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WEAK CONVERGENCE OF PROBABILITY MEASURES ALONG PROJECTIVE SYSTEMS

In this paper the definition and some of the most basic properties, including the counterpart of the Alexandroff's second theorem, of the weak convergence of probability measures along projective systems are given. The concept is motivated by the convergence mode of empirical measures which normally arise as observing finite trajectories of non-stationary stochastic processes.

1. Introduction

In some problems of probability theory and stochastic processes a net $\{\mu_\alpha\}$ of probability measures is not defined on a common measurable space, but rather on a projective system of measurable spaces. Since the probability measures are not necessarily compatible their convergence behavior is different from that of inverse limits of measure spaces and extension to a common underlying space may not be feasible everytime.

In order to tackle with such situations more efficiently, some kind of formalism seems to be desirable.

In section 2, an example of a sequence of probability measures on a standard projective system which has motivated the proposed definition of convergence is discussed. This particular sequence arises in relation with empirical characterisitic functionals and measures in sequence spaces.

In section 3, we give the definition of "weak convergence along a projective system" in a set-up considerably more general than that of the example. Important properties of this type of convergence including the counterpart of the Alexandroff's characterization of the weak convergence (the portmanteau theorem) and the interaction with tightness is discussed.

2. Convergence of empirical measures in sequence spaces

DEFINITION 2.1. The "*characteristic functional*" of a probability distribution μ on $(E, \mathcal{B}_E)$, where E is a real sequence space and $\mathcal{B}_E$ its Borel σ-field is given by

$$\chi^\mu(f) := \int_E \exp(i\langle f, x\rangle) d\mu(x), \qquad f \in F.$$

Here F will be

i) the sequence space G if E satisfies $E = G^*((.)^*$: continuous dual,
ii) E^* if i) does not hold.

Some examples of (E, F) pairs would be $(l_1, c_0), (l_\infty, l_1), (l_p, l_q), (c_0, l_1)$ etc. $(c_0 = \{x \in R^N : \lim_j x_j = 0\}, \frac{1}{p} + \frac{1}{q} = 1)$.

If E is reflexive i) and ii) yield the same F. Also for certain E spaces, the values of χ^μ on G may uniquely determine its values on the whole E^*, (e.g. $E = l_1$).

The canonical projection onto the first n coordinates will be indicated by π_n. The projection of μ on $(R^n, \mathcal{B}^n)$ and its restriction to $\pi_n^{-1}(\mathcal{B}^n)$ are denoted by $\widetilde{\mu}_n$ and $\overline{\mu}_n$ respectively. A superscript $(.)^\circ$ on a finite sequence will indicate argumentation to an infinite sequence will indicate argumentation to an infinite sequences by filling out the rest of the positions by zeros.

Let $(S, \mathcal{F}, P)$ be a probability space and let Θ be an $\mathcal{F} \leftrightarrow \mathcal{B}_E$ measurable mapping of S into E. We may suppose that the random sequence $\Theta(\omega)$ is generated by a non-stationary random process or by any other source which can be observed any number of times independently and under identical conditions. Such observations will yield a tableau of the following form:

$$\begin{array}{ccccc} \eta_{11} & \eta_{12} & & \eta_{1n} & \cdots \\ \eta_{21} & \eta_{22} & & \eta_{2n} & \cdots \\ \vdots & \vdots & & \vdots & \cdots \\ \eta_{m1} & \eta_{m2} & & \eta_{mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{array} \tag{2.1}$$

where the rows represent the observed components of independent random sequences $\Theta_m(\omega)$ $(m = 1, 2, \ldots)$. By assumption Θ and Θ_m $(m = 1, 2, \ldots)$ induce the same probability distribution, say μ on $(E, \mathcal{B}_E)$. More generally we may think of Θ as a mapping from Ω into R^N such that one of the sufficient conditions for the induced probability measure to be concentrated on E is satisfied. (c.f. [8], Theorem 1.3.4).

DEFINITION 2.2. "*The empirical distribution*" $\lambda_{nm}(\omega)$ associated with (2.1) is the random probability measure

$$\lambda_{nm}(\omega) := \frac{1}{m} \sum_{j=1}^{m} \delta(\pi_n^{-1} \pi_n \Theta_j(\omega))$$

defined on $E, \pi_n^{-1}(\mathcal{B}^n))$ and concentrated on m atoms $\pi_n^{-1}\pi_n\Theta_j(\Omega)$ $(j = 1, \dots, m)$. Thus $\widetilde{\lambda}_{nm} := \pi_n \lambda_{nm}$ is concentrated on m points in R^n.

The concept of empirical characteristic functional is introduced in [3].

DEFINITION 2.3. *"The empirical characteristic functional (e.c.f.l.)"* $\widehat{\chi}_{nm}(f,\omega)$ associated with (2.1) is defined by

$$\widehat{\chi}_{nm}(f,\omega) := \int_E \exp(i < (\pi_n f)^\circ, x >)\lambda_{nm}(\omega, dx), \qquad f \in F.$$

The e.c.f.l. can alternatively be expressed as

$$\begin{aligned}(2.2) \qquad \widehat{\chi}_{nm}(f,\omega) &= \int_{R^n} \exp[i((\pi_n f).y)]\widetilde{\lambda_{nm}}(\omega, dy) \\ &= \frac{1}{m}\sum_{j=1}^{m} \exp\left(i\sum_{k=1}^{n} f_k \eta_{jk}(\omega)\right)\end{aligned}$$

where (.,.) denotes the scalar product in finite dimensions.

The characteristic functional of the projecton of μ is accordingly given by

$$\begin{aligned}(2.3) \qquad \chi_n^\mu(f) &:= \int_E \exp(i < (\pi_n f)^\circ, x > d\overline{\mu}_n(x) \\ &= \int_{R^n} \exp[i((\pi_n f).y)]\widetilde{\mu}_n(dy) \qquad f \in F.\end{aligned}$$

We observe that,

$$(2.4) \qquad \chi_n^\mu(f) = \chi^\mu((\pi_n f)^\circ) \quad \text{and} \quad \lim_{n\to\infty} \chi_n^\mu(f) = \chi^\mu(f), \qquad f \in F.$$

Now we can state the following Glivenko-Cantelli type theorem.

