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COMMUTATORS IN ORTHOMODULAR POSETS

Introduction

In the theory of orthomodular lattices (abbreviated OML) a classical theorem, proved by G. Bruns and G. Kalmbach [5], states that every finitely generated OML decomposes into a direct product of a Boolean algebra and a tightly generated OML. This classical decomposition theorem has been generalized to OML's in which the commutator of (not necessarily finite) generating set exists [19], to OML's with a finite set of commutators [13] and to locally modular OML's [6]. Commutators of finitely generated OML's are further studied in [3] and [17]. Relations between commutators and partial compatibility are shown in [18], [19], [22] (see also [7] for another approach).

Relations between commutators and joint distributions of observables are studied in [8]–[11], [20], [22].

In the present paper, orthomodular σ-orthoposets (called logics) are studied and the commutator of a logic is defined. It is shown that provided the centre of a logic is complete, the logic decomposes into a direct product of two factors, one of them being a Boolean algebra and the other having no Boolean factor. Further, a transitive closure of a "c-compatibility" is introduced. It is shown that a logic L with a complete centre decomposes into a direct product of a Boolean algebra and a horizontal sum of so many logics, how many equivalence classes of the transitive closure of c-compatibility (all different from the centre) are contained in L.

1. Definitions and known results

In this part, we introduce some basic definitions and known results about orthomodular posets. These results can be found in [1], [2], [16], [18], [21], [22]. We follow [22] as concerns definitions and notations.

By a (quantum) logic we will mean a σ-orthomodular poset (Def. 1.1. in [22]). That is, L is a partially ordered set with a unary operation $'$ such that the following conditions are satisfied (the symbols $\wedge$ and $\vee$ denote the lattice operations of meet and join induced by $\leq$):

(i) L possesses a least and a greatest elements, 0 and 1, respectively, and $0 \neq 1$;

(ii) $(a')' = a$;

(iii) $a \leq b \Rightarrow a' \geq b'$;

(iv) calling a, b in L orthogonal (written $a \perp b$) if $a \leq b'$, $\bigvee_{i \in \mathbb{N}} a_i$ exists in L for every subset $(a_i)_{i \in \mathbb{N}}$ of pairwise orthogonal elements of L;

(v) $a \leq b$ implies $b = a \vee (a' \wedge b)$.

We note that (i)–(v) imply that $a \vee a' = 1$ for any $a \in L$ (dually $a \wedge a' = 0$). Property (v) is the orthomodular law. In dual form (v) is

(v') $b \leq a \Rightarrow b = a \wedge (a' \vee b)$.

If L is a lattice (i.e., if it is closed under formation of finite suprema and infima), then (iv) implies that L is a σ-lattice (i.e., countable suprema and infima exist in L, see [22], 1.3.9).

Two important examples of logics are Boolean σ-algebra and the lattice $L(H)$ of all closed linear subspaces of a (real or complex) Hilbert space H. Another example, which is not necessarily a lattice, is a concrete logic, that is, a subset of the power set 2^X of a nonempty set X, closed under formation of set-theoretical complements and countable suprema of mutually disjoint elements (see [22]).

In what follows, the symbol L is reserved for a logic. A subset M of L is called a sublogic of L if M is closed under formation of orthocomplements and countable suprema of mutually orthogonal elements. If M is a sublogic of L and M is a Boolean σ-algebra with the operations $'$, $\vee$, $\wedge$ inherited from L, we call M a Boolean sublogic of L. A maximal Boolean sublogic of L is called a block of L.

A pair (a, b), $a, b \in L$, is called a compatible pair (or the elements a, b are called compatible) if there exist three mutually orthogonal elements a_1, b_1, c in L such that $a = a_1 \vee c$, $b = b_1 \vee c$. We write $a \leftrightarrow b$ if (a, b) is a compatible pair. We collect some important properties of the relation $\leftrightarrow$ in the following proposition.

PROPOSITION 1.1. (i) *If $a \leftrightarrow b$ and $a = a_1 \vee c$, $b = b_1 \vee c$ with a_1, b_1, c mutually orthogonal then $a_1 = a \wedge b'$, $b_1 = a' \wedge b$, $c = a \wedge b$, $a \vee b = a_1 \vee b_1 \vee c$.*

(ii) $a \leq b \Rightarrow a \leftrightarrow b$.

(iii) $a \leftrightarrow b \Leftrightarrow a' \leftrightarrow b$.

(iv) *Assume that $b \leftrightarrow a_i$ for all $i \in I$, where I is any set. If the suprema $\bigvee_{i \in I} a_i$, $\bigvee_{i \in I}(a_i \wedge b)$ exist in L, then $b \leftrightarrow \bigvee_{i \in I} a_i$ and $(\bigvee_{i \in I} a_i) \wedge b =$*

$\bigvee_{i\in I}(a_i \wedge b)$. *Dually, if the infima* $\bigwedge_{i\in I} a_i$ *and* $\bigwedge_{i\in I}(a_i \vee b)$ *exist in* L, *then* $b \leftrightarrow \bigwedge_{i\in I} a_i$ *and* $(\bigwedge_{i\in I} a_i) \vee b = \bigwedge_{i\in I}(a_i \vee b)$.

