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ON MULTIDIMENSIONAL WIRTINGER'S TYPE DISCRETE INEQUALITIES

1. Preliminaries

In this paper we give one example of discrete inequality for functions of n independent variables which can be treated as some generalization of the discrete analogy of Wirtinger's inequality. We present one theorem which contains as a special cases both result obtained by Pachpatte in [2].

In the paper we shall use the following notions. Throughout the paper n is fixed positive integer greater than one. N denotes the set of positive integers and R the set of reals.

$$N_t := \{1, \dots, t\} \text{ for any } t \in N,$$
$$\overline{m} := (m_1, m_2, \dots, m_n), \quad m_i \in N, i = 1, \dots, n.$$

In the sequel the following cartesian product will be used .

$$N^k := \bigtimes_{i=1}^{k} N; \quad N_{\overline{m}} := \bigtimes_{i=1}^{n} N_{m_i} \quad (= N_{m_1} \times N_{m_2} \times \dots \times N_{m_n});$$

$$N_{\overline{m+1}} := \bigtimes_{i=1}^{n} N_{m_i+1}; \quad N_{\overline{n}}^{k} := \bigtimes_{i=1}^{k} N_n \quad (= N_n \times N_n \times \dots \times N_n).$$

$$A(n,k) := \{\overline{a} = (a_1, \dots, a_k) \in N_{\overline{n}}^{k} : a_1 < a_2 < \dots < a_k\}.$$

Let $r \in N$ and C be any subset of $N_r \times N_r$. We denote

$$C_{\cdot,s} := \{(i,s) : (i,s) \in C\}; \quad C_{s,\cdot} := \{(s,j) : (s,j) \in C\}$$

(1) $$c_s := \overline{\overline{C_{\cdot,s}}} + \overline{\overline{C_{s,\cdot}}} - \overline{\overline{C_{\cdot,s} \cap C_{s,\cdot}}}$$

for any fixed $s \in N_r$ (here $\overline{\overline{A}}$ denotes the cardinal of the set A).

Let $\overline{x} = (x_1, \dots, x_n)$. For any function $g : N^n \to R$ we define the difference operators $\triangle_i^1$, $i \in N_n$ as follows

$$\triangle_i^1 g(\overline{x}) = g(x_1, \dots, x_i + 1, \dots, x_n) - g(x_1, \dots, x_i, \dots, x_n).$$

Of course $\Delta_i^1 g(\overline{x})$ is some new function, say $G : N^n \to R$, therefore for G we can find Δ_j^1. Let for example $i < j \le n$. Then

$$\begin{aligned}\Delta_j^1 G(\overline{x}) &= G(x_1,\dots,x_j+1,\dots,x_n) - G(x_1,\dots,x_j,\dots,x_n)\\ &= \Delta_i^1\, g(x_1,\dots,x_j+1,\dots,x_n) - \Delta_i^1 g(x_1,\dots,x_j,\dots,x_n)\\ &= [g(x_1,\dots,x_i+1,x_{i+1},\dots,x_j+1,\dots,x_n)\\ &\qquad - g(x_1,\dots,x_i,x_{i+1},\dots,x_j,\dots,x_n)]\\ &\quad - [g(x_1,\dots,x_i+1,x_{i+1},\dots,x_j,\dots,x_n)\\ &\qquad - g(x_1,\dots,x_i,x_{i+1},\dots,x_j,\dots,x_n)].\end{aligned}$$

It is not difficult to define $\Delta_j^1 G(\overline{x})$ in the case $j \le i$. The above operator in relation to the function $g(\overline{x})$ is called second order difference and is denote by $\Delta_{i,j}^2 g(\overline{x})$. Therefore $\Delta_{i,j}^2 g(\overline{x}) = \Delta_j^1(\Delta_i^1 g(\overline{x}))$. It can be observed that $\Delta_{i,j}^2 g(\overline{x}) = \Delta_{j,i}^2 g(\overline{x})$. Following this way we can define higher order differences by the recurrence formula

$$\Delta_{i_1,\dots,i_j}^j g(\overline{x}) = \Delta_{i_1}^1(\Delta_{i_2,\dots,i_j}^{j-1} g(\overline{x}))$$

for $j > 1$ and $i_1,\dots,i_j \in N_n$.

