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SINGULAR LIMIT IN A PARABOLIC EQUATION

1. Introduction

Our aim here is to study the singular limit, when $\varepsilon \to 0$, in a parabolic problem

$$(1)\qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(t,u), & x \in \Omega \subset \mathbb{R}^n,\ t > 0\,, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega\,, \\ Bu = 0 & \text{for } x \in \partial\Omega\,, \end{cases}$$

with a bounded smooth domain Ω, $\varepsilon > 0$ and the boundary operator B of either the Dirichlet type ($B = \mathrm{Id}$) or the Neumann type ($B = \frac{\partial}{\partial n}$, n is the normal vector to $\partial\Omega$). We want to estimate the difference between u and the solution $y = y(t,x)$ of the limit problem

$$(2)\qquad \begin{cases} y_t = f(t,y), & t > 0\,, \\ y(0,x) = u_0(x), & x \in \Omega\,, \end{cases}$$

the central interest being estimates on bounded time interval $[0,T]$. The problem of long time behaviour (when $t \to \infty$) of a pair u, y has been considered previously, e.g. in [4, 7]. This paper has been prompted by recent studies on the so-called "bistable" reaction-diffusion equation and the phenomenon of transition layers (cf. [3, 8, 1, 12]). The simplest non-numerical way to study the preliminary phase of creation of the layer structure for the solution of (1) is to observe this phenomenon for the solution y of (2) and to give the ε-dependent estimates for the difference $(u - y)$ suitable for small ε.

2. Assumptions

The function f considered here is assumed to be of the class $\mathbb{C}^1$ with respect to t and $\mathbb{C}^2$ with respect to u on bounded subsets of $[0,\infty) \times \mathbb{R}$ and to have only three different zeros; $f(t,\pm 1) = f(t,0) = 0$ with $f_u(t,\pm 1) < 0$ and $f_u(t,0) \geq 0$. As a typical example of such f let us consider the function

$\frac{1}{2}(u - u^3)$. We denote

$$m = \sup f_u(t,u), \qquad M = \sup f_{uu}(t,u)\,,$$

where the supremums are taken over $[0,T]\times[-1,1]$. Let $\partial\Omega \in \mathrm{C}^{2+\mu}$ (Hölder space) with some $\mu > 0$. We further assume that the initial function $u_0 \in \mathrm{C}^{2+\mu}(\overline{\Omega})$ satisfies the compatibility conditions:

$$u_0(x) = 0 = \Delta u_0(x) \quad \text{for } x \in \partial\Omega \text{ in the Dirichlet case,}$$
$$\frac{\partial u_0}{\partial n}(x) = 0 \quad \text{for } x \in \partial\Omega \text{ in the Neumann case.}$$

DEFINITION. A function $u \in L^\infty((0,T)\times\Omega) \cap L^2(0,T;V)$ ($V = H_0^1(\Omega)$ in the Dirichlet case, $V = H^1(\Omega)$ in the Neumann case) is said to be a weak solution of (1) in $D = (0,T)\times\Omega$, if the identity

$$\iint_D [u\phi_t - \varepsilon^2\nabla u\nabla\phi + f(t,u)\phi]\,dx\,dt = -\int_\Omega u_0\phi(0,x)\,dx \tag{3}$$

holds for every $\phi \in \mathrm{C}^1(\overline{D})$ with $\phi(T,x) = 0$ and, in the Dirichlet case, $\phi(t,x) = 0$ for $x \in \partial\Omega$. Weak solutions of the problem (2) are defined in a similar way; we formally set $\varepsilon = 0$ in (3).

Remark 1. As long as we consider initial functions u_0 with values in the interval $[-1,1]$, it is a familiar consequence of the Maximum Principle and the condition $f(t,\pm 1) = 0$ that the values of the corresponding solutions u also belong to $[-1,1]$. Thus the form of f outside $[-1,1]$ for such u_0 is invalid.

We will assume in the sequel that the values of u_0 belong to $[-1,1]$. With all this assumptions there is a unique global solution u to (1), moreover with u_t, Δu continuous in $[0,\infty)\times\overline{\Omega}$ (see [5, 11]) and u_{tx_i} belonging to $L^2(D)$ ([11], p. 513).

