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Henryk Dębiński, Izydor Dziubiński

SOME REMARKS ON INFINITE ALGEBRAIC LINEAR EQUATIONS

The purpose of this paper is to prove the existence and some estimations of solutions of the infinite system of algebraic linear equations. We choose these theorems which will be useful in the theory of solving of linear equations in Banach algebras, which have some applications to the linear partial differential equations and linear integral equations of many variables and other.

I.1. Let k be any natural number and $\mathbb{N}^k$- the product of k copies of $\mathbb{N}$. We introduce in the set $\mathbb{N}^k$ an order-relation determined in the following way:

If $(i_1,\ldots,i_k)$, $(j_1,\ldots,j_k) \in \mathbb{N}^k$ and $W(k) = \{1,2,\ldots,k\}$, then:

D.1.1. $(i_1,\ldots,i_k) = (j_1,\ldots,j_k)$ if $\forall p \in W(k)\ i_p = j_p$

D.1.2. $(i_1,\ldots,i_k) < (j_1,\ldots,j_k)$ if one of the condition is fulfilled:

(i) $$\sum_{p=1}^{k} i_p < \sum_{p=1}^{k} j_p,$$

(ii) $$\Big(\sum_{p=1}^{k} j_p = \sum_{p=1}^{k} i_p\Big) \& (\exists m \in W(k) : i_1 = j_1, \ldots, i_{m-1} = j_{m-1},\ i_m < j_m).$$

D.1.3. $(i_1,\ldots,i_k) \le (j_1,\ldots,j_k)$ if one of the following conditions is fulfilled:

(i) $$(i_1,\ldots,i_k) < (j_1,\ldots,j_k),$$

(ii) $$(i_1,\ldots,i_k) = (j_1,\ldots,j_k).$$

Under the above definitions of the order-relation we can uniquely express the set $\mathbb{N}^k$ in the form of the one-dimension sequence $\{\alpha_r\}_{r=1,2,\ldots}$, where $\alpha_r = (i_1, i_2, \ldots, i_k)$. If it is necessary, the position of the element

$(i_1, i_2, \ldots, i_k)$ in the above sequence $\{\alpha_r\}_{r=1,2,\ldots}$ will be denoted by $r = r_{i_1 \ldots i_k}$.

It is easy to prove the following lemmas.

LEMMA 1.1. *Let $k, p \in \mathbb{N}$, $p \geq k$ be arbitrary fixed elements. Then the number of elements $(i_1, \ldots, i_k) \in \mathbb{N}^k$ such that*

$$\sum_{m=1}^{k} i_m = p \quad \text{is equal to} \quad \binom{p-1}{k-1}.$$

LEMMA 1.2. *If $(i_1, \ldots, i_k) \in \mathbb{N}^k$, then the number of the elements $(j_1, \ldots, j_k) \in \mathbb{N}^k$ such that $\sum_{m=1}^{k} j_m < \sum_{m=1}^{k} i_m$ is determined by the formula*

$$\binom{\sum_{m=1}^{k} i_m - 1}{k}. \tag{1.1}$$

LEMMA 1.3. *If $(i_1, \ldots, i_k) \in \mathbb{N}^k$, then the number of the elements $(j_1, \ldots, j_k) \in \mathbb{N}^k$ less than $(i_1, \ldots, i_k)$ such that $\sum_{m=1}^{k} i_m = \sum_{m=1}^{k} j_m$ is determined by the formula:*

$$\sum_{n=1}^{k-1} \sum_{p=0}^{i_n - 2} \binom{\sum_{m=n+1}^{k} i_m + p}{k-n-1} \tag{1.2}$$

where we assumed the convention: if $n < k$, then $\sum_{m=k}^{n} a_m = 0$.

Next we will prove

THEOREM 1.1. *For every $k \in \mathbb{N}$*

$$r_{i_1, \ldots, i_k} = \binom{\sum_{m=1}^{k} i_m - 1}{k} + \sum_{n=1}^{k-1} \sum_{p=0}^{i_n - 2} \binom{\sum_{m=n+1}^{k} i_m + p}{k-n-1} + 1 \tag{1.3}$$

under the convention: if $n < k$, then $\sum_{i=k}^{n} a_m = 0$.