THEOREM 2.4. *Let $f \in F$, then $\lim_{n,m\to\infty} \widehat{\chi}_{nm}^\mu(f) = \chi^\mu(f)$ a.s. holds in the following cases*:

i) $(E, F) = (R^N, R_0^N)$, *(R_0^N: The space of all sequences with finite lenght).*

ii) *The elements of F are absolutely summable and Θ is a.s. bounded in l_∞ norm.*

Besides in both cases the convergence is uniform on compact subsets of F.

Proof. i) Let the lenght of f be ν. Then for $n \geq \nu$, in view of (2.1), (2.2) and (2.4), $\widehat{\chi}_{nm}(f,\omega)$ depends only on m and furthermore $\chi_n^\mu(f) = \chi^\mu(f)$. Hence the result follows from the strong law of large numbers by observing that

$$(2.5)\quad \int_{\Omega} \exp\Big(i\sum_{k=1}^{n} f_k\eta_{jk}(\omega)\Big)\, dP(\omega) = \int_{R^n} \exp(i(f.y))\, d\widetilde{\mu}_n(y) = \chi_n^{\mu}(f)\,.$$

ii) By (2.4) $\exists n_1$ such that $|\chi_n^{\mu}(f) - \chi^{\mu}(f)| < \frac{\epsilon}{3}$ for $n \geq n_1$. On the other hand $|\sum_{k=n}^{\infty} f_k \eta jk(\omega)| \leq \sum_{k=n}^{\infty} |f_k| M$ and by assumptions it can be made arbitrarily small by properly selecting n. Thus there exists n_2 such that except on a negligible set and for $n > n_2$,

$$|\widehat{\chi}_{nm}(f,\omega) - \widehat{\chi}_{n_2 m}(f,\omega)| \leq \frac{1}{m}\sum_{j=1}^{m}\Big|\exp\Big(i\sum_{n_2+1}^{n} f_k\eta_{jk}(\omega)\Big) - 1\Big| < \frac{\epsilon}{3}$$

independently of m. Let $n_0 = \max(n_1, n_2)$. In view of (2.5), $\lim_{m\to\infty} \widehat{\chi}_{n_0 m}(f,\omega) = \chi_{n_0}^{\mu}(f)$ a.s. and we can fix m_0 such that for $m \geq m_0$ $|\widehat{\chi}_{n_0 m}(f,\omega) - \chi_{n_0}^{\mu}(f)| < \frac{\epsilon}{3}$ a.s. Now if $n > n_0$ and $m \geq m_0$:

$$\begin{aligned}|\widehat{\chi}_{nm}(f,\omega) - \chi^{\mu}(f)| &\leq |\widehat{\chi}_{nm}(f,\omega) - \widehat{\chi}_{n_0 m}(f,\omega)| + |\widehat{\chi}_{n_0 m}(f,\omega) - \chi_{n_0}^{\mu}(f)| \\ &\quad + |\chi_{n_0}^{\mu}(f) - \chi^{\mu}(f)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\,.\end{aligned}$$

As the compact sets in R_0^N can be identified by their finite dimensional projections, the uniformity of convergence follows from the theory of finite dimensional empirical characteristic functions.

In case ii) let K be a compact subset of F. As $|\chi_n^{\mu}(f) - \chi^{\mu}(f)|^2 \leq 2|1 - \int_E \cos < h_n, x > |d\mu(x)$ where $h_n = f - \pi_n^{\circ} f$ is the tail of f, the total boundedness of K implies that n_1 can be selected independent of f. The selection of n_2 independently of f is also a consequence of total boundedness of K.

On the other hand $\widehat{\chi}_{nm}(f) \to \chi_n^{\mu}(f)$ (a.s.) as $m \to \infty$ implies by Lévy's continuity theorem that for every n, $\widetilde{\lambda_{nm}} \xrightarrow{w} \widetilde{\mu_n}$ as $m \to \infty$ ($\xrightarrow{w}$ denotes weak convergence). Since π_n is continuous it follows that $\lambda_{nm} = \pi_n^{-1}\widetilde{\lambda_{nm}} \xrightarrow{w} \pi_n^{-1}\widetilde{\mu}_n = \overline{\mu}_n$ a.s. for every n. Thus for $n_0 = \max(n_1, n_2)$, $\{\lambda_{n_0 m}\}_{m=1}^{\infty}$ is weakly relatively compact and $\{\widehat{\chi}_{n_0 m}\}_{m=1}^{\infty}$ is equicontinuous. Hence the convergence $\chi_{n_0 m}(f) \to \chi_{n0}^{\mu}$ is almost surely uniform on K and m_0 can be fixed independent of f □.

Here we are concerned with the convergence behaviour of $\widetilde{\lambda_{nm}}$ whenever we confine ourselves to (R^N, R_0^N) duality.

THEOREM 2.5. *Let μ be a probability measure induced on R^N by Θ and let $(R^N, \mathcal{B}'_{R^N}, \mu')$ and $(R^n, \mathcal{B}^{n'}, \overline{\mu}'_n)$ $(n = 1, 2, \ldots)$ be the completions of probability spaces $(R^N, \mathcal{B}_{R^N}, \mu)$ and $(R^n, \mathcal{B}^n, \overline{\mu}_n)$ respectively. If $m_n \to \infty$ as $n \to \infty$ and $B \in \mathcal{B}_{R^N}$ is a μ-continuity set, then $\lim_{n\to\infty} \widetilde{\lambda}_{nm_n}(\pi_n B) = \mu'(B)$ a.s. where P-null set is independent of B.*

Proof. Let $\overset{*}{\lambda}_{nm_n}$ $(n = 1, 2, \ldots)$ be a set of arbitrary extensions of λ_{nm_n} from $\pi_n^{-1}(\mathcal{B}^n)$ to $\mathcal{B}'_{R^N}$ (such extensions always exist). Since $\widehat{\chi}_{nm_n}(f)$ depend only on finite dimensional restrictions of measures, except a null set we have by Theorem 2.4.

$$\lim_{n\to\infty} \int_{R^N} \exp(i < f, x >)\overset{*}{\lambda}_{nm_n}(\omega, dx) = \chi^\mu(f), \qquad f \in R_0^N .$$

By the analogue of Lévy-Cramer theorem (c.f. [8], Theorem 1.2.8) $\overset{*}{\lambda}_{nm_n} \xrightarrow{w} \mu'$ a.s. as $n \to \infty$. If B is finite dimensional set, then for large n : $\pi_n^{-1}\pi_n B = B$ and $\widetilde{\lambda}_{nm_n}(\pi_n B) = \overset{*}{\lambda}_{nm_n}(B)$ and the conclusion follows immediately.