If we express $\Delta_{i_1,\dots,i_j}^j g(\overline{x})$ as the sum of $2j$ terms everyone of which states function g for suitable values of the variables $x_1,\dots,x_n$ then we shall see that

$$\Delta_{i_1,\dots,i_j}^j g(\overline{x}) = \Delta_{k_1,\dots,k_j}^j g(\overline{x})$$

where $(k_1,\dots,k_j)$ is arbitrary permutation of the sequence $(i_1,\dots,i_j)$.

Let $\overline{a} \in A(n,k)$, instead of $\Delta_{a_1,\dots,a_k}^k g(\overline{x})$ we shall write $\Delta_{\overline{a}}^k g(\overline{x})$

$$\sum_{\overline{y}\in N_{\overline{m}}} g(\overline{y}) := \sum_{y_1=1}^{m_1}\sum_{y_2=1}^{m_2}\cdots\sum_{y_n=1}^{m_n} g(y_1,y_2,\dots,y_n).$$

We shall say, a real valued function g defined on $N_{\overline{m+1}}$ is of the class $P(\overline{m})$ if

$$(2)\qquad \begin{cases} g(1,x_2,\dots,x_n) = \dots = g(x_1,\dots,x_{n-1},1) = 0\\ g(m_1+1,x_2,\dots,x_n) = \dots = g(x_1,\dots,x_{n-1},m_n+1) = 0\end{cases}$$

for all $x_1 \in N_{m_1+1},\dots,x_n \in N_{m_n+1}$.

Suppose $G(\overline{x}) = \Delta_{i_1,\dots,i_j}^j g(\overline{x})$ for some $1 \le i_1 < \dots < i_j \le n$, and let $s \in N_n \setminus \{i_1,\dots,i_j\}$. Then

$$(3)\qquad \begin{aligned}\sum_{x_s=u}^{v} \Delta_s^1 G(\overline{x}) &= \sum_{x_s=u}^{v} [G(x_1,\dots,x_s+1,\dots,x_n) - G(x_1,\dots,x_s,\dots,x_n)]\\ &= G(x_1,\dots,v+1,\dots,x_n) - G(x_1,\dots,u,\dots,x_n).\end{aligned}$$

If $u = 1$ then $G(x_1, \ldots, 1, \ldots, x_n) = 0$ as the result of G is then difference of elements all equal zero. Similarly if $v = m_s$ then $G(x_1, \ldots, m_s + 1, \ldots, x_n) = 0$. Therefore by (3) we get

$$\sum_{x_s=1}^{v} \Delta_s^1 G(\overline{x}) = G(x_1, \ldots, v+1, \ldots, x_n) \tag{4}$$

and

$$\sum_{x_s=u}^{m_s} \Delta_s^1 G(\overline{x}) = -G(x_1, \ldots, u, \ldots, x_n). \tag{5}$$

Furthermore

$$\sum_{x_s=u}^{v} \Delta_s^1 g(\overline{x}) = g(x_1, \ldots, v+1, \ldots, x_n) - g(x_1, \ldots, u, \ldots, x_n)$$

gives

$$\sum_{x_s=1}^{v} \Delta_s^1 g(\overline{x}) = g(x_1, \ldots, v+1, \ldots, x_n) \tag{6}$$

and

$$\sum_{x_s=u}^{m_s} \Delta_s^1 g(\overline{x}) = -g(x_1, \ldots, u, \ldots, x_n). \tag{7}$$

We use the customary convention

$$\sum_{y_1=1}^{x_1} \cdots \sum_{y_t=1}^{x_t} g(\overline{y}) = 0 \tag{8}$$

if for any $i \in N_t$ there is $x_i = 0$.

We denote

$$|D^k g(\overline{x})| := \Big\{ \sum_{\overline{a} \in A(n,k)} |\, \Delta_{\overline{a}}^{k} g(\overline{x})|^2 \Big\}^{1/2}. \tag{9}$$

Now we recall two elementary inequalities which hold for real numbers. Let C be any subset of $N_r \times N_r$, and consider the sum $\sum_{(i,j)\in C} d_i d_j$ where $d_t,\ t = 1, \ldots, r$ are real constants. In this sum every element d_t appears c_t – times, where c_t is defined by (1). For arbitrary product $d_i d_j$ we have $d_i d_j \le \frac{1}{2}(d_i^2 + d_j^2)$. Therefore $\sum_{(i,j)\in C} d_i d_j \le \sum_{(i,j)\in C} \frac{1}{2}(d_i^2 + d_j^2)$. Hence by the remark we have just given, we obtain

$$\sum_{(i,j)\in C} d_i d_j \le \sum_{t=1}^{r} \frac{1}{2} c_t (d_t^2). \tag{10}$$