Proof. According to the order-relation, the set T of the elements $(j_1, \ldots, j_k) \in \mathbb{N}^k$ less than $(i_1, \ldots, i_k) \in \mathbb{N}^k$ is the union of the set T_1 of the elements $(j_1, \ldots, j_k) \in \mathbb{N}^k$ such that $\sum_{m=1}^{k} j_m < \sum_{m=1}^{k} i_m$ and the set T_2 of the elements $(j_1, \ldots, j_k) \in \mathbb{N}^k$ such that $\sum_{m=1}^{k} j_m = \sum_{m=1}^{k} i_m$ and $(j_1, \ldots, j_k) < (i_1, \ldots, i_k)$. Of course, $T_1 \cap T_2 = \emptyset$.

Thus by use of Lemma 1.2 and Lemma 1.3 we obtain

$$\operatorname{card} T = \operatorname{card} T_1 + \operatorname{card} T_2$$

what complete the proof.

2. Let F be any number field (i.e. $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{Z}_p$) and $k \in \mathbf{N}$ — an arbitrary number.

D.1.4. Every function mapping $\mathbb{N}^k$ into F we call a k-dimensional infinite matrix denoted by $\overline{A}$ or $[a_{i_1 \dots i_k}]$, where $i_s = 1, 2, \dots$; for $s = 1, 2, \dots, k$. The number i_s ($s = 1, 2, \dots, k$) is called the index of the matrix $\overline{A}$.

Let $L^k(F)$ be the set of all k-dimensional infinite matrices over F. In the set $L^k(F)$ we introduce addition of elements and multiplication of elements by scalars in the standard way, so that $L^k(F)$ becomes a vector space.

D.1.5. The section of the matrix

$$\overline{A} = [a_{i_1 \dots i_p \dots i_q \dots i_k}] \in L^k(F), \qquad (i_s = 1, 2, \dots, s; \; s = 1, 2, \dots, k)$$

with respect to indices $(p, \dots, q)$ $(1 \le p \le q \le k)$ for the fix set $(j_p, \dots, j_q)$ we shall call the matrix

$$\overline{A}_{j_p, \dots, j_q}^{\;p,q} := [b_{i_1 \dots i_{p-1} i_{q+1} \dots i_k}] \in L^{k-q+p-1}(F)\ ^{1/}$$

where

$$b_{i_1 \dots i_{p-1}\ i_{q+1} \dots i_k} := a_{i_1 \dots i_{p-1}\ j_p \dots j_q\ i_{q+1} \dots i_k}$$

for $i_s = 1, 2, \dots$; $s = 1, \dots, p-1, q+1, \dots, k$. From the above definition we obtain a matrix $\overline{A}_{j_p, \dots, j_q}^{\;p,q}$ from k-dimensional matrix $\overline{A}$ if we leave elements of $\overline{A}$ which have fixed indices $j_p, \dots, j_q$ in positions $p, p+1, \dots, q$.

EXAMPLES. If $\overline{A} = [a_{ij}] \in L^2(F)$, then

$$\overline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

and $\overline{A}_3^{1,1} = [a_{31} a_{32} a_{33} \dots]$, $\overline{A}_4^{2,2} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ \dots \end{bmatrix}$.

If $\overline{A}[a_{i_1 i_2 i_3}] \in L^3(F)$, then

$$\overline{A}_{4,1}^{2,3} = \begin{bmatrix} a_{141} \\ a_{241} \\ a_{341} \\ \dots \end{bmatrix}, \qquad \overline{A}_3^{1,1} = \begin{bmatrix} a_{311} & a_{312} & a_{313} & \dots \\ a_{321} & a_{322} & a_{323} & \dots \\ a_{331} & a_{332} & a_{333} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

