w)/2 + t^3(v^3+3vw+x)/6 + t^4(v^4+6v^2w+4vx+3w^2+y)/24 + t^5(v^5+10v^3w+10v^2x+5v(3w^2+y)+10wx+z)/120.$$

Assume that $rk(E)=2$. Then: $c_t(E) = 1+tv+t^2(v^2+w)/2$ so $c_1(E) = v$, $c_2(E) = (v^2+w)/2$ and

$$\begin{aligned} x &= -v(v^2+3w) \\ y &= 3(v^4+2v^2w-w^2) \\ z &= -6v(v^4-5w^2) \end{aligned}$$

and:

$$\begin{aligned} a &= (v^5+20v^4+70v^3+40v^2w-v(5w^2-210w+96)-20w(w-10))/480 \\ b &= -(v^5+10v^4+10v^3+20v^2w-v(5w^2-30w-24)-10w(w+2))/480 \\ c &= v(v^4-10v^2-5w^2-2(15w+8))/480 \\ d &= -(v^5-10v^4+10v^3-20v^2w-v(5w^2-30w-24)+10w(w+2))/480 \\ e &= (v^5-20v^4+70v^3-40v^2w-v(5w^2-210w+96)+20w(w-10))/480. \end{aligned}$$

For

$$v = 0: \quad \begin{aligned} a &= -w(w-10w)/24, \\ b &= w(w+2)/48, \\ c &= 0 \\ d &= -w(w+2)/48, \\ e &= w(w-10)/24. \end{aligned}$$

For

$$v = 1: \quad \begin{aligned} a &= -(5w^2 - 90w + 1)/96, \\ b &= (w^2 - 2w - 3)/32, \\ c &= -(w^2 + 6w + 5)/96, \\ d &= -(w^2 + 6w + 5)/96, \\ e &= (w^2 - 2w - 3)/32. \end{aligned}$$

It is easy to find all integers a, b, c, d, e of given forms. We have the following theorem:

THEOREM 2.1. *The Chern polynomial of any normalized 2-bundle on P^5 is of the form:*

1. $12nt^2 + 1$, *then*
   $a = -2n(12n - 5), b = n(12n + 1), c = 0,$
   $d = -n(12n + 1), e = 2n(12n - 5).$
2. $6nt^2 + t + 1$ then
   $a = -(15n^2 - 25n + 2)/2, b = 3n(3n - 1)/2,$
   $c = -n(3n + 1)/2, d = -n(3n + 1)/2, e = 3n(3n - 1)/2.$
3. $(6n + 10)t^2 + t + 1$ then
   $a = -(15n^2 + 25n + 2)/2, b = (9n^2 + 27n + 20)/2,$
   $c = -(3n^2 + 11n + 10)/2, d = -(3n^2 + 11n + 10)/2, e = (9n^2 + 27n + 20)/2.$

Moreover such polynomial is reducible if and only if:

1. $n = -3k^2$.
2. $n = -k(3k - 1)/2$ or $n = -k(3k + 1)/2$.
3. $n = -(3k^2 + 3k + 4)/2$.

□

In the next paragraph we show that the integers a, b, c, d, e are correspondingly $-\chi(E(-1)), \chi(E(-2)), -\chi(E(-3)), \chi(E(-4)), -\chi(E(-5))$.

3. Chern character of a 2-bundle on P^5

The explicit formulas for Chern character and Euler characteristic on P^5 are:

$ch_t(E) = (5c_5 - 5c_1c_4 - 5c_2c_3 + 5c_1^2c_3 + 5c_1c_2^2 - 5c_1^3c_2 + c_1^5)t^5/120 + (4c_1c_3 - 4c_4 + 2c_2^2 - 4c_1^2c_2 + c_1^4)t^4/24 + (3c_3 - 3c_1c_2 + c_1^3)t^3/6 + (c_1^2 - 2c_2)t^2/2 + c_1t + rk(E)$

$\chi(E) = (5c_5 - 5c_1c_4 - 5c_2c_3 + 5c_1^2c_3 + 5c_1c_2^2 - 5c_1^3c_2 + c_1^5 + 60c_1c_3 - 60c_4 + 30c_2^2 - 60c_1^2c_2 + 15c_1^4 + 255c_3 - 255c_1c_2 + 85c_1^3 + 225c_1^2 - 450c_2 + 274c_1)/120 + rk(E).$

The Chern characters of $ch_t(\Omega^i(i))$ are:

1. $ch_t(\Omega^1(1)) = -t^5/120 - t^4/24 - t^3/6 - t^2/2 - t + 5$
2. $ch_t(\Omega^2(2)) = 13t^5/60 + 5t^4/12 + t^3/3 - t^2 - 4t + 10$
3. $ch_t(\Omega^3(3)) = -11t^5/20 + t^3 - 6t + 10$
4. $ch_t(\Omega^4(4)) = 13t^5/60 - 5t^4/12 + t^3/3 + t^2 - 4t + 5$
5. $ch_t(\Omega^5(5)) = -t^5/120 + t^4/24 - t^3/6 + t^2/2 - t + 1$

We are able to prove the following formula:

THEOREM 3.1. *For any 2-bundle on P^5.*

$$ch_t(E) = \sum_{i=0}^{n}(-1)^i\chi(E(-i))ch_t(\Omega^i(i)). \tag{1}$$

Proof.

$$\begin{aligned} ch_t(E) = a_0 & \\ & + a_1(-t^5/120 - t^4/24 - t^3/6 - t^2/2 - t + 5) \\ & + a_2(13t^5/60 + 5t^4/12 + t^3/3 - t^2 - 4t + 10) \\ & + a_3(-11t^5/20 + t^3 - 6t + 10) \\ & + a_4(13t^5/60 - 5t^4/12 + t^3/3 + t^2 - 4t + 5) \\ & + a_5(-t^5/120 + t^4/24 - t^3/6 + t^2/2 - t + 1. \end{aligned}$$

On the other hand for any 2-bundle on P^5:
$ch_t(E) = t^5c_1(c_1^4 - 5c_1^2c_2 + 5c_2^2)/120 + t^4(c_1^4 - 4c_1^2c_2 + 2c_2^2)/24 + t^3c_1(c_1^2 - 3c_2)/6 + t^2(c_1^2 - 2c_2)/2 + c_1t + 2$.

The above equations leads to solutions:

$$\begin{aligned} a_0 &= (c_1^5 + 15c_1^4 - 5c_1^3(c_2 - 17) - 15c_1^2(4c_2 - 15) + c_1(5c_2^2 \\ &\quad - 255c_2 + 274) + 30(c_2^2 - 15c_2 + 8))/120 && = \chi(E) \\ -a_1 &= -(c_1^5 + 10c_1^4 - 5c_1^3(c_2 - 7) - 10c_1^2(4c_2 - 5) + +c_1(5c_2^2 \\ &\quad - 105c_2 + 24) + 20c_2(c_2 - 5))/120 && = \chi(E(-1)) \\ a_2 &= (c_1^5 + 5c_1^4 - 5c_1^3(c_2 - 1) - 5c_1^2(4c_2 + 1) + c_1(5c_2^2 \\ &\quad - 15c_2 - 6) + 10c_2(c_2 + 1))/120 && = \chi(E(-2)) \\ -a_3 &= -c_1(c_1^4 - 5c_1^2(c_2 + 1) + 5c_2^2 + 15c_2 + 4)/120 && = \chi(E(-3)) \\ a_4 &= (c_1^5 - 5c_1^4 - 5c_1^3(c_2 - 1) + 5c_1^2(4c_2 + 1) + c_1(5c_2^2 \\ &\quad - 15c_2 - 6) - 10c_2(c_2 + 1))/120 && = \chi(E(-4)) \\ -a_5 &= -(c_1^5 - 10c_1^4 - 5c_1^3(c_2 - 7) + 10c_1^2(4c_2 - 5) + c_1(5c_2^2 \\ &\quad - 105c_2 + 24) - 20c_2(c_2 - 5))/120 && = \chi(E(-5)) \end{aligned}$$

Moreover substituting $c_1 = v$, $c_2 = (v^2 + w)/2$ from §1 we have:
$\chi(E(-1)) = -a_1 = a, \chi(E(-2)) = a_2 = b, \chi(E(-3)) = -a_3 = c,$
$\chi(E(-4)) = a_4 = d, \chi(E(-5)) = -a_5 = e$
$\chi(E) = (c_1^5 + 15c_1^4 - 5c_1^3(c_2 - 17) - 15c_1^2(4c_2 - 15) + c_1(5c_2^2 - 255c_2 + 274) + 30c_2^2 - 450c_2 + 240)/120.$ □

As a completion of the formula (1) we get

$$c_t(E) = \prod_{k=1}^{n} c_t(\Omega^k(k))^{(-1)^i \chi(E(-k))}. \tag{2}$$ □

References

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