If B is an infinite dimensional set, it will heve no interior with respect to Tikhonov's topology, thus $\mu'(B) = 0$. As $\{\pi_n^{-1}\pi_n B\}$ is decreasing sequence of universally measurable sets, letting $C = \bigcap_{n=1}^{\infty} \pi_n^{-1}\pi_n B$, we have $B \subset C \subset \bigcap_{n=1}^{\infty} \pi_n^{-1}\pi_n \overline{B} = \overline{B}$ and thus $\lim_{n\to\infty} \mu'(\pi_n^{-1}\pi_n B) = \mu'(C) = 0$. There exists a sequence $\{n_k\}$ of positive integers and a decreasing sequence $\{C_k\}$ of μ'-continuity sets satisfying:

1) $C_k \supset \pi_{n_k}^{-1}\pi_{n_k} B$,
2) $\lim_{k\to\infty} \mu'(C_k) = 0$.

The double limit $\lim_{n,k\to\infty} \overset{*}{\lambda}_{nm_n}(C_k)$ exists and is equal to zero. For there are positive integers k_0 and n_0 such that for $k \geq k_0$ and $n \geq n_0$ we have $|\mu'(C_k) - \mu'(B)| = \mu'(C_k) < \frac{\epsilon}{2}$ and $|\overset{*}{\lambda}_{nm_n}(C_{k_0}) - \mu'(Ck_0)| < \frac{\epsilon}{2}$, therefore $|\overset{*}{\lambda}_{nm_n}(C_{k_0}) - \mu'(B)| < \epsilon$. But $\overset{*}{\lambda}_{nm_n}(C_k) \leq \overset{*}{\lambda}_{nm_n}(C_{k_0})$ and $\mu'(B) = 0$. Therefore $\lim_{k\to\infty} \overset{*}{\lambda}_{nm_k}(C_k) = 0$ and this implies $\overset{*}{\lambda}_{nm_n}(\pi_n^{-1}\pi_n B) = \overline{\lambda}_{nm_n}(\pi_n B) \to 0 = \mu'(B)$ a.s. □.

Here supressing the almost sure behavior, the type of convergence deserves a special attention. Examples exhibiting the same type of convergence can be constructed also in the domain of cylindrical measures.

3. Weak convergence of probability measures along a projective system

Following the notation and the terminology of [7], we consider projective systems of Hausdorff topological spaces of the form

$$\{(\Omega_\alpha, \pi_{\alpha\beta})_{\alpha\preceq\beta} : \alpha, \beta \in D\}$$

having the projective limit $\Omega = \varprojlim(\Omega_\alpha, \pi_{\alpha\beta})$ with continuous canonical mappings $\pi_\alpha: \Omega \to \Omega_\alpha$. The right-filtering partially ordered set D and all other symbols are assumed to have their usual meaning and properties.

In relation with such a projective system we consider two hypotheses:

a) *Hypothesis* R_1: π_α^{-1}, $\alpha \in D$ commute with the operation of forming the rim, i.e. if $\pi_\alpha^{-1}(r(A)) = r(\pi_\alpha^{-1}(A))$ $(r(A) = A \cap \overline{A^c})$.

b) *Hypothesis* R_2: For every $\alpha \in D$, $\pi_\alpha \mathcal{B} \subset \mathcal{B}_\alpha$, where $\mathcal{B}$ and $\mathcal{B}_\alpha$ are Borel σ-fields in Ω and Ω_α respectively, the former being with respect to the projective limit topology.

Hypothesis R_1 is possessed by many important projective systems including those where each π_α $(\alpha \in D)$ is an open mapping. (In this case hypothesis R_1 is actually equivalent to the stronger property $\pi_\alpha^{-1}(\delta A) = \delta(\pi_\alpha^{-1} A)$ ($\delta(.)$: boundary)). This would be the case if for instance the projective limit topology coincides with the product topology. Some examples:

1. $\{\prod_{t \in A} \Omega_t, \pi_{\alpha\beta})_{\alpha \subset \beta}: \alpha, \beta \in D\}$ where D is the family of all finite subsets of an index set T directed by inclusion, $\Omega_t (t \in T)$ are Hausdorff spaces and $\pi_{\alpha\beta}$ the canonical projections.
2. $\{(N', \pi_{N_1 N_2})_{N_1 \subset N_2}: N_1, N_2 \in D\}$ where D is the directed set of all finite dimensional subspaces N of a topological vector space F and N' is the (algebraic) dual. (The natural set-up for cilindrical measures).

Hypothesis R_2 would be ensured if for instance Ω and Ω_α $(\alpha \in D)$ are Polish spaces and $\mathcal{B}$ and $\mathcal{B}_\alpha$ are replaced by σ-fields of sets which are measurable for the completion of probability measures on Borel sets, thus containing all universally measurable sets (c.f. [5], pp. 391).