Let now d_i, $i = 1, \ldots, r$ be any nonnegative real numbers, and $p > 1$. If we put in the Hölder's inequality

$$\sum_{t=1}^{r} d_t b_t \le \Big(\sum_{t=1}^{r} d_t^p\Big)^{1/p} \Big(\sum_{t=1}^{r} b_t^q\Big)^{1/q}$$

$b_t \equiv 1$, and next we use monotonicity property of the function $f(x) = x^p$ we obtain

$$\Big(\sum_{t=1}^{r} d_t\Big)^p \le r^{p/q}\Big(\sum_{t=1}^{r} d_t^p\Big).$$

Since $p/q = p - 1$ therefore we have

$$\Big(\sum_{t=1}^{r} d_t\Big)^p \le r^{p-1} \sum_{t=1}^{r} (d_t)^p. \tag{11}$$

Let us see that inequality (11) remains true for the case $p = 1$. If $p \in (0, 1)$ then the function $f(x) = x^p$ is a concave function on the set $[0, \infty)$ therefore by Petrović's inequality (see e.g. [1]) we have

$$\sum_{t=1}^{r} (d_t)^p \ge \Big(\sum_{t=1}^{r} d_t\Big)^p + (r-1)0^p.$$

Hence

$$\Big(\sum_{t=1}^{r} d_t\Big)^p \le \sum_{t=1}^{r} (d_t)^p. \tag{12}$$

2. The main result

THEOREM. *Let*

p_s, $s = 1, \ldots, r$ be positive real numbers,
$C \subset N_r \times N_r$ any fixed subset, $\overline{m} = (m_1, \ldots, m_n)$,
f_s, $s = 1, \ldots, r$ belong to the class $P(\overline{m})$.

Then

$$\sum_{\overline{x} \in N_{\overline{m}}} \sum_{(i,j) \in C} |f_i(\overline{x})|^{p_i} |f_j(\overline{x})|^{p_j} \le \tag{13}$$

$$\le \sum_{q=1}^{r} \frac{1}{2} c_q \left[2^k \binom{n}{k}\right]^{-2p_q} A_{k,q} \sum_{\overline{a} \in A(n,k)} \sum_{\overline{x} \in N_{\overline{m}}} |\Delta_{\overline{a}}^{k} f_q(\overline{x})|^{2p_q} \le$$

$$\le \sum_{q=1}^{r} \frac{1}{2} c_q \left[2^k \binom{n}{k}\right]^{-2p_q} B_{k,q} \sum_{\overline{x} \in N_{\overline{m}}} |D^k f_q(\overline{x})|^{2p_q}$$

for all $\overline{x} \in N_{\overline{m}}$, *and arbitrary* $k \in N_n$, *where*

$$(14)\qquad \begin{cases} e_k = \max\limits_{\overline{a}\in A(n,k)} \prod\limits_{s=1}^{k} m_{a_s}, \\ A_{k,q} = \begin{cases} \left[\binom{n}{k}\right]^{-2p_q-1} (e_k)^{2p_q} & \text{if } p_q \geq 1/2 \\ e_k & \text{if } p_q \in (0,1/2), \end{cases} \end{cases}$$

$$B_{k,q} = \begin{cases} A_{k,q} & \text{if } p_q \geq 1 \\ \left[\binom{n}{k}\right]^{1-p_q} A_{k,q} & \text{if } p_q \in (0,1) \end{cases}$$

c_q, $|D^k(.)|$ *are defined by* (1) *and* (9) *respectively.*

Proof. Let us take arbitrary $q \in N_r$, $k \in N_n$, $\overline{a} \in A(n,k)$, $\overline{x} = (x_1,\ldots,x_n) \in N_{\overline{m}}$. For any $\overline{y} = (y_1,\ldots,y_n)$ we denote

$$\overline{xy}_{\overline{a}} = (x_1,\ldots,x_{a_1-1},y_{a_1},x_{a_1+1},\ldots,x_{a_k-1},y_{a_k},x_{a_k+1},\ldots,x_n).$$

