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ON THE REFLEXIVITY SYMMETRY AND TRANSITIVITY OF THE TANGENCY RELATION OF SETS OF THE CLASS $A^*_{P,K}$ IN GENERALIZED METRIC SPACES

Introduction

In the present paper we consider the problem of the equivalence of the tangency relation $T_l(a,b,k,p)$ of sets of the class $A^*_{p,k}$ in generalized metric spaces.

Some sufficient conditions for reflexivity, symmetry and transitivity of this relation have been given here.

Let E_0 denote the family of all non-empty subsets of certain set E and l an arbitrary, non-negative real function defined on the Cartesian product $E_0 \times E_0$.

The pair (E,l) will be called a generalized metric space (see [9]). Let a,b are an arbitrary, non-negative real functions defined in the right-hand side neighbourhood of the point 0 such that

$$a(r) \underset{r\to 0^+}{\longrightarrow} 0 \quad\text{and}\quad b(r) \underset{r\to 0^+}{\longrightarrow} 0. \tag{1}$$

In [9] W. Waliszewski has introduced the following definition of the tangency relation in the space (E,l):

$$\begin{aligned} T_l(a,b,k,p) = \{(A,B) : A,B \in E_0, &\text{ the pair } (A,B) \text{ is } (a,b)\text{-clustered} \\ &\text{at the point } p \text{ of the space} (E,l) \text{ and} \\ &\frac{1}{r} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0\}, \end{aligned} \tag{2}$$

where k is a positive real number and the set $S_l(p,r)_u$ is the so-called u-neighbourhood of the sphere $S_l(p,r)$ with the centre at the point p and the radius r in the space (E,l).

Let ϱ be a metric on the set set E and A,B arbitrary sets of the family

E_0. We shall denote

$$\varrho(A,B) = \inf\{\varrho(x,y) : x \in A, y \in B\}, \quad \operatorname{diam}_\varrho A = \sup\{\varrho(x,y) : x,y \in A\}. \tag{3}$$

Let F_ϱ (see [6]) be the class of functions l fulfilling the conditions:

(a) $l : E_0 \times E_0 \to \langle 0, \infty)$,

(b) there exist numbers m, M such that $0 < m \leq M < \infty$ and

$$m\varrho(A,B) \leq l(A,B) \leq M \operatorname{diam}_\varrho(A \cup B) \quad \text{for } A, B \in E_0,$$

(c) the function l generates on the set E the metric l_0 defined by formula: $l_0(x,y) = l(\{x\},\{y\})$ for $x, y \in E$.

Let F_ϱ^* denote the class of function l fulfilling the conditions (a) and (b) for $m = M = 1$.

It is easy to see that very function $l \in F_\varrho^*$ generates on the set E the metric ϱ.

From the above definitions it follows that $F_\varrho^* \subset F_\varrho$.

We shall say that functions $l_1, l_2 \in F_\varrho$ fulfil in the set $A \in E_0$ the condition of the proximity of the spheres of order k at the point $p \in E$ with regard to the metric ϱ if

$$\frac{1}{r^k}\varrho(A \cap S_{l_1}(p,r), A \cap S_{l_2}(p,r)) \underset{r \to 0^+}{\longrightarrow} 0. \tag{4}$$

Let $p \in A'$, where A' is the set of all cluster points of the set $A \in E_0$.

We say that the set $A \in E_0$ has the Darboux property at the point p of the metric space (E, ϱ), if there exists a number $\tau > 0$ such that for an arbitrary $r \in (0, \tau)$ the set $A \cap S_\varrho(p,r) \neq \emptyset$.