D.1.6. Let $p, q \in W(k)$ and $p \le q$. The order of the matrix $\overline{A} = [a_{i_1 \dots i_p \dots i_q \dots i_k}] \in L^k(F)$, where $i_s = 1, 2, \dots$; $s = 1, 2, \dots, k$; with respect to index $(p, \dots, q)$ will be called the maximum number $R_A(p, q)$ of the linearly independent sections $\overline{A}_{j_p, \dots, j_q}^{\;p,q}$ with respect to indices $(j_p, \dots, j_q) \in$

$^{1/}$ If $p = 1$ or $q = k$ the above denotation means here matrices $\overline{A}_{j_1 \dots j_q}^{\;1,q} = [b_{i_{q+1} \dots i_k}]$ or $\overline{A}_{j_p \dots j_k}^{\;q,k} = [b_{i_1 \dots i_{p-1}}]$

$\mathbb{N}^{q-p+1}$ in the space $L^{k-q+p-1}(F)$ if it is finite; if it is infinite we assume $R_A(p,q) = +\infty$. Of course, the sections $\overline{A}_{j_p,\dots,j_q}^{\,p,q}$ of the matrix $\overline{A}$ belong to the linear space $L^{k-q+p-1}(F)$ for $(j_p,\dots,j_q) \in \mathbb{N}^{q-p+1}$.

D.1.7. Let $(j_p,\dots,j_q) \in \mathbb{N}^{q-p+1}$. We call a section $\overline{A}_{j_p,\dots,j_q}^{\,p,q}$ of the matrix $\overline{A}$ the null section with respect to indices $(j_p,\dots,j_q)$, if

$$\forall (i_1 \dots i_{p-1}, i_{q+1} \dots i_k) \in \mathbb{N}^{k-q+p-1} \qquad a_{i_1 \dots j_p \dots j_q \dots j_k} = 0 .$$

It means that the matrix $\overline{A}_{j_p,\dots,j_q}^{\,p,q}$ is the zero vector as an element of the space $L^{k-q+p-1}(F)$.

D.1.8. Let $p,q \in W(k)$ $(p \le q)$. We call a matrix $\overline{A} = [a_{i_1 \dots i_p \dots i_q \dots i_k}] \in L^k(F)$ the $(p,\dots,q)$-finite index matrix, if

$$\forall (j_p,\dots,j_q) \in \mathbb{N}^{q-p+1} \exists (j_1,\dots,j_{p-1},j_{q+1},\dots,j_k) \in \mathbb{N}^{k-q+p-1}$$
$$\forall (i_1,\dots,i_{p-1},i_{q+1},\dots,i_k) > (j_1,\dots,j_{p-1},j_{q+1},\dots,j_k)$$
$$a_{i_1 \dots i_{p-1}\, j_p \dots j_q\, i_{q+1} \dots i_k} = 0 .$$

D.1.9. Let $p,q \in W(k)$, $(p \le q)$ and let a matrix $\overline{A} = [a_{i_1 \dots i_p \dots i_q \dots i_k}] \in L^k(F)$ be $(p,\dots,q)$-finite index, then we call $(p,\dots,q)$-finite index of the order r if each section $\overline{A}_{j_p,\dots,j_q}^{\,p,q}$ for $(j_p,\dots,j_q) \in \mathbb{N}^{q-p+1}$ of $\overline{A}$ with respect to $(j_p,\dots,j_q)$ has at least r elements different from zero and there exists at least one section which has exactly r elements different from zero.

3. Let $F = \mathbb{R}$, $(\mathbb{C})$ and $k \in \mathbb{N}$ be an arbitrary number.

D.1.10. Let $\overline{A} = [a_{i_1 \dots i_k}] \in L^k(F)$, $\overline{B} = [b_{i_1 \dots i_k}] \in L^k(F)$. If the series

$$\overline{A} \circ \overline{B} := \sum_{i_1 \dots i_k = 1}^{\infty} a_{i_1 \dots i_k} \cdot \overline{b}_{i_1 \dots i_k}$$

is absolutely convergent,then $\overline{A} \circ \overline{B}$ is said to be the inner product of the matrices $\overline{A}$ and $\overline{B}$. In the other case the inner product $\overline{A} \circ \overline{B}$ is not defined. In the linear space $L^k(F)$ we introduce the norm. If the inner product $\overline{A} \circ \overline{A}$ exists, then the number

$$\|\overline{A}\|_2 := \sqrt{\overline{A} \circ \overline{A}} \quad \text{we call the norm of } \overline{A}.$$

In other case we assume $\|\overline{A}\|_2 = \infty$.