(v) $a = (a \wedge b) \vee (a \wedge b') \Rightarrow a \leftrightarrow b$.

Proof. Proofs of (i), (ii) and (iii) can be found in [22], 1.3.2, 1.3.4 and 1.3.5. (iv) can be proved in the same way as 1.3.8 in [22] (where the set I is supposed to be countable). (v) Put $c = a \wedge b$, $a_1 = a \wedge b'$. From $c \leq b$ and orthomodularity we have $b = c \vee (c' \wedge b)$. Hence with $b_1 = c' \wedge b$, a_1, b_1, c are pairwise orthogonal and $a = a_1 \vee c$, $b = b_1 \vee c$.

For a subset A of L we write

$$A^c = \{b \in L : b \leftrightarrow a \quad \text{for all } a \in A\}.$$

Clearly, $A \subset A^{cc}$ (where $A^{cc} = (A^c)^c$), $A \subset B \Rightarrow B^c \subset A^c$, $A^{ccc} = A^c$ for any $A, B \subset L$. The set L^c is the centre of L. The centre L^c of L is a Boolean sublogic of L (it is equal to the intersection of all blocks, [22], 1.3.17). Moreover, $L = L^c$ if and only if L is a Boolean σ-algebra.

If L is a lattice, then every pairwise compatible subset of L can be embedded into a Boolean sublogic of L. If L is not a lattice, the following definition determines those subsets of L which can be enlarged to Boolean sublogics of L:

DEFINITION 1.2. A subset A of L is called compatible (or the elements of A are called compatible) if for any finite subset F of A, $F = \{a_1, \ldots, a_n\}$, there is a finite subset $G = \{o_1, \ldots, o_m\}$ of pairwise orthogonal elements of L such that every element in F is the supremum of some elements of G. The set G is called an orthogonal covering of F.

It can be shown that if $F = \{a_1, a_2\}$, then F is compatible in the sense of Definition 1.2 if and only if $a_1 \leftrightarrow a_2$ ([22], 1.3.19).

For any $a \in L$, $a \neq 0$, the interval $L_{[0,a]} = \{b \in L : b \leq a\}$ with partial order inherited from L and with the relative orthocomplementation $b \to b'^a = b' \wedge a$ is a logic ([22], 1.3.12). It is easy to see that the following statements hold.

PROPOSITION 1.3. *Let* $a \in L$, $a \neq 0$. *A finite subset* $F = \{b_1, \ldots, b_n\}$ *of* L, *where* $b_i \leq a$, $i \leq n$, *is compatible in* L *if and only if* F *is compatible in* $L_{[0,a]}$.

For $a \in L$, let us write $a^1 = a$, $a^{-1} = a'$. Put $D = \{-1, 1\}$. Let $F = \{a_1, \ldots, a_n\}$ be a finite subset of L. If for any $d \in D^n$, $d = \{d_1, \ldots, d_n\}$, the infimum

$$F(d) = a_1^{d_1} \wedge \cdots \wedge a_n^{d_n}$$

exists in L, we call the element $\bigvee_{d\in D^n} F(d)$ the commutator of F and write

$$\operatorname{com} F = \bigvee_{d\in D^n} a_1^{d_1} \wedge \cdots \wedge a_n^{d_n}.$$

PROPOSITION 1.4. *Let F be a finite subset of L. Then F is compatible if and only if all the elements $F(d)$, $d \in D^n$ exist in L and $\bigvee_{d\in D^n} F(d) = 1$ (i.e., $\operatorname{com} F = 1$). If F is compatible then $\{F(d) : d \in D^n\}$ is the coarsest covering of the set $F \cup F'$, where $F' = \{a' : a \in F\}$.*

Proof. See [22], 1.3.22.

PROPOSITION 1.5. *Let A be a compatible subset of L. Then there is a Boolean sublogic L_0 of L such that $A \subset L_0$.*

Proof. See [22], 1.3.23.

Clearly, in a Boolean logic every finite subset is compatible, hence the converse of Proposition 1.5 also holds.

A logic L is called regular if for every pairwise compatible elements a, b, c in L we have $a \leftrightarrow b \vee c$. By Proposition 1.1 (iv), every lattice logic is regular. The following proposition characterizes those logics in which pairwise compatibility coincides with compatibility.

PROPOSITION 1.6. *A logic L is regular if and only if every pairwise compatible subset of L can be embedded into a Boolean sublogic of L.*

Proof. See [22], 1.3.29.

2. Partial compatibility and commutators

In this section, we will introduce the notion of partial compatibility and prove some basic statements. Then we will introduce the notion of a commutator of a logic L with complete centre and prove a decomposition theorem.

DEFINITION 2.1. We say that a subset M of L is partially compatible with respect to an element a of L if

(i) $M \leftrightarrow a$ (i.e., $m \leftrightarrow a$ for all $m \in M$),

and

(ii) $M \wedge a = \{m \wedge a : m \in M\}$ is a compatible set.

We will write m is p.c. a if M is partially compatible with respect to a.