Let (see [6])

$$A_{p,k}^* = \{A \in E_0 : p \in A' \text{ and there exists a number } \lambda > 0 \text{ such that} \quad \limsup_{[A,p;k] \ni (x,y) \to (p,p)} \frac{\varrho(x,y) - \lambda\varrho(x,A)}{\varrho^k(x,p)} \leq 0\}, \tag{5}$$

where $\varrho(x,A) = \inf\{\varrho(x,y) : y \in A\}$, k is an arbitrary fixed positive real number and

$$[A,p;k] = \{(x,y) : x \in E, y \in A \text{ and } \varrho(x,A) < \varrho^k(x,p) = \varrho^k(y,p)\}. \tag{6}$$

We assume by definition

$$A_{p,k}^{*D} = \{A \in E_0 : A \in A_{p,k}^* \text{ and } A \text{ has the Darboux property at the point } p \text{ of the space } (E,\varrho)\}. \tag{7}$$

1. Reflexivity of the tangency relation

Let a, b be non-negative, real functions defined in the right-hand side neighbourhood of the point 0 fulfilling the condition (1).

THEOREM 1.1. *If*

$$\frac{1}{r^k}a(r) \underset{r\to 0^+}{\longrightarrow} 0, \quad \frac{1}{r^k}b(r) \underset{r\to 0^+}{\longrightarrow} 0, \tag{1.1}$$

then for an arbitrary function $l \in F_\varrho$ *the tangency relation* $T_l(a,b,k,p)$ *is reflexive in the class of sets* $A^{*D}_{p,k}$, *i.e.* $(A, A) \in T_1(a,b,k,p)$ *for* $A \in A^{*D}_{p,k}$.

Proof. Let us denote $\alpha = \max(a, b)$. From here and from (1.1) it follows that

$$\frac{1}{r^k}\alpha(r) \underset{r\to 0^+}{\longrightarrow} 0. \tag{1.2}$$

Hence and from Lemma 1 of the paper [6] we have

$$\frac{1}{r^k}\operatorname{diam}_l(A \cap S_l(p,r)_{\alpha(r)}) \underset{r\to 0^+}{\longrightarrow} 0 \quad \text{for } A \in A^{*D}_{p,k}. \tag{1.3}$$

Because

$$m\varrho(x,y) \le l(\{x\},\{y\}) \quad \text{for } x,y \in E,\ l \in F_\varrho,$$

then

$$m \operatorname{diam}_\varrho(A \cap S_l(p,r)_{\alpha(r)}) \le \operatorname{diam}_l(A \cap S_l(p,r)_{\alpha(r)}). \tag{1.4}$$

From the fact that $l \in F_\varrho$ and from (1.4) we obtain

$$\begin{aligned}
&\frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, A \cap S_l(p,r)_{b(r)}) \\
&\quad \le \frac{M}{r^k}\operatorname{diam}_\varrho((A \cap S_l(p,r)_{a(r)}) \cup (A \cap S_l(p,r)_{b(r)})) \\
&\quad \le \frac{M}{r^k}\operatorname{diam}_\varrho(A \cap S_l(p,r)_{\alpha(r)}) \le \frac{M}{m}\frac{1}{r^k}\operatorname{diam}_l(A \cap S_l(p,r)_{\alpha(r)}).
\end{aligned}$$

From the last inequality and from (1.3) it follows that

$$\frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, A \cap S_l(p,r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0. \tag{1.5}$$

From the Darboux property of the set $A \in A^{*D}_{p,k}$ at the point p of the space (E, l), (see Lemma 2 of the paper [6]) it follows that the pair (A, A) is (a, b)-clustered at the point p of this space. From here and from (1.5) we have that $(A, A) \in T_l(a,b,k,p)$ for $A \in A^{*D}_{p,k}$.

2. Symmetry of the tangency relation

We shall say that the function $l \in F_\varrho$ is symmetric if $l(A, B) = l(B, A)$ for an arbitrary sets $A, B \in E_0$.

Let l^* denote a certain symmetric function of the class F_ϱ.