Further on, we denote by $L^k(F)$ the linear space for which the norm $\|.\|_2$ is defined.

D.1.11. We denote by $L_2^k(F)$ the set:

$$L_2^k(F) := \{\overline{A} \in L^k(F) : \|\overline{A}\|_2 < \infty\}.$$

Of course $L_2^k(F)$ is a linear subspace of $L^k(F)$.

D.1.12. The symbol $L_0^k(F)$ will denote the set of all matrices $\overline{A} \in L^k(F)$ such that for any $\overline{A} \in L_0^k(F)$ there exists a number $M > 0$ such that

$\forall(i_1,\dots,i_k)\in\mathbb{N}^k\ |a_{i_1\dots i_k}|\le M$. We introduce a norm in the space $L_0^k(F)$. If $\overline{A}=[a_{i_1\dots i_k}]\in L_0^k(F)$, then $\|\overline{A}\|_1:=\sup_{i_1,\dots,i_k\in\mathbb{N}^k}|a_{i_1\dots i_k}|$.

Further on by $L_0^k(F)$ we understand the space determined in D.1.12. with the norm $\|.\|_1$.

It is easy to show

THEOREM 1.2. *There take place the following inclusions:*

$$L_2^k(F)\subset L_0^k(F)\subset L^k(F)\,.$$

4. Let F be an arbitrary number field and let $n,m\in\mathbb{N}$ be arbitrary natural numbers.

D.1.13. If $\overline{A}=[a_{i_1\dots i_m\ j_1\dots j_n}]\in L^{m+n}(F)$ and $\overline{B}=[b_{j_1\dots j_n}]\in L^n(F)$, where $i_1,\dots,i_m,j_1,\dots,j_n=1,2,\dots$; then we call a right-sided product of the matrices $\overline{A}$ and $\overline{B}$ the matrix $\overline{A}\cdot\overline{B}=[c_{i_1\dots i_m}]$, where

$$c_{i_1\dots i_m}=\sum_{j_1\dots j_n=1}^{\infty}a_{i_1\dots i_m\ j_1\dots j_n}\cdot b_{j_1\dots j_n};\ i_s=1,2,\dots;\ s=1,\dots,m;$$

under the assumption that the above series are unconditionally convergent. If at least one series is not convergent, the right-sided product $\overline{A}$ and $\overline{B}$ is not defined.

D.1.14. If $\overline{A}=[a_{i_1\dots i_m}]\in L^m(F)$ and $\overline{B}=[b_{i_1\dots i_m\ j_1\dots j_n}]\in L^{m+n}(F)$, then we call a left-sided product of the matrices $\overline{A}$ and $\overline{B}$ the matrix $\overline{A}\cdot\overline{B}=[c_{j_1\dots j_n}]$, where

$$c_{j_1\dots j_n}=\sum_{i_1\dots i_m=1}^{\infty}a_{i_1\dots i_m}\cdot b_{i_1\dots i_m\ j_1\dots j_n};\ j_s=1,2,\dots;\ s=1,\dots,n;$$

under the assumption that the above series are unconditionally convergent. If at least one series is not convergent, the left-sided product $\overline{A}$ and $\overline{B}$ is not defined.

Let us denote $S_p^2(m,n):=\{\overline{A}\in L^{m+n}(F):\forall\overline{B}\in L^n(F)$ the right-sided product $\overline{A}\cdot\overline{B}$ exists$\}$, $S_l^2(m,n):=\{\overline{B}\in L^{m+n}(F):\forall\overline{A}\in L^m(F)$ the left-sided product $\overline{A}\cdot\overline{B}$ exists$\}$.