PROPOSITION 2.2. *Let $F = \{a_1, \ldots, a_n\}$ be a finite subset of L. Then F is p.c. a ($a \in L$) if and only if all elements $F(d) \wedge a = a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a$ ($d \in D^n$) exist and*

$$(*) \qquad \bigvee_{d\in D^n} F(d) \wedge a = a.$$

A subset M of L is p.c. a if and only if every finite subset F of M is p.c. a.

Proof. Let F be p.c. a. Then $F \wedge a$ is compatible in the logic $L_{[0,a]}$ (see Proposition 1.3). Hence, by Proposition 1.4, $\mathrm{com}_{[0,a]}(F \wedge a) = a$ (where $\mathrm{com}_{[0,a]}$ denotes the commutator computed in $L_{[0,a]}$). An easy computation, using $F \leftrightarrow a$, shows that $\mathrm{com}_{[0,a]}(F \wedge a) = \bigvee_{d \in D^n} F(d) \wedge a$. Conversely, assume that $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a$ ($d \in D^n$) exist and (*) holds. (We note that the existence of the element $F(d) \wedge a = a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a$ does not mean that the element $F(d) = a_1^{d_1} \wedge \ldots \wedge a_n^{d_n}$ exists). By Proposition 1.1, $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a \leftrightarrow a_j^j$ for all $i \leq n$ and $j \in \{-1, 1\}$. Using Proposition 1.1 (iv), we get from (*) that $a_i^j \wedge a = \bigvee_{d \in D^n} F(d) \wedge a \wedge a_i^j = \bigvee_{\{d \in D^n : d_i = j\}} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a$, and again by (*), $(a_i \wedge a) \vee (a_i' \wedge a) = \bigvee_{\{d \in D^n : d_i = 1\}} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a \vee \bigvee_{\{d \in D^n : d_i = -1\}} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a) = \bigvee_{d \in D^n} F(d) \wedge a = a$. Hence by Proposition 1.1 (v), $a_i \leftrightarrow a$ ($i \leq n$).

This gives

$$\underset{[0,a]}{\mathrm{com}}(F \wedge a) = \bigvee_{d \in D^n} F(d) \wedge a$$

and (*) implies that $\mathrm{com}_{[0,a]}(F \wedge a) = a$. By Proposition 1.4, $F \wedge a$ is compatible in $L_{[0,a]}$, and by Proposition 1.3 this is equivalent to the compatibility in L. Consequently, F is p.c. a. The remaining part of the proof follows directly by the definition of partial campatibility.

If L is a lattice then $\mathrm{com}\, F$ exists for every finite set F ($F \subset L$). By [22] 5.1.8, $\mathrm{com}\, F$ in a lattice logic is the greatest element with respect to which F is partially compatible. If L is not a lattice, then the commutator of a finite subset of L need not exist. As an example, consider the Greechie diagram

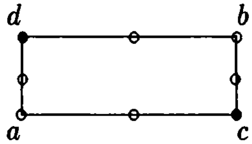

Fig. 1

(We recall that, in a Greechie diagram, points represent atoms of L, smooth lines join atoms belonging to a block of L and angles represent "pasting" of two blocks in a common atom, see [12], [14], [22]). Let $M = \{a, b\}$. The set of all lower bounds of a' and b' is $\{c, d\}$, but c and d are noncomparable (because they are not contained in the same block). Therefore $a' \wedge b'$ does not exist, and hence also $\mathrm{com}(a, b)$ does not exist.

In a special case when the commutator exists, it plays a similar role in a σ-orthomodular poset as in an OML. This will be shown in the following proposition.

PROPOSITION 2.3. *Let $F = \{a_1, \ldots, a_n\}$ be a finite subset of L and let* $\operatorname{com} F$ *exist. Then the following hold:*

(i) *If F is p.c. a ($a \in L$) then $a \leq \operatorname{com} F$.*
(ii) *F is p.c. $\operatorname{com} F$.*
(iii) *If L is regular, then $a \leq \operatorname{com} F$ and $a \leftrightarrow F$ imply that F is p.c. a.*

Proof. (i) If F is p.c. a, then by $(*)$ of Proposition 2.2,

$$a \geq (\operatorname{com} F) \wedge a \geq \bigvee_{d \in D^n} F(d) \wedge a = a,$$

hence $a \leq \operatorname{com} F$.

(ii) Since $a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \leftrightarrow a_i$ ($i \leq n$, $d \in D^n$), by Proposition 1.1 (iv) we get $\operatorname{com} F \leftrightarrow F$ and

$$\operatorname{com} F = \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge \operatorname{com} F,$$

hence by Proposition 2.2, F is p.c. $\operatorname{com} F$.