Hence by (4)

$$\sum_{y_{a_1}=1}^{x_{a_1}-1}\cdots\sum_{y_{a_k}=1}^{x_{a_k}-1} \Delta^k_{\overline{a}} f_q(\overline{xy}_{\overline{a}})$$

$$= \sum_{y_{a_1}=1}^{x_{a_1}-1}\cdots\sum_{y_{a_{k-1}}=1}^{x_{a_{k-1}}-1} \Big[\sum_{y_{a_k}=1}^{x_{a_k}-1} \Delta^1_{a_k}(\Delta^{k-1}_{a_1,\ldots,a_{k-1}} f_q(\overline{xy}_{\overline{a}}))\Big]$$

$$= \sum_{y_{a_1}=1}^{x_{a_1}-1}\cdots\sum_{y_{a_{k-1}}=1}^{x_{a_{k-1}}-1} \Delta^{k-1}_{a_1,\ldots,a_{k-1}} f_q(x_1,\ldots,y_{a_{k-1}},\ldots,x_{a_k},\ldots,x_n).$$

Following this way by (4) and (6) we obtain

$$(15)\qquad \sum_{y_{a_1}=1}^{x_{a_1}-1}\cdots\sum_{y_{a_k}=1}^{x_{a_k}-1} \Delta^k_{\overline{a}} f_q(\overline{xy}_{\overline{a}}) = f_q(\overline{x}).$$

Let us observe that by (8) equality (15) holds if $x_{a_i} = 1$ for some a_i.

Let now $b = (b_1,\ldots,b_z)$, $z \leq k$ be such that

$$a_1 \leq b_1 < b_2 < \ldots < b_z \leq a_k$$

and for every $s \in N_z$ there exists $j \in N_k$ such that $b_s = a_j$. Let us take $b_1 = a_{i_1}$, $b_2 = a_{i_2},\ldots,b_z = a_{i_z}$. We calculate the sum

$$(16)\qquad \sum_{y_{a_1}=1}^{x_{a_1}-1}\cdots\sum_{y_{b_1}=x_{b_1}}^{m_{b_1}}\cdots\sum_{y_{b_z}=x_{b_z}}^{m_{b_z}}\cdots\sum_{y_{a_k}=1}^{x_{a_k}-1} \Delta^k_{\overline{a}} f_q(\overline{xy}_{\overline{a}}).$$

For this by (4) we have

$$\sum_{y_{a_k}=1}^{x_{a_k}-1} \Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}}) = \Delta_{a_1,\ldots,a_{k-1}}^{k-1} f_q(x_1,\ldots,y_{b_z}\ldots,y_{a_{k-1}},x_{a_{k-1}+1},\ldots,x_n).$$

Summing as long as the first inner sum will be $\sum_{y_{b_z}=x_{b_z}}^{m_{b_z}}$ we obtain the following form of the summation expression

$$\text{(17)}\qquad \sum_{y_{a_k}=1}^{x_{a_k}-1} \cdots \sum_{y_{b_z}=x_{b_z}}^{m_{b_z}} \Delta_{a_1,\ldots,a_{i_z}}^{i_z} f_q(x_1,\ldots,y_{a_{i_z}},x_{a_{i_z}+1},\ldots,x_n).$$

Of course if $b_z = a_k$ then $i_z = k$ and till now we make no summation, so in this case (17) coincides with (16). Now by (5) we obtain

$$\sum_{y_{b_z}=x_{b_z}}^{m_{b_z}} \Delta_{a_1,\ldots,a_{i_z}}^{i_z} f_q(\ldots,y_{a_{i_z}},\ldots) = -\,\Delta_{a_1,\ldots,a_{i_z-1}}^{i_z-1} f_q(\ldots,x_{b_z},\ldots).$$

Proceeding this way, every time we must calculate the sum $\sum_{y_{b_i}=x_{b_i}}^{m_{b_i}}$ we get by (5) the factor (-1) before the main expression, if the summation is of the form $\sum_{y_{a_i}=1}^{x_{a_i}-1}$ the sign does nor change. In both the cases the order of difference decreases and the last variable of the function f_q which was till denoted by y_i would be now x_i. Since we have z sum of the type with m_{b_i} as the upper limit of summation therefore we get finally

$$\sum_{y_{a_1}=1}^{x_{a_1}-1} \cdots \sum_{y_{b_z}=x_{b_z}}^{m_{b_z}} \cdots \sum_{y_{a_k}=1}^{x_{a_k}-1} \Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}}) = (-1)^z f_q(\overline{x}).$$

From this we have

$$\text{(18)}\qquad |f_q(\overline{x})| \le \sum_{y_{a_1}=1}^{x_{a_1}-1} \cdots \sum_{y_{b_z}=x_{b_z}}^{m_{b_z}} \cdots \sum_{y_{a_k}=1}^{x_{a_k}-1} |\,\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})|.$$

Inequality (18) holds for each sequence b of the properties described above.