THEOREM 1.3. *The matrix $\overline{A}\in L^{m+n}(F)$ is $(1,\dots,m)$-finite index if and only if $\overline{A}\in S_p^2(m,n)$.*

Proof. Let $\overline{A}=[a_{i_1\dots i_m\ j_1\dots j_n}]\in L^{m+n}(F)$ and let us suppose that $\overline{A}$ is $(1,\dots,m)$-finite index. Let $\overline{B}=[b_{j_1\dots j_n}]\in L^n(F)$ be an arbitrary fixed matrix. Then for each $(i_1,\dots,i_m)\in\mathbb{N}^m$ the series

$$\sum_{j_1\dots j_n=1}^{\infty}a_{i_1\dots i_m\ j_1\dots j_n}\cdot b_{j_1\dots j_n}=\sum_{(j_1,\dots,j_n)=(1,\dots,1)}^{(k_1,\dots,k_n)}a_{i_1\dots i_m\ j_1\dots j_n}\cdot b_{j_1\dots j_n}$$

is convergent, so $\overline{A} \in S^2_p(m,n)$. Next we assume that $\overline{A} \in S^2_p(m,n)$ and let us suppose that $\overline{A}$ is not $(1,\dots,m)$-finite index. Then there exists a set of indices $(i^0_1,\dots,i^0_m) \in \mathbb{N}^m$ and a sequence $(j^s_1,\dots,j^s_n)$, $s=1,2,\dots$ such that $a_{i^0_1\dots i^0_m\ j^s_1\dots j^s_n} \neq 0$, for $s=1,2,\dots$ let $b_{j_1\dots j_n} = 1/a_{i^0_1\dots i^0_m\ j^s_1\dots j^s_n}$ for $(j_1,\dots,j_n)=(j^s_1,\dots,j^s_n)$, and $b_{j_1\dots j_n}=0$ for $(j_1,\dots,j_n)\neq(j^s_1,\dots,j^s_n)$, $s=1,2,\dots$; The series $\sum_{j_1,\dots,j_n=1}^{\infty} a_{i_1\dots i_m\ j_1\dots j_n}\cdot b_{j_1\dots j_n}$ is not convergent, which contradicts the assumption that $\overline{A}\in S^2_p(m,n)$.

THEOREM 1.4. *The matrix $\overline{B} \in L^{m+n}(F)$ is $(m+1,\dots,m+n)$ finite index if and only if $\overline{B} \in S^2_1(m,n)$.*

The proof is analogous to the proof of Theorem 1.3.

II.1. Let F be an arbitrary number field and $m,n \in N$ are fixed. Let us consider the matrix equation

$$\overline{A}\cdot x = \overline{B}, \tag{2.1}$$

where $\overline{A} = [a_{i_1\dots i_m\ j_1\dots j_n}] \in S^2_p(m,n)$, $\overline{B}\in L^m(F)$ and $x \in L^n(F)$ is unknown.

From Theorem 1.3 it follows, that $\overline{A}$ is $(1,\dots,m)$-finite index, D.1.8 gives us that for every system of indices $(i_1,\dots,i_m)$ there exists the system of indices $(k_1,\dots,k_n)$ such that

$$\forall (j_1,\dots,j_n) > (k_1,\dots,k_n) \qquad a_{i_1\dots i_m\ j_1\dots j_n} = 0\,.$$

Let $(i^0_1,\dots,i^0_m) \in \mathbb{N}^m$ be an arbitrary fixed system of indices and $\overline{B} = [b_{i_1\dots i_m}]$, where $i_s = 1,2,\dots$ for $s=1,2,\dots,m$. We will consider two cases:

(i) $$\forall (j_1,\dots,j_n)\in\mathbb{N}^n \qquad a_{i^0_1\dots i^0_m j_1\dots j_n} = 0\,.$$

If $b_{i^0_1\dots i^0_m} \neq 0$, then the equation (2.1) is contradictory and if $b_{i^0_1\dots i^0_m} = 0$, then the system of indices $(i^0_1,\dots,i^0_m)$ can be omitted without the loss of generality.