DEFINITION 3.1. Let $\mathcal{P} = \{(\Omega_\alpha, \pi_{\alpha\beta})_{\alpha \preceq \beta}: \alpha, \beta \in D\}$ be a projective system of metrizable spaces with the projective limit $\Omega = \varprojlim(\Omega_\alpha, \pi_{\alpha\beta})$ furnished with the projective limit topology and let the σ-fields $\mathcal{B}_\alpha$ and $\mathcal{B}$, in Ω_α and Ω respectively, satisfy hypothesis R_2. If $\{\mu_\alpha, \alpha \in D\}$ is a net of probability measures defined on measurable spaces $(\Omega_\alpha, \mathcal{B}_\alpha)$, $\alpha \in D$, we say that "*μ_α converges weakly along the projective system $\mathcal{P}$ to a probability measure μ on $\mathcal{B}$*" and denote $\mu_\alpha \xrightarrow{w-\mathcal{P}} \mu$ if for every μ-continuity set $B \in \mathcal{B}$

$$\lim_\alpha \mu_\alpha(\pi_\alpha B) = \mu(B)$$

holds.

Note. In the case of a Polish projective system with a Polish projective limit, $\mathcal{B}$ and $\mathcal{B}_\alpha$ may be chosen as the σ-fields obtained by the completion of Borel probability measures μ and $\pi_\alpha(\mu)$ respectively.

LEMMA 3.2. *For the projective system of probability spaces described in Definition* 3.1.:

i) *If* $B \in \mathcal{B}$ *is closed, then* $\bigcap_{\alpha \in D} \pi_\alpha^{-1}\pi_\alpha B = B$.

ii) *Consider a family* $\{C_t\}_{t\in T}$ *of compact measurable sets in* Ω *where* T *is an arbitrary partially ordered index set such that* $C_t \downarrow C \neq \emptyset$. *Then*

$$\lim_t \mu(C_t) = \inf_t \mu(C_t) = \mu(C).$$

iii) *If* $B \in \mathcal{B}$ *then* $\bigcap_{\alpha \in D} \overline{\pi_\alpha^{-1}\pi_\alpha B} = \overline{B} = \overline{\bigcap_{\alpha \in D} \pi_\alpha^{-1}\pi_\alpha B}$.

iv) *If hypothesis* R_1 *is satisfied, then* $(\pi_\alpha^{-1}A)^\circ = \pi_\alpha^{-1}(A^\circ)$, $A \in \mathcal{B}_\alpha$.

Proof. i), ii): c.f. [7] pp. 127–128.

iii) The second equality follows from $B \subset \overline{\cap_\alpha \pi_\alpha^{-1}\pi_\alpha B} \subset \cap_\alpha \pi_\alpha^{-1}\pi_\alpha \overline{B} = \overline{B}$. For the first equality, the inclusion $\overline{B} \subset \cap_\alpha \overline{\pi_\alpha^{-1}\pi_\alpha B}$ is obvious. In order to prove the opposite inclusion let $\omega \in \overline{B}^c$. Then there exists a neighborhood N of the form $N = \bigcap_{i=1}^{k} \pi_{\alpha_i}^{-1} S_{\alpha_i}$, S_{α_i} being open spheres in Ω_{α_i} $(i = 1, 2, \ldots, k)$ such that $\omega \in N$ and $N \cap B = \emptyset$. Let $\beta \in D$ be any index satisfying $\beta \succeq \alpha_i$ $(i = 1, 2, \ldots, k)$. We show that $N \cap \pi_\beta^{-1}\pi_\beta B$ is also empty. Otherwise if we let $z \in N \cap \pi_\beta^{-1}\pi_\beta B$, then $\pi_\beta\{z\} \in \pi_\beta B$, thus there exists $v \in B$ such that $\pi_\beta\{v\} = \pi_\beta\{z\}$. Now for $i = 1, \ldots, k$, $\pi_{\alpha_i}\{v\} = \pi_{\alpha_i\beta}\pi_\beta\{v\} = \pi_{\alpha_i\beta}\pi_\beta\{z\} = \pi_{\alpha_i}\{z\} \in S_{\alpha_i}$. But then v is also in N which is a contradiction. Hence $N \cap \pi_\beta^{-1}\pi_\beta B = \emptyset$ so that $\omega \in (\overline{\pi_\beta^{-1}\pi_\beta B})^c \subset (\bigcap_{\alpha \in D} \overline{\pi_\alpha^{-1}\pi_\alpha B})^c$.

iv) $\pi_\alpha^{-1}A = r(\pi_\alpha^{-1}A) \cup (\pi_\alpha^{-1}A)^\circ$, on the other hand $\pi_\alpha^{-1}A = \pi_\alpha^{-1}(rA) \cup \pi_\alpha^{-1}(A^\circ) = r(\pi_\alpha^{-1}A) \cup (\pi_\alpha^{-1}(A^\circ))$. Comparison of the two expressions yields the result.

PROPOSITION 3.3. *Let* μ *be a tight probability measure on* $(\Omega, \mathcal{B})$ *of Definition* 3.1.

i) *If* $B \in \mathcal{B}$ *is either closed or a* μ*-continuity set, then*

$$\lim_\alpha \mu(\pi_\alpha^{-1}\pi_\alpha B) = \mu(B)\,.$$

ii) *If* $B \in \mathcal{B}$ *is a* μ*-continuity set, then*

$$\lim_\alpha \mu[r(\pi_\alpha^{-1}\pi_\alpha B)] = 0$$

($r(.)$: *the rim of a set*).