Let us remark that k-dimensional cube $\{1,\ldots,m_{a_1}\}\times\ldots\times\{1,\ldots,m_{a_k}\}$ can be decomposed by the point $\overline{x}$ into 2^k separable cubes

$$\{1,\ldots,x_{a_1}-1\}\times\{1,\ldots,x_{a_2}-1\}\times\ldots\times\{1,\ldots,x_{a_k}-1\},$$
$$\{x_{a_1},\ldots,m_{a_1}\}\times\{1,\ldots,x_{a_2}-1\}\times\ldots\times\{1,\ldots,x_{a_k}-1\},\ldots,$$
$$\{x_{a_1},\ldots,m_{a_1}\}\times\{x_{a_2},\ldots,m_{a_2}\}\times\ldots\times\{1,\ldots,x_{a_k}-1\},\ldots,$$
$$\{x_{a_1},\ldots,m_{a_1}\}\times\{x_{a_2},\ldots,m_{a_2}\}\times\ldots\times\{x_{a_k},\ldots,m_{a_k}\}$$

and on every of these cubes the estimation (18) holds. Some of these cubes can be void sets if any of x_{a_i} is equal 1. In this case the estimation (18) remains true, because by (8) the sum on the right side of (18) is equal zero and the same is with the left hand side (because if $x_{a_i} = 1$ then $f_q \in P(\overline{m})$ implies $f_q(\overline{x}) = 0$). Therefore we have

$$2^k|f_q(\overline{x})| \leq \sum_{y_{a_1}=1}^{m_{a_1}} \sum_{y_{a_2}=1}^{m_{a_2}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})|. \tag{19}$$

The cardinal of the set $A(n,k)$ is $\binom{n}{k}$. For every point $\overline{a} \in A(n,k)$ estimation (19) holds. Hence we get

$$\binom{n}{k} 2^k|f_q(\overline{x})| \leq \sum_{\overline{a}\in A(n,k)} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})|. \tag{20}$$

for all $\overline{x} \in N_{\overline{m}}$. From (20) there follows

$$|f_q(\overline{x})|^{p_q} \leq \left[2^k \binom{n}{k}\right]^{-p_q} \left\{ \sum_{\overline{a}\in A(n,k)} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})| \right\}^{p_q}. \tag{21}$$

Applying (21) we get from (10)

$$\begin{aligned} \sum_{(i,j)\in C} |f_i(\overline{x})|^{p_i} |f_j(\overline{x})|^{p_j} &\leq \sum_{q=1}^{r} \frac{1}{2} c_q |f_q(\overline{x})|^{2p_q} \\ &\leq \sum_{q=1}^{r} \frac{1}{2} c_q \left[2^k \binom{n}{k}\right]^{-2p_q} \left\{ \sum_{\overline{a}\in A(n,k)} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})| \right\}^{2p_q}. \end{aligned} \tag{22}$$

Applying $k+1$ times (12) in the case $2p_q \in (0,1)$ and (11) in the case $2p_q \geq 1$ we get

$$\begin{aligned} &\left\{ \sum_{\overline{a}\in A(n,k)} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})| \right\}^{2p_q} \\ &\qquad \leq \sum_{\overline{a}\in A(n,k)} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})|^{2p_q} \end{aligned}$$

or

$$\begin{aligned} &\left\{ \sum_{\overline{a}\in A(n,k)} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})| \right\}^{2p_q} \\ &\leq \binom{n}{k}^{2p_q-1} \sum_{\overline{a}\in A(n,k)} \left(\prod_{s=1}^{k} m_{a_s}\right)^{2p_q-1} \sum_{y_{a_1}=1}^{m_{a_1}} \cdots \sum_{y_{a_k}=1}^{m_{a_k}} |\Delta_{\overline{a}}^{k} f_q(\overline{x}\overline{y}_{\overline{a}})|^{2p_q} \end{aligned}$$