(ii) $$\exists (k_1,\dots,k_n)\in\mathbb{N}^n \forall (j_1,\dots,j_n) > (k_1,\dots,k_n)$$
$$a_{i^0_1\dots i^0_m\ j_1\dots j_n} = 0 \text{ and } a_{i^0_1\dots i^0_m\ k_1\dots k_n} \neq 0\,.$$

In this case each $(i^0_1,\dots,i^0_m)$ corresponds explicitly to the element $(k_1,\dots$ $\dots,k_n)\in\mathbb{N}^n$ which we denote by $P_{i^0_1\dots i^0_m}$ or $P(i^0_1,\dots,i^0_m)$. Hence, it follows that the sequence $(P_{i_1,\dots,i_m})$ of indices explicitly corresponds to the matrix $\overline{A}$. We denote this sequence by $\widehat{P}_A$, the set of elements of $\widehat{P}_A$ by P_A. Of course, for each $(i_1,\dots,i_m)\in\mathbb{N}^m$ $P_{i_1,\dots,i_m}\in\mathbb{N}^n$.

The above assumption implies that the equation (2.1) is equivalent to the system of equations of the form

$$\sum_{(j_1,\dots,j_n)=(1,\dots,1)}^{P(i_1,\dots,i_m)} a_{i_1\dots i_m\ j_1\dots j_n} \cdot x_{j_1\dots j_n} = b_{i_1\dots i_m} \tag{2.2}$$

where $i_s = 1, 2, \dots$; $s = 1, 2, \dots, m$;

THEOREM 2.1. *The equation* (2.1) (*the system* (2.2)) *has a solution belonging to* $L^n(F)$, *if and only if one from the following conditions hold*

(a) *for any system of indices* $(k_1, \dots, k_m) \in \mathbb{N}^m$ *the system of equations*

$$\sum_{(j_1,\dots,j_n)=(1,\dots,1)}^{P(i_1,\dots,i_m)} a_{i_1\dots i_m\ j_1\dots j_n} \cdot x_{j_1\dots j_n} = b_{i_1\dots i_m}\ ; \tag{2.3}$$

$(i_1, \dots, i_m) \le (k_1, \dots, k_m)$ *has a solution in* $L^n(F)$,

(b) *every finite subsystem of* (*2.2*) *has a solution in* $L^n(F)$.

Proof. It is enough to observe that we can replace the equation (2.1) by the equation of the form: $\widehat{A} \cdot \widehat{x} = \widehat{B}$, where $\widehat{A} \in L^2(F)$, $x \in S$, $\widehat{B} \in S$, S-the space of all one-index sequences, whose elements belong to F. Next we may use the theorem from [1].

From the above we immediately obtain:

COROLLARY. *If for any set of indices* $(k_1, \dots, k_m) \in \mathbb{N}^m$, *the rank of the matrix* $[a_{i_1\dots i_m\ j_1\dots j_n}]$, $(i_1, \dots, i_m) \le (k_1, \dots, k_m)$ *equals* $r_{k_1,\dots,k_m}$ *with respect to indices* $(1, \dots, m)$, *then the system* (2.2) *has a solution with respect to indices* $(1, \dots, m)$ *in* $L^n(F)$ *for any* $\overline{B} \in L^m(F)$.

D.2.1. The symbol $\overline{S}_p^2(m,n)$ denotes the set of matrices $\overline{A} \in S_p^2(m,n)$ which do not include null section $\overline{A}_{i_1,\dots i_m}^{\ 1,m}$ for indices $(i_1, \dots, i_m) \in \mathbb{N}^m$ and for which the sequence $\widehat{P}_A : \mathbb{N}^m \mapsto P_A$ $(P_A \subset \mathbb{N}^n)$ is a one-to-one function.

D.2.2. If $\overline{A} \in S_p^2(m,n)$ and $P_A = \mathbb{N}^n$, then $\overline{A}$ will be called a simple matrix.

THEOREM 2.2. *If* $\overline{A} \in \overline{S}_p^2(m,n)$, *then the equation* (2.1) *has a solution in* $L^n(F)$ *for any* $\overline{B} \in L^m(F)$.