Proof. i) Given $\epsilon > 0$, let Ω' be a compact subset of Ω satisfying $\mu(\Omega') > 1 - \frac{\epsilon}{2}$. By Lemma 3.2 -ii) and iii) μ is right-continuous along

$\overline{\pi_\alpha^{-1}\pi_\alpha B} \cap \Omega'$. Thus there exists $\alpha_0 \in D$ such that for $\alpha \succeq \alpha_0$:

$$0 \le \mu(\overline{\pi_\alpha^{-1}\pi_\alpha B} \cap \Omega') - \mu(\overline{B} \cap \Omega') < \frac{\epsilon}{2}.$$

As $0 \le \mu(\overline{\pi_\alpha^{-1}\pi_\alpha B}) - \mu(\overline{\pi_\alpha^{-1}\pi_\alpha B} \cap \Omega') < \frac{\epsilon}{2}$ and $0 \le \mu(\overline{B}) - \mu(\overline{B} \cap \Omega') < \frac{\epsilon}{2}$, we conclude that for $\alpha \succeq \alpha_0: 0 \le \mu(\overline{\pi_\alpha^{-1}\pi_\alpha B}) - \mu(\overline{B}) < \epsilon$, showing $\mu(\overline{\pi_\alpha^{-1}\pi_\alpha B}) \downarrow \mu(\overline{B})$. Now consider the inequalities

$$\mu(B) \le \mu(\pi_\alpha^{-1}\pi_\alpha B) \le \mu(\overline{\pi_\alpha^{-1}\pi_\alpha B}), \quad \alpha \in D$$

which imply the desired conclusion if either B is closed or $\mu(\overline{B}) = \mu(B)$.

ii) By part i):

$$\begin{aligned}\mu(\pi_\alpha^{-1}\pi_\alpha B) &= \mu[r(\pi_\alpha^{-1}\pi_\alpha B) \cup (\pi_\alpha^{-1}\pi_\alpha B)^\circ] \\ &= \mu[r(\pi_\alpha^{-1}\pi_\alpha B)] + \mu[(\pi_\alpha^{-1}\pi_\alpha B)^\circ] \downarrow \mu(B).\end{aligned}$$

But since $(\pi_\alpha^{-1}\pi_\alpha B)^\circ \supset B^\circ$ and $\mu(B) = \mu(B^\circ)$ it follows that $\lim_\alpha [r(\pi_\alpha^{-1}\pi_\alpha B)] = 0$.

Now we state the following version of Alexandroff's second theorem.

THEOREM 3.4. *Let $\{\mu_\alpha, \alpha \in D\}$ be a net of probability measures on a projective system $\mathcal{P}$ as described in Definition 3.1, satisfying furthermore hypothesis R_1. If Ω is sufficiently rich (i.e. $\pi_\alpha \Omega = \Omega_\alpha$) and μ is a tight measure on Ω, then the following are equivalent:*

i) $\mu_\alpha \xrightarrow{w-\mathcal{P}} \mu$, $\alpha \in D$.

ii) Let $f \in C(\Omega_\beta)$, $\beta \in D$ *and if* $\alpha \in D$, $\alpha \succeq \beta$, *let* F^α *and* F^Ω *be the lifts of* f *to* Ω_α *and* Ω *respectively*, (*i.e.* $f^\alpha(x) = f(\pi_{\beta_\alpha} x)$, $f^\Omega(x) = f(\pi_\beta x)$). *Then* $\lim_{\alpha \succeq \beta} (f^\alpha) = \mu(f^\Omega)$.

iii) *The same conditions as in* ii), $C(\Omega_\beta)$ *being replaced by the set of bounded uniformly continuous functions.*

iv) *For* $\beta \in D$, *let* $F \in \mathcal{B}_\beta$ *be a closed subset of* Ω_β, *then*

$$\limsup_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta_\alpha}^{-1} F) \le \mu(\pi_\beta^{-1} F).$$

v) *For* $\beta \in D$, *let* $G \in \mathcal{B}_\beta$ *be an open subset of* Ω_β, *then*

$$\liminf_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta_\alpha}^{-1} G) \ge \mu(\pi_\beta^{-1} G).$$

Proof.

a) i)⇒ii): Proceeding as in the proof of Theorem 3.1. [1], it is sufficient to consider $f \in C(\Omega_\beta)$ satisfying $0 \le f(x) \le 1$, $x \in \Omega_\beta$. Let $t_0 < t_1 < \ldots < t_m$ be fixed in such a way that $t_0 < \inf_{\Omega_\beta} f(x)$, $t_m > \sup_{\Omega_\beta} f(x)$ and $t_i - t_{i-1} < \epsilon$ $(i = 1, \ldots, m)$. For $\alpha \succeq \beta$ and the closed sets $F_{\alpha i} = \{x \in \Omega_\alpha : f^\alpha(x) \ge t_i\}$, we have

$$\sum_{i=1}^{m}(t_i - t_{i-1})\mu_\alpha(F_{\alpha i}) \leq \mu_\alpha(f^\alpha) \leq \sum_{i=1}^{m}(t_i - t_{i-1})\mu_\alpha(F_{\alpha,i-1})$$

where the upper and lower sums differ by less than ϵ. On the other hand it is possible to determine t_i so that the Borel sets $F_i = \{\omega \in \Omega : f^\Omega(\omega) \geq t_i\}$ become μ-continuous. As $\lim_\alpha \mu_\alpha(F_{\alpha i}) = \lim_\alpha \mu_\alpha(\pi_\alpha F_i) = \mu(F_i)$ and $\lim_\alpha \mu_\alpha(F_{\alpha,i-1}) = \lim_\alpha \mu_\alpha(\pi_\alpha F_{i-1}) = \mu(F_{i-1})$, the implication follows from the fact that $\sum(t_i - t_{i-1})\mu(F_{i-1})$ and $\sum(t_i - t_{i-1})\mu(F_i)$ which are upper and lower sums for $\mu(f^\Omega)$ also differ by less than ϵ.

b) ii)$\Rightarrow$iii) is obvious.

c) iii)$\Rightarrow$iv): As F is a G_δ set there exists a sequence of open sets G_n^β in Ω_β such that $G_n^\beta \downarrow F$. If $G_n = \pi_\alpha^{-1}(G_n^\beta)$, then $G_n \downarrow \pi_\beta^{-1}F$, so that there is n_0 satisfying $\mu(G_n \setminus \pi_\beta^{-1}F) < \epsilon$ for $n \geq n_0$. Let n be any positive integer greater than n_0. There exists a uniformly continuous function f on Ω_β connecting F and $\Omega_\beta \setminus G_n^\beta$, i.e.:

$$f(x) = \begin{cases} 1 & \text{if } x \in F \\ \in [0,1] & \text{if } x \in G_n^\beta \setminus F \\ 0 & \text{if } x \in \Omega_\beta \setminus G_n^\beta. \end{cases}$$