3. Introductory observation

Our first goal is to show that when ε tends to zero in (1) the corresponding weak solution u^ε tends to the weak solution y of (2). Let us first observe that for all ε, $|u^\varepsilon(t,x)| \le 1$. Next, by the Schwarz inequality

$$\varepsilon \iint_D \varepsilon\nabla u^\varepsilon\nabla\phi\,dx\,dt \le \varepsilon\|\varepsilon\nabla u^\varepsilon\|_{L^2(D)}\|\nabla\phi\|_{L^2(D)}\,,$$

hence the middle component in (3) will vanish when $\varepsilon \to 0$, provided we justify uniform in ε estimate of $\|\varepsilon\nabla u^\varepsilon\|_{L^2(D)}$.

LEMMA 1. *The quantity $\|\varepsilon\nabla u^\varepsilon\|_{L^2(D)}$ is bounded uniformly in ε for $\varepsilon \in [0,\varepsilon_0]$.*

Proof. To simplify notation we put $u^\varepsilon = u$ from now on. Multiplying the equation in (1) by Δu and integrating over $D_t = (0,t) \times \Omega$ we get

$$\iint u_t \Delta u \, dx\, ds = \varepsilon^2 \iint (\Delta u)^2 \, dx\, ds + \iint f(s,u) \Delta u \, dx\, ds$$

or integrating by parts, using the boundary condition and the estimate $f_u \le m$,

$$\int_\Omega \sum_i u_{x_i}^2(t,x)\, dx - \int_\Omega \sum_i u_{0x_i}^2 \, dx + 2\varepsilon^2 \iint (\Delta u)^2 \, dx\, ds \le 2m \iint \sum_i u_{x_i}^2(s,x)\, dx\, ds, \quad t \in (0,T]. \tag{4}$$

As a first consequence of (4), using the Gronwall inequality, we get

$$\int_\Omega \sum_i u_{x_i}^2(t,x)\, dx \le \int_\Omega \sum_i u_{0x_i}^2 \, dx e^{2mt}. \tag{5}$$

Next, from (4) and (5) we get

$$\begin{aligned} 2\varepsilon^2 \int_0^T \int_\Omega (\Delta u)^2 \, dx\, d\tau &\le \int_\Omega \sum_i u_{0x_i}^2 \, dx \left(1 + 2m \frac{e^{2mT} - 1}{2m}\right) \\ &= \int_\Omega \sum_i u_{0x_i}^2 \, dx e^{2mT}. \end{aligned}$$

Basing on the property of the Newtonian potential [9], p. 235;

$$\Big(\sum_{i,j} \|h_{x_i x_j}\|_{L^2(\Omega)}^2\Big)^{1/2} = \|\Delta h\|_{L^2(\Omega)},$$

valid for $h \in H_0^2(\Omega)$ and the estimate of intermediate derivatives (e.g. [9], p. 171) we find that the expression $(\|\Delta h\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2)^{1/2}$ defines an equivalent norm on $H_0^2(\Omega)$. For arbitrary $h \in H_0^2(\Omega)$ we thus have an estimate

$$\int_\Omega \sum_i h_{x_i}^2 \, dx \le C\Big(\int_\Omega (\Delta h)^2 \, dx + \int_\Omega h^2 \, dx\Big) \tag{6}$$

($C = C(n, \Omega)$) and, continuing our previous calculations, we find that

$$\begin{aligned} \varepsilon^2 \int_0^T \int_\Omega \sum_i u_{x_i}^2 dx\, dt &\le C\varepsilon^2 \int_0^T \int_\Omega (\Delta u)^2 \, dx\, dt + C\varepsilon^2 \int_0^T \int_\Omega u^2 dx\, dt \\ &\le \frac{C}{2} \int_\Omega \sum_i u_{0x_i}^2 \, dx e^{2mT} + C\varepsilon_0^2 T |\Omega| \end{aligned} \tag{7}$$

($|\Omega|$ denotes the measure of Ω). The proof is completed.

Remark 2. For the case of the Neumann boundary condition, instead of (6) we will use an estimate ([4], p. 12)

$$\lambda\|\nabla h\|^2_{L^2(\Omega)} \le \|\Delta h\|^2_{L^2(\Omega)} \tag{6'}$$

valid for every $h \in H^2(\Omega)$ with $\frac{\partial h}{\partial n} = 0$ on $\partial\Omega$ and the smallest positive eigenvalue λ of $-\Delta$ with homogeneous Neumann boundary condition.