(iii) Let L be regular and let $a \leftrightarrow F$ and $a \leq \operatorname{com} F$. Let $b \in F$, Then a, b, $\operatorname{com} F$ are pairwise compatible. By regularity of L, $a \leftrightarrow b \wedge \operatorname{com} F$, $a \leftrightarrow b' \wedge \operatorname{com} F$. Since $\{b \wedge \operatorname{com} F : b \in F \cup F'\}$ is a compatible set, regularity implies that $a \leftrightarrow a_1^{d_1} \wedge \operatorname{com} F \wedge \ldots \wedge a_n^{d_n} \wedge \operatorname{com} F = F(d)$ for any $d \in D^n$. By Proposition 1.1 (iv),

$$a = a \wedge \operatorname{com} F = \bigvee_{d \in D^n} F(d) \wedge a,$$

and by Proposition 2.2, F is p.c. a.

PROPOSITION 2.4 (see also [18]). *Let F be a finite subset of L and let $a \in L$. Then if F is p.c. a, also F^{cc} is p.c. a.*

Proof. Let $F = \{a_1, \ldots, a_n\}$ be p.c. a. Let $b \in F^{cc}$ ($b \notin F$). We prove that $F \cup \{b\}$ is p.c. a. Owing to Proposition 2.2, we have to prove that

$$a = \bigvee_{d \in D^{n+1}} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge b^{d_{n+1}} \wedge a,$$

$d = (d_1, \ldots, d_n, d_{n+1})$. Since F is p.c. a, we have

$$(*) \qquad a = \bigvee_{d \in D^{n+1}} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \wedge a = \bigvee_{d \in D^n} F(d) \wedge a.$$

As $F(d) \wedge a \in F^c$ and $b \in F^{cc}$, we have $F(d) \wedge a \leftrightarrow b$ ($d \in D^n$). By Proposition 1.1. (iv),

$$\Big(\bigvee_{d\in D^n} F(d)\wedge a\Big)\wedge b^j = \bigvee_{d\in D^n} F(d)\wedge a\wedge b^j,\ j\in\{-1,1\}.$$

From this, using $(*)$ and the equality $a = (a\wedge b)\vee(a\wedge b')$ we derive the desired result.

Now we will proceed by induction. Assume that $F\cup\{b_1,\ldots,b_n\}$ is p.c. a, where $b_1,\ldots,b_n\in F^{cc}$. Let $b_{n+1}\in F^{cc}$. Since $F^{cc} = (F\cup\{b_1,\ldots,b_n\})^{cc}$, by the first part of the proof we obtain that $F\cup\{b_1,\ldots,b_n,b_{n+1}\}$ is p.c. a. Consequently, every finite subset of F^{cc} is p.c. a, and hence F^{cc} is p.c. a.

For a subset M of L define

$$\mathcal{P}(M) = \{a\in L : M \text{ is p.c. } a\}.$$

Clearly, $\mathcal{P}(M)\subset M^c$ and for every $a\in\mathcal{P}(M)$, $M\wedge a$ is compatible.

PROPOSITION 2.5. *Let $(q_i)_{i\in\mathbb{N}}$ be pairwise orthogonal sequence of elements of L such that $q_i\in\mathcal{P}(M)$ $(i\in\mathbb{N})$. Then $\bigvee_{i\in\mathbb{N}} q_i\in\mathcal{P}(M)$.*

Proof. Since M^c is a sublogic of L (see [22] 1.3.16), and $\mathcal{P}(M)\subset M^c$, we obtain that $\bigvee_{i\in\mathbb{N}} q_i\in M^c$. Let $F=\{a_1,\ldots,a_n\}\subset M$. Since F is p.c. q_i $(i\in\mathbb{N})$, $(*)$ of Proposition 2.2. holds with a replaced by q_i $(i\in\mathbb{N})$. From $F\leftrightarrow q_i$ and Proposition 1.1. (iv) we have $a_j\wedge(\bigvee_{i\in\mathbb{N}} q_i) = \bigvee_{i\in\mathbb{N}}(q_i\wedge a_j)$. Using $(*)$ we get

$$\begin{aligned} a_j\wedge\Big(\bigvee_{i\in\mathbb{N}} q_i\Big) &= \bigvee_{i\in\mathbb{N}} a_j\wedge\Big(\bigvee_{d\in D^n} F(d)\wedge q_i\Big)\\ &= \bigvee_{i\in\mathbb{N}}\ \bigvee_{\{d\in D^n : d_j=1\}} a_1^{d_1}\wedge\ldots\wedge a_n^{d_n}.\wedge q_i, \end{aligned}$$

where the last equality follows by the fact that $F(d)\wedge q_i\leftrightarrow a_j$ $(d\in D^n)$. Hence the set $\{a_1^{d_1}\wedge\ldots\wedge a_n^{d_n}\wedge q_i : d\in D^n,\ i\in\mathbb{N}\}$ forms an orthogonal covering of the set $\{a_j\wedge(\bigvee_{i\in\mathbb{N}} q_i) : j\le n\}$, and hence the latter set is compatible. This proves that F is p.c. $\bigvee_{i\in\mathbb{N}} q_i$ for every finite subset F of M, which concludes the proof.

Recall that a σ-ideal in a Boolean algebra B is a subset P of B such that (i) $a\in P$ and $b\le a$ imply $b\in P$, and (ii) $a_i\in P$ $(i\in\mathbb{N})\Rightarrow\bigvee_{i\in\mathbb{N}} a_i\in P$.