Let f^α and f^Ω be the lifts of f to any Ω_α $(\alpha \succ \beta)$ and Ω respectively, denote also $\pi_{\beta\alpha}^{-1}F$ by F^α. Then $\mu_\alpha(f^\alpha) \geq \mu_\alpha(1_{F^\alpha} f^\alpha) = \mu_\alpha(F^\alpha)$ and $\limsup_{\alpha \succeq \beta} \mu_\alpha(F^\alpha)$ $\leq \limsup_{\alpha \succeq \beta} \mu_\alpha(f^\alpha) = \mu(f^\Omega) \leq \mu((1_{\pi_\beta^{-1}F}) + \mu(1_{G_n \setminus \pi_\beta^{-1}F}) \leq \mu(\pi_\beta^{-1}F) + \epsilon.$

d) iv)$\Leftrightarrow$v): Obvious.

e) v)$\Rightarrow$i): Let B be a μ-continuity set in $\mathcal{B}$. By Proposition 3.3. i)-ii) there exists $\beta_0 \in D$ such that for $\beta \succeq \beta_0$, $|\mu(\pi_\beta^{-1}\pi_\beta B) - \mu(B)| < \frac{\epsilon}{2}$ and $\mu[r(\pi_\beta^{-1}\pi_\beta B)] < \frac{\epsilon}{2}$ are simultaneously satisfied. Now the subsequent applications of hypothesis R_2, Lemma 3.2. iv)-v) and hypothesis R_1 yield

$$\begin{aligned}\mu(B^\circ) &\leq \mu[(\pi_\beta^{-1}\pi_\beta B)^\circ] = \mu[\pi_\beta^{-1}((\pi_\beta B)^\circ)] \leq \liminf_{\alpha \succeq \beta} \mu_\alpha[\pi_{\beta\alpha}^{-1}((\pi_\beta B)^\circ)] \\ &\leq \liminf_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta\alpha}^{-1}\pi_\beta B) \leq \limsup_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta\alpha}^{-1}\pi_\beta B) \\ &\leq \limsup_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta\alpha}^{-1}\overline{\pi_\beta B}) \leq \mu[\pi_\beta^{-1}(\overline{\pi_\beta B})] = \mu[\pi_\beta^{-1}((\pi_\beta B)^\circ \cup r(\pi_\beta B))] \\ &\leq \mu(\pi_\beta^{-1}\pi_\beta B) + \mu[r(\pi_\beta^{-1}\pi_\beta B)] < \mu(B) + \frac{\epsilon}{2} + \frac{\epsilon}{2}.\end{aligned}$$

Hence for $\beta \succeq \beta_0$:

$$\mu(B) \leq \liminf_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta\alpha}^{-1}\pi_\beta B) \leq \limsup_{\alpha \succeq \beta} \mu_\alpha(\pi_{\beta\alpha}^{-1}\pi_\beta B) \leq \mu(B) + \epsilon,$$

as $\epsilon > 0$ is arbitrary and π_α are onto, this implies

$$\lim_{\alpha\succeq\beta} \mu_\alpha(\pi_\alpha\pi_\beta^{-1}\pi_\beta B) = \mu(B) \qquad \text{for all } \beta \succeq \beta_0\,.$$

For $\delta > 0$, there exists $\alpha_0 \succeq \beta_0$ such that if $\alpha \succeq \alpha_0 \succeq \beta \succeq \beta_0$

$$\mu(B) \le \mu_\alpha(\pi_\alpha\pi_\beta^{-1}\pi_\beta B) < \mu(B) + \delta\,,$$

as $\mu_\alpha(\pi_\alpha\pi_\beta^{-1}\pi_\beta B)$ is decreasing in β

$$\mu(B) \le \mu_\alpha(\pi_\alpha\pi_\beta^{-1}\pi_\beta B) = \mu_\alpha(\pi_\alpha B) < \mu(B) + \delta\,.$$

THEOREM 3.5. *Let the projective system $\mathcal{P}$ as described in Definition* 3.1 *have a separable, metrizable projective limit. Further assume that for each $\epsilon > 0$ there exists a compact subset K_ϵ of Ω such that $\mu_\alpha(\pi_\alpha K_\alpha) \ge 1 - \epsilon$ for every $\alpha \in D$ is satisfied by a net $\{\mu_\alpha\}_{\alpha\in D}$ of probability measures. Then $\{\mu_\alpha\}$ has a subnet converging along the projective system $\mathcal{P}$.*

Proof. Let $\overline{\mu}_\alpha$ be the image of μ_α on $\pi_\alpha^{-1}\mathcal{B}_\alpha$, i.e. $\overline{\mu}_\alpha \circ \pi_\alpha^{-1} = \mu_\alpha$ and let $\overline{\mu}_\alpha^*$ ($\alpha \in D$) be any set of extensions of $\overline{\mu}_\alpha$ to $(\Omega, \mathcal{B})$. Such extensions always exist but may consist of measures which are only finitely additive. On the other hand Ω can be imbedded topologically into a compact metric space $\widehat{\Omega}$. For any $\overline{\mu}_\alpha^*$, let m_α be the measure on $\widehat{\Omega}$ defined by $m_\alpha(B) = \overline{\mu}_\alpha^*(B \cap \Omega)$ for all Borel subset of $\widehat{\Omega}$. The net $\{m_\alpha\}$ has a subnet, say $\{m_{N_\alpha}\}_{\alpha\in D}$ converging weakly to a σ-additive measure ν on $\widehat{\Omega}$. For any index α, denote $C_{\epsilon,\alpha} = \pi_\alpha^{-1}\pi_\alpha K_\epsilon$ which is compact in $\widehat{\Omega}$. Let β be a fixed index, then by the ordinary weak convergence of measures and the fact that $C_{\epsilon,\alpha}\downarrow$:

$$\begin{aligned}\nu(C_{\epsilon,\beta}) &\ge \limsup_{\alpha\succeq\beta} m_{N_\alpha}(C_{\epsilon,\beta}) \ge \limsup_{\alpha\succeq\beta} m_{N_\alpha}(C_{\epsilon,N_\alpha})\\ &= \limsup_{\alpha\succeq\beta} \mu_{N_\alpha}(\pi_{N_\alpha}K_\epsilon) \ge 1 - \epsilon\,.\end{aligned}$$