THEOREM 2.1. *If*

$$\frac{1}{r^k}a(r) \underset{r\to 0^+}{\longrightarrow} 0, \qquad \frac{1}{r^k}b(r) \underset{r\to 0^+}{\longrightarrow} 0, \tag{2.1}$$

and the functions $l, l^* \in F_\varrho$ fulfil in the sets $A, B \in A^{*D}_{p,k}$ the condition of the proximity of the spheres of order k at the point $p \in E$ with regard to the metric ϱ, then the tangency relation $T_l(a,b,k,p)$ is symmetric in the class of sets $A^{*D}_{p,k}$, i.e. if $(A,B) \in T_l(a,b,k,p)$ then $(B,A) \in T_l(a,b,k,p)$ for $A, B \in A^{*D}_{p,k}$.

Proof. Let $(A,B) \in T_l(a,b,k,p)$ for $A, B \in A^{*D}_{p,k}$. Hence and from Theorem 2 of the paper [6] it follows that $(A,B) \in T_l(b,a,k,p)$. From here and from Theorem 1 of this paper we obtain that $(A,B) \in T_{l^*}(b,a,k,p)$. Hence

$$\frac{1}{r^k}l^*(A \cap S_{l^*}(p,r)_{b(r)}, B \cap S_{l^*}(p,r)_{a(r)}) \underset{r\to 0^+}{\longrightarrow} 0. \tag{2.2}$$

From (2.2) and by symmetry of the function l^* we have

$$\frac{1}{r^k}l^*(B \cap S_{l^*}(p,r)_{a(r)}, A \cap S_{l^*}(p,r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0. \tag{2.3}$$

From the Darboux property of sets $A, B \in A^{*D}_{p,k}$ at the point p of the space (E,ϱ) and the same of the space (E, l^*) it follows that the pair (B,A) is (a,b)-clustered at the point p of the space (E,l^*). From above and from (2.3) it follows that $(B,A) \in T_{l^*}(a,b,k,p)$. Hence and from Theorem 1 of the paper [6] we get that $(B,A) \in T_l(a,b,k,p)$ for $A, B \in A^{*D}_{p,k}$.

From the above theorem we have

COROLLARY 2.1. *If the functions a, b fulfil the condition* (2.1), *then for an arbitrary symmetric function $l \in F_\varrho$ the tangency relation $T_l(a,b,k,p)$ is symmetric in the class of sets $A^{*D}_{p,k}$.*

Theorem 2.1 holds with sufficiently strong assumptions. These assumptions one can weaken by restriction of the class of function F_ϱ to the class F^*_ϱ. Then we shall obtain

THEOREM 2.2. *If the functions a, b fulfil the condition* (2.1), *then for an arbitrary function $l \in F^*_\varrho$ the tangency relation $T_l(a,b,k,p)$ is symmetric in the class of sets $A^{*D}_{p,k}$.*

Proof. Let $(A,B) \in T_l(a,b,k,p)$ for $A, B \in A^{*D}_{p,k}$. Hence and from Theorem 2 of the paper [6] we get that $(A,B) \in T_l(b,a,k,p)$.

Because the function ϱ defined by formula (3) belongs to the class F^*_ϱ then from here and from Theorem 3 of the paper [6] it follows that $(A,B) \in$

$T_\varrho(b, a, k, p)$. Hence

$$(2.4)\qquad \frac{1}{r^k}\varrho(A \cap S_\varrho(p, r)_{b(r)}, B \cap S_\varrho(p, r)_{a(r)}) \underset{r\to 0^+}{\longrightarrow} 0.$$

From the symmetry of the function ϱ and from (2.4) we get

$$(2.5)\qquad \frac{1}{r^k}\varrho(B \cap S_\varrho(p, r)_{a(r)}, A \cap S_\varrho(p, r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0.$$

Because by assumption the sets $A, B \in A^{*D}_{p,k}$ have the Darboux property at the point p of the space (E, ϱ), then the pair (B, A) is (a, b)-clustered at the point p of this space.