At this point the usual procedure would be to use the compactness argument for the family $\{u^\varepsilon\}$ to be able to pass to the limit over a sequence $\varepsilon_n \to 0$. Instead we will give a proportional to ε estimate in $L^2(D)$ (and other L^p spaces) for the difference $u^\varepsilon - y$, to justify the limit passage generally when $\varepsilon \to 0$.

We shall start with some remarks concerning the properties and smoothnes of the solution $y(t,x)$.

Boundary condition for y. It is easy to see that the (zero order) compatibility conditions required for u_0 are automatically preserved by equation (2) for y.

For the Dirichlet condition; $u(x_0) = 0$, $x_0 \in \partial\Omega$, with our assumption $f(t,0) = 0$ we find that the solution of (2) satisfies $y(t,x_0) = 0$. For the Neumann case, when $x_0 \in \partial\Omega$, the normal derivative of y satisfies the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial n}(t,x_0)\right) = f_y(t,y(t,x_0))\frac{\partial y}{\partial n}(t,x_0)$$

or

$$\frac{\partial y}{\partial n}(t,x_0) = \frac{\partial y}{\partial n}(0,x_0)\exp\Big(\int_0^t f_y(s,y(s,x_0))\,ds\Big) = 0 \tag{8}$$

which is equal to zero because $\frac{\partial y}{\partial n}(0,x_0) = 0$.

The equation for y. It is clear that the dependence of y on x entres through the dependence of $y(0,x)$ on x. There is no other connection between $y(t,x_1)$ and $y(t,x_2)$ for $t > 0$.

We will now calculate the Laplacian of y. For f twice differentiable with respect to y we have

$$y_{tx_i} = f_y(t,y)y_{x_i}, \qquad y_{tx_ix_i} = f_{yy}(t,y)y^2_{x_i} + f_y(t,y)y_{x_ix_i},$$

so that

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_i y^2_{x_i} = f_y(t,y)\sum_i y^2_{x_i}, \tag{9}$$

$$\frac{\partial}{\partial t}\Delta y = f_{yy}(t,y)\sum_i y^2_{x_i} + f_y(t,y)\Delta y. \tag{10}$$

Solving this system of equations we find an expression for $\sum_i y_{x_i}^2$:

$$\sum_i y_{x_i}^2(t,x) = \sum_i u_{0x_i}^2(x) \exp\Big(2 \int_0^t f_y(\tau, y(\tau,x))\, d\tau\Big), \tag{11}$$

and, with the use of (11), also for Δy

$$\Delta y(t,x) = \Big(\Delta u_0(x) + \int_0^t f_{yy}(\tau, y(\tau,x)) \sum_i u_{0x_i}^2(x) \cdot \exp\Big(\int_0^\tau f_y(s, y(s,x))\, ds\Big) d\tau\Big) \exp\Big(\int_0^t f_y(s, y(s,x))\, ds\Big). \tag{12}$$

Observe, that the solution y of (2) solves simultaneously the problem

$$\begin{cases} y_t = \varepsilon^2 \Delta y + f(t, y(t,x)) - \varepsilon^2 \Delta y, \\ y(0,x) = u_0(x), \quad x \in \Omega, \\ By = 0, \quad x \in \partial\Omega. \end{cases} \tag{13}$$

The "free term" $-\varepsilon^2 \Delta y$ is given by (12).

We are now ready to formulate the estimate of $w = u - y$ in L^p spaces.

LEMMA 2. *For every $k \in \mathbb{N}$ the following estimate holds*

$$\|w(t,\cdot)\|_{L^{2k}(\Omega)} \le \varepsilon \|\nabla u_0\|_{L^{2k}(\Omega)} \Big(\frac{2k-1}{2} t\Big)^{1/2} e^{mt}. \tag{14}$$

Proof. The function w solves the problem

$$\begin{cases} w_t = \varepsilon^2 \Delta w + f_u(t, \widetilde{u}) w + \varepsilon^2 \Delta y, \\ w(0,x) = 0, \quad x \in \Omega, \\ Bw = 0, \quad x \in \partial\Omega. \end{cases} \tag{15}$$

Multiplying the equation in (15) by e^{-mt} ($f_u \le m$) we get the equation for $W = we^{-mt}$

$$W_t = \varepsilon^2 \Delta W + (f_u - m) W + \varepsilon^2 \Delta(ye^{-mt}). \tag{16}$$