By considering a sequence $\epsilon_n \downarrow 0$, this set of inequalities implies along the same lines as in the proof of Theorem 6.7., [6], that there exists a measure μ on Ω such that $\nu(B) = \mu(B \cap \Omega)$ for any Borel subset $B \subset \widehat{\Omega}$. Let now F be any closed subset of Ω_β. There exists a closed set D in $\widehat{\Omega}$ such that $\pi_\beta^{-1}F = D \cap \Omega$. As $m_{N_\alpha} \xrightarrow{w} \nu$ on $\widehat{\Omega}$, we have $\limsup_\alpha m_{N_\alpha}(D) \le \nu(D)$. This is the same thing as stating: $\limsup_\alpha \overline{\mu}_{N_\alpha}^*(\pi_\beta^{-1}F) \le \mu(\pi_\beta^{-1}F)$.

Now for $N_\alpha \succeq \beta^\alpha$:

$$\begin{aligned}\limsup_\alpha \overline{\mu}_{N_\alpha}^*(\pi_\beta^{-1}F) &= \limsup_\alpha \overline{\mu}_{N_\alpha}(\pi_{N_\alpha}^{-1}\pi_{\beta N_\alpha}^{-1}F)\\ &= \limsup_{N_\alpha\succeq\beta} \mu_{N_\alpha}(\pi_{\beta N_\alpha}^{-1}F) \le \mu(\pi_\beta^{-1}F)\,.\end{aligned}$$

Then by Theorem 3.4. -iv) $\mu_{N_\alpha} \xrightarrow{w-\mathcal{P}} \mu$.

THEOREM 3.6. *Let each Ω_α be a separable metric space in the projective system $\mathcal{P}$ described in Definition* 3.1 *and let also μ be a tight measure on Ω. Then $\mu_\alpha \xrightarrow{w-\mathcal{P}} \mu$ if and only if*:

$$\lim_{\alpha\succeq\beta} \sup_{f\in\Lambda} \left| \int_{\Omega_\alpha} f^\alpha \, d\mu_\alpha - \int_{\Omega} f^\Omega \, d\mu \right| = 0$$

for every $\beta \in D$ and every uniformly bounded family of functions $\Lambda \subset C(\Omega_\beta)$ which is equicontinuous at each point of Ω_β.

Proof. 'If' part is obvious by taking $\Lambda = \{f\}$ and applying Theorem 3.4. -ii). For the 'only if' part we first note that given $\beta \in D$, there exists an infinite $\mathcal{B}_\beta$-partition $\{A_j\}_{j=1}^\infty$ of Ω_β such that $\{\pi_\beta^{-1}A_j\}_{j=1}^\infty$ is a $\mathcal{B}$-partition consisting of μ-continuity sets, furthermore $x, y \in A_j$, $f \in \Lambda \Rightarrow |f(x) - f(y)| < \epsilon$ $(j = 1, 2, \ldots)$. To see this, let $x \in \Omega_\beta$ and let $S_r(x)$ be an open sphere centered at x with boundary $B_r(x)$ and the radius $r > 0$ being fixed by using the equicontinuity of the family Λ, so that $|f(x) - f(y)| < \epsilon$ for all $y \in S_r(x)$ and $f \in \Lambda$. As $\pi_\beta^{-1}S_r(x) = \bigcup_{0\le r'<r} \pi_\beta^{-1}(B_{r'}(x))$ there exists $0 < \delta(x) < r$ such that $\pi_\beta^{-1}(S_\delta(x))$ is a μ-continuity set in Ω. By this observation and proceeding as in the proof of Lemma 6.5, [6], we arrive at the conclusion that a partition $\{A_j\}_{j=1}^\infty$ with the asserted properties exist. We also note that the family of functions $\{f^\alpha : f \in \Lambda\}$, $\{f^\Omega : f \in \Lambda\}$ obtained by lifting Λ from Ω_β to Ω_α $(\alpha \succeq \beta)$ and Ω respectively, are also uniformly bounded and equicontinuous in their domains.

Let $x_j \in \pi_\beta^{-1}A_j$ $(j = 1, 2, \ldots)$ be any fixed sequence of points. For $\alpha \succeq \beta$ let $\mu_{\alpha,D}$ be the discrete measure concentrated in the set $\{\pi_\alpha x_j : j = 1, 2, \ldots\}$ and given by $\mu_{\alpha,D}(E) = \sum_j I_E(\pi_\alpha x_j)\mu_\alpha(\pi_{\beta\alpha}^{-1}A_j)$, $E \in \mathcal{B}_\alpha$ (I_E: the indicator of E). Similarly we define $\mu_d(B) = \sum_j I_B(x_j)\mu(\pi_\beta^{-1}(A_j)$.

Since $\pi_\beta^{-1}A_j$ are μ-continuity sets and $\mu_\alpha \xrightarrow{w-\mathcal{P}} \mu$:

$$\mu_\alpha(\pi_\alpha\pi_\beta^{-1}A_j) = \mu_\alpha(\pi_{\beta\alpha}^{-1}A_j) \to \mu(\pi_\beta^{-1}A_j), \qquad (\alpha \succeq \beta);$$

so that

$$\limsup_{\alpha} \sup_{f\in\Lambda} \left| \int_{\Omega_\alpha} f^\alpha \, d\mu_{\alpha,D} - \int_{\Omega} f^\Omega \, d\mu_D \right|$$
$$\le M \limsup_{\alpha} \sum_j |\mu_\alpha(\pi_{\beta\alpha}^{-1}A_j) - \mu(\pi_\beta^{-1}A_j)| = 0\,.$$