Proof. Let $\overline{A} \in \overline{S}_p^2(m,n)$, so that the map $\widehat{P}_A : \mathbb{N}^m \mapsto P_A$ is one-to-one function and it has no null sections with respect to indices $(i_1, \dots, i_m) \in \mathbb{N}^m$, so elements of the sequence $\widehat{P}_A$ are different. Hence, for any set of indices $(k_1, \dots, k_n) \in \mathbb{N}^n$ the rank of the matrix $\overline{A} = [a_{i_1\dots i_m\ j_1\dots j_n}]$, $(i_1, \dots, i_m) \le (k_1, \dots, k_m)$ with respect to $(1, \dots, m)$ is equal to the number of equations in the system (2.3). From Theorem 1.1 it follows that the number of equations in the system (2.3) equals $r_{k_1,\ldots,k_m}$. Next, from the corollary we infer that the system (2.2) has a solution in $L^n(F)$ for any $\overline{B} \in L^n(F)$. Since (2.2) is equivalent to equation (2.1) we obtain the conclusion of the theorem.

Let $L^n(\overline{B},\overline{A}) = \{x \in L^n(F) : \overline{A}\cdot x = \overline{B}\}$.
We can easily prove

LEMMA 2.1. *$L^n(0,\overline{A})$ is a linear subspace of $L^n(F)$.*

THEOREM 2.3. *If x_0 is a solution of the equation (2.1), then $L^n(\overline{B},\overline{A}) = x_0 + L^n(0,\overline{A})$, so $L^n(\overline{B},\overline{A})$ is a linear variety in $L^n(F)$ or $L^n(\overline{B},\overline{A})$ is empty if the equation (2.1) is contradictory. We omit the easy proof.*

Let L be a linear subspace of the space X. Let us determine $\dim L$ in Hamel sense.

$$\dim L := \begin{cases} k, & \text{if there exists maximally } k \text{ lineary independent elements in } L. \\ \aleph_0, & \text{if for any } k \in \mathbb{N} \text{ we can choose } k \text{ linearly independent elements in } L. \end{cases}$$

D.2.3. If T is a linear variety in L then

$$\dim T := \begin{cases} \dim L, & \text{if } T = x_0 + L, \\ -1, & \text{if } T = \emptyset. \end{cases}$$

From above and Theorem 2.3 we have

$$\dim L^n(\overline{B},\overline{A}) = \begin{cases} \dim L^n(0,\overline{A}), & \text{if } L^n(\overline{B},\overline{A}) \neq \emptyset, \\ -1, & \text{if } L^n(\overline{B},\overline{A}) = \emptyset. \end{cases}$$

THEOREM 2.4. *If $\overline{A} \in L^{m+n}(F)$ and $\overline{A}$ is a simple matrix, $\overline{B} \in L^m(F)$, then* $\dim L^n(\overline{B},\overline{A}) = 0$.

Proof. From Theorem 2.2 we infer the existence of a solution $x_0 \in L^n(F)$ of the equation (2.1). Thus $L^n(\overline{B},\overline{A}) \neq \emptyset$ and $\dim L^n(\overline{B},\overline{A}) = \dim L^n(0,\overline{A})$. Let us suppose that there exist two solutions x and y ($x \neq y$), which belong to $L^n(0,\overline{A})$. Let $x = [x_{j_1\ldots j_n}]$, $y = [y_{j_1\ldots j_n}]$, where $j_s = 1,2,\ldots$; $s = 1,2,\ldots,n$.

In accordance with the order relation defined in (Ch.I, §1) we can find a set of indices such that $x_{k_1\ldots k_n} \neq y_{k_1\ldots k_n}$ and $\forall (j_1,\ldots,j_n) < (k_1,\ldots,k_n)$ $x_{j_1\ldots j_n} = y_{j_1\ldots j_n}$ or $(k_1,\ldots,k_n) = (1,\ldots,1)$ and $x_{1\ldots 1} = y_{1\ldots 1}$ since $\overline{A}$ is a simple matrix, thus there exists $(i_1,\ldots,i_m) \in \mathbb{N}^n$, such that $P(i_1,\ldots,i_m) = (k_1,\ldots,k_n)$.