Multiplying (16) by W^{2k-1} and integrating over Ω, we find

$$\frac{1}{2k} \frac{d}{dt} \int_\Omega W^{2k}\, dx \le -\varepsilon^2 \frac{2k-1}{k^2} \int_\Omega \sum_i (W^k)_{x_i}^2\, dx - \varepsilon^2 \frac{2k-1}{k} \int_\Omega \sum_i (ye^{-mt})_{x_i} W^{k-1} (W^k)_{x_i}\, dx =: -J_1 - J_2.$$

Now, from the Cauchy inequality

$$|J_2| \le \varepsilon^2 \frac{2k-1}{k^2} \int_\Omega \sum_i (W^k)_{x_i}^2 dx + \frac{\varepsilon^2}{4}(2k-1) \int_\Omega \sum_i ((ye^{-mt})_{x_i})^2 W^{2k-2} dx,$$

and further, from the Hölder inequality ($p = \frac{k}{k-1}$, $q = k$) and (11)

$$|J_2| \le J_1 + \frac{\varepsilon^2}{4}(2k-1)\Big(\int_\Omega \Big(\sum_i u_{0x_i}^2\Big)^k dx\Big)^{1/k}\Big(\int_\Omega W^{2k}\,dx\Big)^{1-1/k}.$$

We achieve an estimate

$$\frac{d}{dt}\int_\Omega W^{2k}(t,x)\,dx \le \frac{\varepsilon^2}{2}k(2k-1)\|\nabla u_0\|^2_{L^{2k}(\Omega)}\Big(\int_\Omega W^{2k}(t,x)\,dx\Big)^{1-1/k}$$

or, when integrated

$$\|W(t,\cdot)\|^2_{L^{2k}(\Omega)} \le \frac{\varepsilon^2}{2}(2k-1)\|\nabla u_0\|^2_{L^{2k}(\Omega)}t,$$

where we note that $W(0,x) = 0$ in Ω. The proof is completed.

For $k = 1$, integrating (14) over $[0,T]$, we obtain proportional to ε estimate of w in $L^2(D)$. This estimate together with Lemma 1 justify the limit passage in (3) with ε to 0. Weak solution of (1) tends to the weak solution of (2) (being also strong solution of (2)).

4. Main estimates

We are ready to give estimates of the difference $u - y$ more accurate than in Lemma 2.

THEOREM 1. *For the difference w the following estimate holds*

$$\|w(t,\cdot)\|_{L^\infty(\Omega)} \le \varepsilon^2\Big\||\Delta u_0| + \sum_i u_{0x_i}^2 M\frac{e^{mt}-1}{m}\Big\|_{L^\infty(\Omega)} te^{mt}, \tag{17}$$

with $f_u \le m$ and $f_{uu} \le M$ for $(t,x) \in [0,T]\times[-1,1]$ (when $m = 0$ then the term $\frac{e^{mt}-1}{m}$ in (17) should be replaced by t).

The proof follows directly from Proposition 1 if we apply it to W solving (16) and use the immediate estimate of $\varepsilon^2\Delta y$ following from (12).

Application. Consider the standard nonlinearity $f(z) = \frac{1}{2}(z - z^3)$. We may ask how large τ should be to allow $y(t,x_0)$ to grow from value 0,1 for $t = 0$ to value 0,9 for $t = \tau$. The problem

$$y_t = \frac{1}{2}(y - y^3), \quad y(0,x_0) = 0,1,$$

is solvable by

$$y(t,x_0) = \Big(\frac{0,01e^t}{0,99 + 0,01e^t}\Big)^{1/2},$$

hence $\tau \cong 6,045$. The behaviour of y reflects the behaviour of u (for our special f). For this f we have $m = \frac{1}{2}$, $M = 3$, hence for $t = 6$, as a consequence of (17):

$$(18)\qquad |u(6,x_0)-y(6,x_0)| \le \varepsilon^2 \Big\| |\Delta u_0| + \sum_i u_{0x_i}^2 6(e^3-1)\Big\|_{L^\infty(\Omega)} 6e^3$$

($6e^3 \cong 120,5$). For $\varepsilon \le 10^{-3}$ it is reasonable to have the right side in (18) of the order 0,1 (this is, of course, a requirement on u_0). Hence, for such ε and u_0 we have $u(6,x_0) \ge y(6,x_0) - 0,1 \ge 0,8$, so that $u(t,x_0)$ growth from value 0,1 for $t=0$ to (at least) value 0,8 for $t=6$. Similar estimates will be made for the points x_1 at which u_0 is negative. We thus see how, in a relatively short time ($\tau = 6$), the initial variation $u_0(x_0) - u_0(x_1) = 0,2$ has been increased to the value $u(6,x_0) - u(6,x_1) \ge 1,6$. This is just a phenomenon of creation of layer structure for the solution u. The graph of $u(t,\cdot)$ consists of parts either close to $+1$ or close to -1 and a number of sharply sloping connections between these different parts.