THEOREM 2.6. *For any logic L, the set $\mathcal{P}(L)$ is a σ-ideal in L^c. In addition, if $a\in\mathcal{P}(L)$ and $b\le a$, $b\in L$, then $b\in\mathcal{P}(L)$.*

Proof. We have $\mathcal{P}(L)\subset L^c$. Let $a\in\mathcal{P}(L)$ and $b\le a$, $b\in L$. Since $L\wedge a$ is compatible and $b\wedge a=b$, we have $b\leftrightarrow c\wedge a$ for any $c\in L$. Since $a\in L^c$, we have $c=(c\wedge a)\vee(c\wedge a')$. Now $c\wedge a'\le a'\le b'$ imply that $c\wedge a'\leftrightarrow b$. Therefore by Proposition 1.1 (iv), $b\leftrightarrow c$ for every $c\in L$, i.e. $b\in L^c$. Since $L\wedge b = L\wedge b\wedge a\subset L\wedge a$, and $a\in\mathcal{P}(L)$, $L\wedge b$ is compatible. Hence $b\in\mathcal{P}(L)$. Now let $(a_i)_{i\in\mathbb{N}}\subset\mathcal{P}(L)$. Put $b_1=a_1$, $b_{n+1}=a_{n+1}\wedge(\bigvee_{i\le n} a_i)'$. By the first part of the proof, $b_i \in \mathcal{P}(L)$ ($i \in \mathbb{N}$). In addition, $\bigvee_{i\in\mathbb{N}} b_i = \bigvee_{i\in\mathbb{N}} a_i$, and since $(b_i)_{i\in\mathbb{N}}$ are mutually orthogonal, by Proposition 2.5, $\bigvee_{i\in\mathbb{N}} a_i = \bigvee_{i\in\mathbb{N}} b_i \in \mathcal{P}(L)$.

Recall that in an orthomodular lattice L an element $a \in L$ is called central abelian if $a \in L^c$ and the interval $L_{[0,a]}$ is a Boolean algebra. In analogy with this, we will call the elements in $\mathcal{P}(L)$ central abelian elements of the logic L.

A logic L is separable if every set of pairwise orthogonal elements in L is at most countable. A separable logic which is a lattice is a complete lattice (see [22] 2.5.2 f). In what follows we will consider a logic L with a complete centre L^c, i.e., L^c is a complete Boolean algebra. For example, if L is separable, or if L^c is separable, then L^c is complete.

THEOREM 2.7. *Let L be a logic with complete centre L^c. Then the supremum $\bigvee \mathcal{P}(L)$ exists and belongs to $\mathcal{P}(L)$. Moreover, L is isomorphic to the direct product $L_{[0,c]} \times L_{[0,c']}$ where $c = \bigvee \mathcal{P}(L)$. The logic $L_{[0,c]}$ is a Boolean algebra, and $L_{[0,c']}$ has no nonzero central abelian element.*

Proof. As L^c is complete, the supremum $c = \bigvee \mathcal{P}(L)$ exists and belongs to L^c. Let $a \in L$ be fixed. We have $a \leftrightarrow d$ for all $d \in \mathcal{P}(L)$. Moreover, $a \wedge d \leq d$, and hence $a \wedge d \in \mathcal{P}(L) \subset L^c$ by Theorem 2.6. As L^c is complete, we have $\bigvee_{d\in\mathcal{P}(L)} a \wedge d$ exists and belongs to L^c. By Proposition 1.1 (iv), $a \wedge c = \bigvee_{d\in\mathcal{P}(L)} a \wedge d \in L^c$. Hence $L \wedge c \subset L^c$, which entails that $c \in \mathcal{P}(L)$. From $c \in L^c$ it follows that L is isomorphic to $L_{[0,c]} \times L_{[0,c']}$. Now $L \wedge c = L_{[0,c]}$ is compatible in $L_{[0,c]}$, hence it is a Boolean algebra. Assume that $0 \neq c_1$ is a central abelian element in $L_{[0,c']}$. We show that $c_1 \in \mathcal{P}(L)$ and since $c_1 \perp c$, we have a contradiction. As $c_1 \leq c'$ and $c_1 \leftrightarrow d$ for any $d \in L_{[0,c']}$, it follows that $c_1 \in L^c$. As c_1 is central abelian in $L_{[0,c']}$, the set $L \wedge c_1 = L \wedge c' \wedge c_1$ is compatible. This implies that $c_1 \in \mathcal{P}(L)$.

If L is a logic with complete centre L^c, we will define the commutator of L by $\operatorname{com} L = \bigvee \mathcal{P}(L)$. This definition is a generalization of the commutator in an orthomodular lattice. We note that if M is a proper subset of a logic L, then the supremum of $\mathcal{P}(L)$ need not exist even if L is finite. Let us consider again the logic L with the Greechie diagram on Fig. 1. We can easily see that $\mathcal{P}(\{a,b\}) = \{c, d\}$, but $c \vee d$ does not exist in L. By Proposition 2.3, if the commutator of a finite set F in L exists, then $\operatorname{com} F = \bigvee \mathcal{P}(F)$.