Thus the following estimates are easily obtained as in the proof of Theorem 6.8., [6]:

$$\limsup_{\alpha\succeq\beta}\sup_{f\in\Lambda}\left|\int_{\Omega} f^\alpha\,d\mu_\alpha - \int_{\Omega} f^\Omega\,d\mu\right|$$
$$\leq \limsup_{\alpha\succeq\beta}\left\{\sup_{f\in\Lambda}\left|\int_{\Omega_\alpha} f^\alpha\,d\mu_\alpha - \int_{\Omega_\alpha} f^\alpha\,d\mu_{\alpha,D}\right|\right.$$
$$+ \sup_{f\in\Lambda}\left|\int_{\Omega} f^\Omega\,d\mu - \int_{\Omega} f^\Omega\,d\mu_D\right|$$
$$\left.+ \sup_{f\in\Omega}\left|\int_{\Omega_\alpha} f^\alpha\,d\mu_{\alpha,D} - \int_{\Omega} f^\Omega\,d\mu_D\right|\right\} \leq 2\epsilon .$$

4. Some applications

Since probability distributions in abstract spaces can be characterized solely by their characteristic functionals (Fourier transforms), it is natural to expect that their empirical versions (i.e. e.c.f.l. of Definition 2.3.) should provide tools necessary for dealing with the estimation problems in such spaces.

The convergence property in Theorem 2.4 can be rephrased as: "$\widetilde{\lambda}_{nm_n}$ converges weakly to μ' along the projective system $\mathcal{P} = \{R^n, \pi_{n_1n_2}: n_1, n_2 \in N\}$". The empirical measures and e.c.f.l. may arise in connection with the partially observed trajectories of identical, discrete-time non-stationary stochastic processes as described by (2.1) in Section 2. A process of this kind will induce in general, an unknown probability measure on the set of trajectories which is usually a suitable sequence space. in order to test certain hypotheses that can be postulated about the unknown probability measure (such as being Gaussian etc.), some functionals of e.c.f.l. can be introduced. Two examples of such functionals would be

$$(4.1)\qquad Y_{nm}(f) := m^{1/2}(\widehat{\chi}_{nm}(\pi_n f) - \chi_n^\mu(\pi_n f)), \qquad f \in F,$$
$$(4.2)\qquad Z_{nm}(f) := m^{1/2}\{|\widehat{\chi}_{nm}(S_{nm}^{-1/2}(\pi_n f))|^2 - e^{-\langle\pi_n f, \pi_n f\rangle}\}, \qquad f \in F.$$

where in (4.2), S_{nm} is the simple covariance matrix and $\widehat{\chi}(S_{nm}^{-1/2}(\pi_n f))$ the Mahalonobis transform of $\widehat{\chi}_{nm}$. Y_{nm} and Z_{nm} can be regarded as double sequences of stochastic precesses (fields) with the generalized index set F, the first one being complex- valued and the latter real-valued. Such a stochastic process interpretation leads to convergence results (invariance principles) which are stronger than those given in Theorem 2.4.

Under suitable conditions and for fixed dimension n, the process Y_{nm} converges weakly (in the ordinary sense) as $m \to \infty$ to a complex- valued n-variate Gaussian random field Y_n, which is the Fourier transform of an n-variate Brownian bridge process, (c.f. [2]). Similarly Z_{nm} converges weakly to a real-valued continuous-path Gaussian process Z_n, (c.f. [2]). In both of these weak convergences f (or $\pi_n f$ if $F \neq R_0^N$) is considered to be restricted to some compact sets K_n of the form $K_n = [-T,T]^n$ $(T > 0)$.

Now if both the dimension n and the number of observations m increase without bound, we have to consider the weak convergences of the processes described above within the framework of a projective system, i.e. in the sense of weak convergence along projective systems. Specifically we define for the multicubes $K_n = [-T,T]^n$, $(T > 0)$ the projection mappings $\gamma_{n_1n_2}: \mathcal{C}(K_{n_2}) \to \mathcal{C}(K_{n_1})$ $(n_1 < n_2)$ by:

$$(\gamma_{n_1n_2}g)(x_1,\ldots,x_{n_1}) = g(x_1,,\ldots,x_{n_1},0,\ldots,0) \quad \text{for all } g \in \mathcal{C}(K-n_2)$$

(a similar definition applies to complex-valued functions to yield $\gamma_{n_1n_2}$: $\mathcal{C}^2(K_{n_2}) \to \mathcal{C}^2(K_{n_1})$). Let ν_n^Y (resp. ν_n^Z) be the probability measure induced by the process Y_n (resp. Z_n) on $\mathcal{C}^2(K_n)$ (resp. $\mathcal{C}(K_n)$). Let also $(\mathcal{C}^2(K_\infty), \mathcal{B}_{\mathcal{C}^2(K_\infty)}, \nu^Y)$ (resp. $\mathcal{C}(K_\infty), \mathcal{B}_{\mathcal{C}(K_\infty)}, \nu^Z)$ be the unique projective limit of the following projective system of Gaussian probability spaces:

$$\mathcal{P}^Y = \{(\mathcal{C}^2(K_n), \mathcal{B}_{\mathcal{C}^2(K_n)}, \nu_n^Y, \gamma_{n_1n_2})_{n_1<n_2}: n_1n_2 \in N\}$$
$$(\text{resp. } \mathcal{P}^Z = \{(\mathcal{C}(K_n), \mathcal{B}_{\mathcal{C}(K_n)}, \nu_n^Z, \gamma_{n_1n_2})_{n_1<n_2}: n_1n_2 \in N\}).$$

Then utilizing Theorem 3.5, it can be shown that for any sequence m_n such that $m_n \to \infty$ as $n \to \infty$, the probability measures induced on $\mathcal{C}^2(K_n)$ (resp. $\mathcal{C}(K_n)$ by the process Y_{nm_n} (resp. Z_{nm_n}) have subsequences converging weakly to ν^Y (resp. ν^Z) along the projective system $\mathcal{P}^Y$ (resp. $\mathcal{P}^Z$)), (c.f. [4]).

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