We proceed now to the proof of a version of the Maximum Principle used in Theorem 1. The presented proof is a simplification and generalization of that in [6], p. 500.

Consider the equation

$$(19)\qquad v_t = \sum_{i,j}(a_{ij}(t,x)v_{x_j})_{x_i} + a(t,x)v + f(t,x)$$

$x \in \Omega$, $t \in (0,T]$ ($T \le +\infty$), with continuous coefficients fulfilling the requirements; $a(t,x) \le \beta$ ($\beta \ge 0$ is a constant), f bounded in $[0,T]\times\overline{\Omega}$ and the main part in (19) weakly elliptic

$$(20)\qquad \forall_{(t,x)}\ \forall_{\xi\in\mathbb{R}^n}\ \sum_{i,j} a_{ij}(t,x)\xi_i\xi_j \ge 0\,.$$

Let v satisfy the Dirichlet or Neumann condition

$$Bv = 0 \quad \text{for } x\in\partial\Omega, \qquad v(0,x) = v_0(x) \quad \text{for } x \in \Omega\,.$$

We may then formulate

PROPOSITION 1. *The following estimate for the solution v of* (19) *holds*

$$(21)\qquad \|v(t,\cdot)\|_{L^\infty(\Omega)} \le (\|v_0\|_{L^\infty(\Omega)} + \sup_{0\le s\le t} \|f(s,\cdot)e^{-\beta s}\|_{L^\infty(\Omega)} t)e^{\beta t}\,.$$

Proof. If $\beta > 0$, then we may multiply (19) by $e^{-\beta t}$ to get the equation for the function $w(t,x) = v(t,x)e^{-\beta t}$

$$(22)\qquad w_t = \sum_{i,j}(a_{ij}(t,x)w_{x_j})_{x_i} + (a(t,x)-\beta)w + f(t,x)e^{-\beta t}\,.$$

Multiplying (22) by w^{2k-1} ($k = 1,2,3,\ldots$), integrating over Ω and by parts in the first component, we get

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega w^{2k}\,dx = -(2k-1)\int_\Omega w^{2k-2}\sum_{i,j} a_{ij}(t,x)w_{x_i}w_{x_j}\,dx + \int_\Omega (a(t,x)-\beta)w^{2k}\,dx + \int_\Omega f(t,x)e^{-\beta t}w^{2k-1}\,dx\,.$$

Omitting non-positive components and using the Hölder inequality ($p = 2k$, $q = \frac{2k}{2k-1}$) we find that

$$\frac{d}{dt}\int_\Omega w^{2k}(t,x)\,dx \le 2k\|f(t,\cdot)e^{-\beta t}\|_{L^\infty(\Omega)}|\Omega|^{1/2k}\Big(\int_\Omega w^{2k}(t,x)\,dx\Big)^{1-1/2k}.$$

Solving this differential inequality we obtain

$$\|w(t,\cdot)\|_{L^{2k}(\Omega)} \le \|v_0\|_{L^{2k}(\Omega)} + M(t)|\Omega|^{1/2k}t\,, \tag{23}$$

where $M(t) = \sup_{0\le s\le t}\|f(s,\cdot)e^{-\beta s}\|_{L^\infty(\Omega)}$. Passing in (23) with k to $+\infty$ and returning to the proper function v we get (21). The proof is finished.

Remark 3. It is possible to get uniform in ε estimates for various norms of the solution u of (1). We will propose an example of such estimate.

Assuming convexity of Ω (for the significance of this assumption compare [2], p. 65) and the Neumann condition for u, then using the inequality

$$\sum_{i,j} u_{x_ix_j}u_{x_j}\cos(n,x_i) \le 0 \quad \text{for } x\in\partial\Omega \tag{24}$$

proved in [2], p. 59, it is simple to reach the following estimate

$$\int_\Omega \Big(\sum_i u^2_{x_i}(t,x)\Big)^k dx \le \int_\Omega \Big(\sum_i u^2_{0x_i}\Big)^k dx\,e^{2kmt} \tag{25}$$

$k = 1,2,3,\ldots$ (cf. [5], Lemma 2.5 for similar proof).

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