3. Transitive compatibility and horizontal sums of logics

In this section, we will study relations between horizontal sums and equivalence classes of the transitive closure of so-called c-compatibility.

DEFINITION 3.1. Let L be a logic and let the centre L^c of L be trivial, i.e. $L^c = \{0,1\}$. We say that L admits a decomposition into horizontal sum (or a horizontal decomposition) if there is a system $\{L_i : i \in I\}$ satisfying the following conditions:

(i) $L = \bigcup_{i\in I} L_i$,
(ii) L_i is a sublogic of L $(i \in L)$,
(iii) $L_i \neq L^c$ $(i \in I)$,
(iv) $L_i \cap L_j = \{0,1\}$ $i \neq j$ $(i,j \in I)$,
(v) $0 \leq a \leq 1$ for any $a \in L$ and for any $a, b \in L \setminus \{0,1\}$ we have $a \leq b$ if and only if there is $i_0 \in I$ such that $a, b \in L_{i_0}$ and $a \leq b$ in L_{i_0}.

If L admits a decomposition $\{L_i : i \in I\}$ into a horizontal sum, we shall write $L = \boxplus_{i\in I} L_i$.

LEMMA 3.2. *Let $L = \boxplus_{i\in I} L_i$. The sets $L_i \setminus \{0,1\}$ $(i \in I)$ form a partition of $L \setminus \{0,1\}$ (in the usual set-theoretical sense).*

The proof follows directly from the definition.

Let $\{L_i : i \in I\}$ and $\{M_j : j \in J\}$ be two horizontal decompositions of a logic L. We will say that the decomposition $\{L_i : i \in I\}$ is a refinement of the decomposition $\{M_j : j \in J\}$ if the partition $\{L_i \setminus \{0,1\} : i \in I\}$ is a refinement of the partition $\{M_j \setminus \{0,1\} : j \in J\}$.

Now we introduce a definition of a c-compatibility as follows: the elements a, b of L are c-compatible (written $a \overset{c}{\leftrightarrow} b$) if either (i) $a, b \in L^c$ or (ii) $a \notin L^c, b \notin L^c$ and $a \leftrightarrow b$. Clearly, the relation $\overset{c}{\leftrightarrow}$ is reflexive and symmetric. Let $\sim$ denote the transitive closure of c-compatibility, i.e., $a \sim b$ if there are elements $e_1, \dots, e_n$ in L such that $e_1 = a$, $e_n = b$ and $e_i \overset{c}{\leftrightarrow} e_{i+1}$ $(i < n)$. Then $\sim$ is an equivalence relation. Clearly, one of the equivalence classes is the centre L^c of L. In what follows we will denote by $\mathcal{T}$ the family of all equivalence classes of the relation $\sim$ different from L^c.

PROPOSITION 3.3. *Let $\mathcal{T} \cup \{L^c\}$ be the family of all equivalence classes of the relation $\sim$ in L. Then for every subfamily $\mathcal{S}$ of $\mathcal{T}$ the set $L_1 = \bigcup_{T\in\mathcal{S}} T \cup L^c$ is a sublogic of L.*

Proof. We have to prove that (i) $a \in L_1$ implies $a' \in L_1$, (ii) $\{a_i : i \in \mathbb{N}\} \subset L_1$ and $a_i \perp a_j$, $i \neq j$ $(i,j \in \mathbb{N})$ imply $\bigvee_{i\in\mathbb{N}} a_i \in L_1$. (i) If $a \in L_1$, $a \notin L^c$, then $a' \notin L^c$, hence $a \sim a'$, and therefore $a' \in L_1$. If $a \in L^c$ then also $a' \in L^c$, i.e., $a' \in L_1$.

(ii) Let $\{a_i : i \in \mathbb{N}\}$ be a sequence of pairwise orthogonal elements in L_1. Two cases can occur: (a) $\bigvee_{i\in\mathbb{N}} a_i \in L^c$ and (b) $\bigvee_{i\in\mathbb{N}} a_i \notin L^c$. In case (a) there is nothing to prove. In case (b) there must be at least one $j \in \mathbb{N}$

such that $a_j \notin L^c$. Since $a_j \leq \bigvee_{i\in\mathbb{N}} a_i$, we obtain that $a_j \overset{c}{\leftrightarrow} \bigvee_{i\in\mathbb{N}} a_i$, so that $\bigvee_{i\in\mathbb{N}} a_i \in L_1$.

PROPOSITION 3.4. *Let L be a logic with complete centre L^c. Let there be $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$. Then for any $a \in T_1$, $b \in T_2$ the elements $a \vee b$ and $a \wedge b$ exist in L and belong to L^c.*

Proof. Let d be an upper bound of a and b. Assume that $d \notin L^c$. Then $d \geq a$, $d \geq b$ imply that $d \overset{c}{\leftrightarrow} a$, $d \overset{c}{\leftrightarrow} b$, which contradics the supposition $a \in T_1$, $b \in T_2$. Hence all upper bounds of a and b are in L^c. Since L^c is a complete Boolean sublogic of L, the infimum of all upper bounds exists and belongs to L^c. The proof for $a \wedge b$ is dual.