Hence and from (2.5) we get that $(B, A) \in T_\varrho(a, b, k, p)$. From the above and from Theorem 3 of the paper [6] it follows that $(B, A) \in T_l(a, b, k, p)$ for $A, B \in A^{*D}_{p,k}$.

3. Transitivity of the tangency relation

We shall say that the function $l \in F_\varrho$ have the triangle property or that fulfils the triangle inequality (condition) if $l(A, C) \le l(A, B) + l(B, C)$ for an arbitrary sets $A, B, C \in E_0$.

Let l^* be a certain function of the class F_ϱ having the triangle property.

THEOREM 3.1. *If*

$$(2.1)\qquad \frac{1}{r^k}a(r) \underset{r\to 0^+}{\longrightarrow} 0, \qquad \frac{1}{r^k}b(r) \underset{r\to 0^+}{\longrightarrow} 0,$$

and the functions $l, l^* \in F_\varrho$ fulfil in the sets $A, B, C \in A^{*D}_{p,k}$ the condition of the proximity of the spheres of order k at the point $p \in E$ with regard to the metric ϱ, then the tangency relation $T_l(a, b, k, p)$ is transitive in the class of sets $A^{*D}_{p,k}$, i.e. if $(A, B) \in T_l(a, b, k, p)$ and $(B, C) \in T_l(a, b, k, p)$ then $(A, C) \in T_l(a, b, k, p)$ for $A, B, C \in A^{*D}_{p,k}$.

Proof. Let $(A, B) \in T_l(a, b, k, p)$ and $(B, C) \in T_l(a, b, k, p)$. Hence and from Theorem 1 of the paper [6] it follows that

$$(3.2)\qquad (A, B) \in T_{l^*}(a, b, k, p),$$

and

$$(3.3)\qquad (B, C) \in T_{l^*}(a, b, k, p),$$

for an arbitrary sets $A, B, C \in A^{*D}_{p,k}$. From the condition (3.2) and from Theorem 2 of the paper [6] we get

$$(3.4)\qquad (A, B) \in T_{l^*}(a, a, k, p).$$

Hence and from (3.3) we have

$$(3.5)\qquad \frac{1}{r^k}l^*(A \cap S_{l^*}(p, r)_{a(r)}, B \cap S_{l^*}(p, r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0,$$

and

$$\frac{1}{r^k} l^*(B \cap S_{l^*}(p,r)_{a(r)}, C \cap S_{l^*}(p,r)_{b(r)}) \underset{r\to 0+}{\longrightarrow} 0. \tag{3.6}$$

From the triangle inequality for the function l^* it follows that

$$\begin{aligned} l^*(A \cap S_{l^*}(p,r)_{a(r)}, C \cap S_{l^*}(p,r)_{b(r)}) & \\ \le l^*(A \cap S_{l^*}(p,r)_{a(r)}, B \cap S_{l^*}(p,r)_{a(r)}) + & \\ + l^*(B \cap S_{l^*}(p,r)_{a(r)}, C \cap S_{l^*}(p,r)_{b(r)}). & \end{aligned}$$

Hence and from (3.5) and (3.6) we get

$$\frac{1}{r^k} l^*(A \cap S_{l^*}(p,r)_{a(r)}, C \cap S_{l^*}(p,r)_{b(r)}) \underset{r\to 0+}{\longrightarrow} 0. \tag{3.7}$$

From the Darboux property of the sets $A, C \in E_0$ at the point p of the space (E, ϱ) it follows the Darboux property of these sets at the point p of the space (E, l^*) (see Lemma 2 of the paper [6]), and the same the fact that the pair (A, C) is (a, b)-clustered at the point p of this space. Hence and from Theorem 1 of the paper [6] it follows that $(A, C) \in T_l(a, b, k, p)$ for $A, C \in A^{*D}_{p,k}$.