respectively. Therefore by (22) we obtain

$$(23)\quad \sum_{\overline{x}\in N_{\overline{m}}}\sum_{(i,j)\in C}|f_i(\overline{x})|^{p_i}|f_j(\overline{x})|^{p_j}$$
$$\le \sum_{q=1}^{r}\frac{1}{2}c_q\left[2^k\binom{n}{k}\right]^{-2p_q}\sum_{\overline{x}\in N_{\overline{m}}}\sum_{\overline{a}\in A(n,k)}\sum_{y_{a_1}=1}^{m_{a_1}}\cdots\sum_{y_{a_k}=1}^{m_{a_k}}|\Delta_{\overline{a}}^{k}f_q(\overline{x}\overline{y}_{\overline{a}})|^{2p_q}$$

or by (14)

$$(24)\quad \sum_{\overline{x}\in N_{\overline{m}}}\sum_{(i,j)\in C}|f_i(\overline{x})|^{p_i}|f_j(\overline{x})|^{p_j}$$
$$\le \sum_{q=1}^{r}\frac{1}{2}c_q\left[2^k\binom{n}{k}\right]^{-2p_q}\binom{n}{k}^{2p_q-1}(e_k)^{2p_q-1}$$
$$\times\sum_{\overline{x}\in N_{\overline{m}}}\sum_{\overline{a}\in A(n,k)}\sum_{y_{a_1}=1}^{m_{a_1}}\cdots\sum_{y_{a_k}=1}^{m_{a_k}}|\Delta_{\overline{a}}^{k}f_q(\overline{x}\overline{y}_{\overline{a}})|^{2p_q}$$

in the both considered cases respectively. Let us see that

$$\sum_{\overline{x}\in N_{\overline{m}}}\sum_{\overline{a}\in A(n,k)}\sum_{y_{a_1}=1}^{m_{a_1}}\cdots\sum_{y_{a_k}=1}^{m_{a_k}}|\Delta_{\overline{a}}^{k}f_q(\overline{x}\overline{y}_{\overline{a}})|^{2p_q}$$
$$=\sum_{\overline{a}\in A(n,k)}\left[\prod_{s=1}^{k}m_{a_s}\right]\sum_{\overline{x}\in N_{\overline{m}}}|\Delta_{\overline{a}}^{k}f_q(\overline{x})|^{2p_q}.$$

This equality together with (14) used in (23) and (24) leads us to the first estimation in (13). To obtain the second one we shall apply inequalities (11) or (12) to the sum $\sum_{\overline{a}\in A(n,k)}|\Delta_{\overline{a}}^{k}f_q(\overline{x})|^{2p_q}$ as follows

$$\sum_{\overline{a}\in A(n,k)}|\Delta_{\overline{a}}^{k}f_q(\overline{x})|^{2p_q}=\left\{\left[\sum_{\overline{a}\in A(n,k)}|\Delta_{\overline{a}}^{k}f_q(\overline{x})|^{2p_q}\right]^{1/p_q}\right\}^{p_q}$$
$$\le\left\{\binom{n}{k}^{-1+1/p_q}\sum_{\overline{a}\in A(n,k)}|\Delta_{\overline{a}}^{k}f_q(\overline{x})|^{2}\right\}^{p_q}=\left[\binom{n}{k}\right]^{-1+1/p_q}|D^kf_q(\overline{x})|^{2p_q}$$

in the case $p_q\in(0,1)$. Similary we proceed in the case $p_q\ge 1$. Q.E.D.

Remark. Note that the presented result contains as special cases both inequalities obtained by Pachpatte in [2]. Namely if $n=2$, $r=3$, $C=\{(1,2),(2,3),(3,1)\}$, $k=2$, $p_1,p_2,p_3\ge 1$ we obtain Theorem 1; if $r=3$, $k=1$, $C=\{(1,2),(2,3),(3,1)\}$, $p_1,p_2,p_3\ge 1$ Theorem 2 of the mentioned work. Furthermore the proof seems to be clear than this in [2]. For discrete inequalities of this type for function of one independent variable see references contained therein.

References

[1] D. S. Mitrinović, *Nejednakosti*, Gradevinska Knjiga, Beograd 1965.
[2] B. G. Pachpatte, *On certain multidimensional discrete inequalities*, Chinese J. Math. 14,4 (1986), 185-195.

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