COROLLARY 3.5. *If the centre L^c of L is complete and $a \in T_1$, $b \in T_2$, where T_1 and T_2 are distinct elements of $\mathcal{T}$, then* $\operatorname{com}\{a, b\}$ *exists and belongs to L^c.*

PROPOSITION 3.6. *Let the centre L^c of L be complete and let $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$. Then* $\operatorname{com} L = \operatorname{com}\{a_1, a_2\}$, *where $a_i \in T_i$ $(i = 1, 2)$ are arbitrary elements.*

Proof. If $a_i \in T_i$, $i = 1, 2$, then $\{a_1, a_2\}^c = L^c$. By Proposition 2.3 (ii), $\{a_1, a_2\}$ is p.c. $\operatorname{com}\{a_1, a_2\}$ and by Proposition 2.4, $L = \{a_1, a_2\}^{cc}$ is p.c. $\operatorname{com}\{a_1, a_2\}$. Hence $\operatorname{com}\{a_1, a_2\} \in \mathcal{P}(L)$. Now if $d \in \mathcal{P}(L)$, then $d \in \mathcal{P}(\{a_1, a_2\})$, and by Proposition 2.3 (i), $d \leq \operatorname{com}\{a_1, a_2\}$. Hence $\operatorname{com}\{a_1, a_2\} = \bigvee \mathcal{P}(L) = \operatorname{com}\ L$.

PROPOSITION 3.7. *Let L be a logic and let $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$. If for any $a \in T_1$, $b \in T_2$ $a \vee b = 1$ (or, dually, $a \wedge b = 0$) then $L^c = \{0, 1\}$.*

Proof. Let there be a $c \in L^c$, $c \neq 0$, $c \neq 1$. Let $a \vee b = 1$ for any $a \in T_1$, $b \in T_2$. We have, by Proposition 3.3, $a \vee c \in T_1 \cup L^c$, $b \vee c \in T_2 \cup L^c$. The following cases occur:

(a) $a \wedge c \in L^c$, $b \wedge c \in L^c$. Then $a \wedge c' \in T_1$, $b \wedge c' \in T_2$. Indeed, if $a \wedge c \in L^c$, then $a = (a \wedge c) \vee (a \wedge c') \in L^c$, a contradiction. The equality $(a \wedge c') \vee (b \wedge c') = 1$ implies that $c' = 1$, which contradicts the supposition.

(b) $a \wedge c \in T_1$, $b \wedge c \in L^c$ (or, symmetrically, $a \wedge c \in L^c$, $b \wedge c \in T_2$). Then $b \wedge c' \in T_2$ and therefore $(a \wedge c) \vee (b \wedge c') = 1$. This implies that that $c = c \wedge ((a \wedge c) \vee (b \wedge c')) = a \wedge c \in T_1$, which contradicts $c \in L^c$.

(c) $a \wedge c \in T_1$, $b \wedge c \in T_2$. Then $1 = (a \wedge c) \vee (b \wedge c) \leq c$, a contradiction.

The case when $a \wedge b = 0$ for any $a \in T_1$, $b \in T_2$ can be proved dually.

THEOREM 3.8. *Let L $(L \neq \{0, 1\})$ be a logic with $L^c = \{0, 1\}$. Let $\{L^c\} \cup \{T_i : i \in I\}$ be the partition of L induced by the relation $\sim$ of transitive closure of c-compatibility. Put $L_i = L^c \cup T_i$ $(i \in I)$. Then $\{L_i : i \in I\}$ is a* *horizontal decomposition of L. Moreover, for any horizontal decomposition $\{M_j : j \in J\}$ of L, $\{L_i : i \in I\}$ is a refinement of $\{M_j : j \in J\}$.*

Proof. First we prove that $L = \boxplus_{i \in I} L_i$. Evidently, (i), (iii) and (iv) of Definition 3.1 hold. Further, Proposition 3.3 implies (ii). To prove (v), let $a, b \in L \setminus \{0,1\}$ and $a \leq b$. Then $a \overset{c}{\leftrightarrow} b$, and therefore there is a $i_0 \in I$ such that $a, b \in T_{i_0}$. The rest of (v) follows by Proposition 3.3.

Now let $\{M_j : j \in J\}$ be another decomposition of L into a horizontal sum. If $J = \{j_1\}$, the statement is obvious. If there are at least two elements j_1, j_2 in J, then there are $a, b \in L$ such that $a \in M_{j_1} \setminus \{0,1\}$, $b \in M_{j_2} \setminus \{0,1\}$. Assume that $a \sim b$. Then there are $e_1, \dots, e_n$ in L such that $e_1 = a$, $e_n = b$ and $e_i \overset{c}{\leftrightarrow} e_{i+1}$ $(i < n)$. But $e_i \overset{c}{\leftrightarrow} e_{i+1}$ $(i < n)$ implies, by induction using (v) of Definition 3.1, that $(e_i)_{i \leq n}$ are all contained in $M_{j_1} \setminus \{0,1\}$ in contradiction with $b \in M_{j_2} \setminus \{0,1\}$.