From this theorem there follows immediately

COROLLARY 3.1. *If the functions a, b fulfil the condition* (3.1), *then for an arbitrary function $l \in F_\varrho$ having the triangle property the tangency relation $T_l(a, b, k, p)$ is transitive in the class of sets $A^{*D}_{p,k}$.*

If in consideration of the problem of the transitivity of tangency relation $T_l(a, b, k, p)$ we shall confine ourself to the class F^*_ϱ, then we get

THEOREM 3.2. *If the functions a, b fulfil the condition* (3.1), *then for an arbitrary function $l \in F^*_\varrho$ the tangency relation $T_l(a, b, k, p)$ is transitive in the class of sets $A^{*D}_{p,k}$.*

Proof. Let us assume that

$$(A, B) \in T_l(a, b, k, p), \tag{3.8}$$

and

$$(B, C) \in T_l(a, b, k, p), \tag{3.9}$$

for $A, B, C \in A^{*D}_{p,k}$.

From Theorem 2 of the paper [6] and from (3.9) it follows that

$$(B, C) \in T_l(b, b, k, p). \tag{3.10}$$

Let us denote

$$d_\varrho(A, B) = \operatorname{diam}_\varrho(A \cup B) \text{ for } A, B \in E_0. \tag{3.11}$$

The function d_ϱ evidently belongs to the class F^*_ϱ. Hence and from (3.8), (3.10) and from Theorem 3 of the paper [6] we get that $(A,B)\in T_{d_\varrho}(a,b,k,p)$ and $(B,C)\in T_{d_\varrho}(b,b,k,p)$.

From above it follows that

$$\frac{1}{r^k} d_\varrho(A\cap S_\varrho(p,r)_{a(r)}, B\cap S_\varrho(p,r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0. \tag{3.12}$$

and

$$\frac{1}{r^k} d_\varrho(B\cap S_\varrho(p,r)_{b(r)}, C\cap S_\varrho(p,r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0. \tag{3.13}$$

Because

$$d_\varrho(A,C)\le d_\varrho(A,B)+d_\varrho(B,C) \quad \text{for } A,B,C\in E_0 \text{ and } B\ne\emptyset,$$

therefore from here and from the Darboux property of the set $B\in E_0$ we have

$$\begin{aligned} d_\varrho(A\cap S_\varrho(p,r)_{a(r)}, C\cap S_\varrho(p,r)_{b(r)}) &\le \\ \le d_\varrho(A\cap S_\varrho(p,r)_{a(r)}, B\cap S_\varrho(p,r)_{b(r)}) &+ \\ + d_\varrho(B\cap S_\varrho(p,r)_{b(r)}, C\cap S_\varrho(p,r)_{b(r)}). \end{aligned} \tag{3.14}$$

Hence, from (3.12) and (3.13) we get

$$\frac{1}{r^k} d_\varrho(A\cap S_\varrho(p,r)_{a(r)}, C\cap S_\varrho(p,r)_{b(r)}) \underset{r\to 0^+}{\longrightarrow} 0. \tag{3.15}$$

From the Darboux property of the sets $A,C\in E_0$ it follows that the pair (A,C) is (a,b)-clustered at the point p of the space (E,ϱ). From above and from (3.15) we get that $(A,C)\in T_{d_\varrho}(a,b,k,p)$. Hence and from Theorem 3 of [6] it follows that $(A,C)\in T_l(a,b,k,p)$ for $A,C\in A^{*D}_{p,k}$.

From Theorems 1.1, 2.2 and 3.2 we get immediately

COROLLARY 3.2. *If*

$$\frac{1}{r^k} a(r) \underset{r\to 0^+}{\longrightarrow} 0 \quad \text{and} \quad \frac{1}{r^k} b(r) \underset{r\to 0^+}{\longrightarrow} 0,$$

then for an arbitrary function $l\in F^*_\varrho$ *the tangency relation* $T_l(a,b,k,p)$ *is equivalence relation, i.e. is reflexive, symmetric and transitive in the class of sets* $A^{*D}_{p,k}$.

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