LEMMA 3.9. *Let $L = B \times L_0$, where B is a Boolean algebra. Then the following hold:*

(i) $(a,b) \perp (c,d)$ $((a,b),(c,d) \in L) \Leftrightarrow a \perp c$ *and* $b \perp d$.
(ii) $(a,b) \leftrightarrow (c,d) \Leftrightarrow b \leftrightarrow d$.
(iii) $(a,b) \in L^c \Leftrightarrow b \in L_0^c$.
(iv) $(a,b) \overset{c}{\leftrightarrow} (c,d) \Leftrightarrow b \overset{c}{\leftrightarrow} d$ *in* L_0.
(v) $(a,b) \sim (c,d) \Leftrightarrow b \sim d$ *in* L_0.

Proof. (i), (ii) and (iii) follow directly from the definition of direct product and compatibility of all elements in B. (iv) If $(a,b) \in L^c$, $(c,d) \in L^c$, the statements follows from (iii). If $(a,b) \notin L^c$, $(c,d) \notin L^c$, we use (ii) and (iii). (v) follows from (iv).

THEOREM 3.10. *Let L be a logic with complete centre L^c. Then $L \cong L_{[0,c]} \times L_{[0,c']}$, where c is the commutator of L. The factor $L_{[0,c]}$ is a Boolean σ-algebra and the factor $L_{[0,c']}$ admits a horizontal decomposition $\{L_i : i \in I\}$, where $L_i = (L^c \cup T_i) \wedge c'$, and $\{T_i : i \in I\} \cup L^c$ is the family of all equivalence classes of the transitive closure of c-compatibility in L.*

Proof. By Theorem 2.7, $L \cong L_{[0,c]} \times L_{[0,c']}$, and $L_{[0,c]}$ is a Boolean σ-algebra. By Lemma 3.9, the system $\{T_i \wedge c' : i \in I\} \cup \{L^c \wedge c'\}$ is the family of all equivalence classes of the relation $\sim$ in $L_{[0,c']}$.

If I contains only one element, the statement of the theorem holds. Let there be at least two elements $a, b \in L_{[0,c']}$ such that $a \sim b$ does not hold. Then $\mathrm{com}_{[0,c']}(L_{[0,c']}) = \mathrm{com}_{[0,c']}\{a,b\} = 0$, since by Theorem 2.7, $L_{[0,c']}$ has no nontrivial central abelian element. Therefore for any $a, b \in L_{[0,c']}$ such that $a \sim b$ does not hold, we have $a \wedge b = 0$. By Proposition 3.7, the centre of $L_{[0,c']}$ is $L^c \wedge c' = \{0, c'\}$, and by Theorem 3.8, $L_{[0,c']}$ admits the horizontal decomposition $\{L_i : i \in I\}$.

EXAMPLES. Let H be a Hilbert space and let $L(H)$ denote the OML of all closed subspaces of H. Let X be a finite set with even cardinality and denote by $L_{\mathrm{even}}(X)$ the concrete logic of all subsets of X with even cardinality.

1. Let $L = L(H)$, dim $H = 2$. Then $L^c = \{0,1\}$ and $a \sim b$ implies $b = a'$. Hence L is the horizontal sum of an infinite number of 4-element Boolean algebras $\{0,1,a,a'\}$.

2. Let $L = L(H)$, dim $H \geq 3$. Then $L^c = \{0,1\}$ and $a \sim b$ holds for any $a, b \in L(H)$. Indeed, assume that $a, b \in L \setminus L^c$ and let x, y be vectors in H such that $x \in a$, $y \in b$. Let c be the subspace of H generated by x and y. Then $c \notin L^c$, $c \wedge a \neq 0$, $c \wedge b \neq 0$ and $a \stackrel{c}{\leftrightarrow} a \wedge c$, $a \wedge c \stackrel{c}{\leftrightarrow} c$, $c \stackrel{c}{\leftrightarrow} b \wedge c$, $b \wedge c \stackrel{c}{\leftrightarrow} b$.

3. Let $L = L_{\mathrm{even}}(X)$. Then $a \leftrightarrow b\,(a, b \in L)$ if and only if $a \cap b \in L_{\mathrm{even}}(X)$ (see [22]). If card $X \geq 4$, then $L^c = \{0,1\}$. If card $X = 4$, then it is easy to see that $a \sim b$ if and only if $a \leftrightarrow b$ for $a, b \notin L^c$. From this we conclude that L is the horizontal sum of three 4-element Boolean algebras.

4. Let $L = L_{\mathrm{even}}(X)$, card $X \geq 6$. Then $L^c = \{0,1\}$ and $a \sim b$ for any $a, b \in L \setminus L^c$. Indeed, if $a \cap b$ consits of two elements, then $a \leftrightarrow b$, hence $a \sim b$. If $a \cap b$ contains only one element, say x, then we take $y \in a \setminus (a \cap b)$, $z \in b \setminus (a \cap b)$ and $v \in X \setminus (a \cup b)$. Then $c = \{x, y, z, v\} \in L$, and $a \sim b$ via c. If $a \cap b$ contains at least three elements, $a \sim b$ through a two-element subset of $a \cap b$.

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