Let us consider the equation

$$\sum_{(j_1,\ldots,j_n)=(1,\ldots,1)}^{(k_1,\ldots,k_n)} a_{i_1\ldots i_m\, j_1\ldots j_n} \cdot [x_{j_1\ldots j_n} + (-y_{j_1\ldots j_n})] = 0\,. \tag{2.4}$$

Since $x_{j_1\dots j_n} = y_{j_1\dots j_n}$ for every $(j_1,\dots,j_n) < (k_1,\dots,k_n)$ thus we have from (2.4) $a_{i_1\dots i_m\ k_1\dots k_n}\cdot[x_{k_1\dots k_n}+(-y_{k_1\dots k_n})] = 0$. Of course $P(i_1,\dots,i_m) = (k_1,\dots,k_n)$ so $a_{i_1\dots i_m\ k_1\dots k_n} \neq 0$. Hence $x_{k_1\dots k_n}+(-y_{k_1\dots k_n}) = 0$ and $x_{k_1\dots k_n} = y_{k_1\dots k_n}$, which contradicts our assumption.

Hence we infer that the set $L^n(0,\overline{A})$ includes one solution of the homogeneous equation only, and of course it is the element 0, so $\dim L^n(0,\overline{A}) = 0$ and it completes the proof.

Let $Q_A := \mathbb{N}^n \setminus P_A$.

THEOREM 2.5. *If* $\overline{A} \in \overline{S}^2_p(m,n)$ *and* $\overline{B} \in L^m(F)$, *then*

$$\dim L^n(\overline{B},\overline{A}) = \operatorname{card} Q_A\,.$$

Proof. It follows from Theorem 2.2 that for any $\overline{B} \in L^m(F)$ we have $L^n(\overline{B},\overline{A}) \neq \emptyset$. From D.2.3. we infer that $\dim L^n(\overline{B},\overline{A}) = \dim L^n(0,\overline{A})$ for any $\overline{B} \in L^m(F)$. Hence it is sufficient to show that $\dim L^n(0,\overline{A}) = \operatorname{card} Q_A$.

Let us consider three cases:

a) $Q_A = \emptyset$. In this case $\overline{A}$ is a simple matrix and from Theorem 2.4 we obtain the conclusion.

b) $\operatorname{card} Q_A = \aleph_0$ and $Q_A = \{q_1,q_2,\dots\}$, where $q_1 < q_2 < \dots$; $q_s \in \mathbb{N}^n$, for $s = 1,2,\dots$; $L^n(0,\overline{A})$ is the set of solutions of the system of equations given by

$$\sum_{(j_1,\dots,j_n)=(1,\dots,1)}^{P(i_1,\dots,i_m)} a_{i_1\dots i_m\ j_1\dots j_n}\cdot x_{j_1\dots j_n} = 0\,, \tag{2.5}$$

where $i_s = 1,\dots$; $s = 1,\dots,m$;

Let us assume that $x_{q_k} = 1$ for fixed $k \in \mathbb{N}$ and $x_{q_s} = 0$ for $s \neq k$, $s = 1,2,\dots$; Then the system (2.5) we can rewrite in the form

$$\sum_{(j_1,\dots,j_n)=P_A}^{P(i_1,\dots,j_m)} a_{i_1\dots i_m\ j_1\dots j_n}\cdot x_{j_1\dots j_n} = -a_{i_1\dots i_m,q_k}\,, \tag{2.6}$$

where $i_s = 1,2,\dots$; $s = 1,2,\dots,m$. The matrix $[a_{i_1\dots i_m\ j_1\dots j_n}]$ is a simple matrix.

From Theorem 2.4 it follows that the system (2.6) has exactly one solution, which we denote by $\widehat{x} = [\widehat{x}_{j_1\dots j_n}]$, $(j_1,\dots,j_n) \in P_A$.

Let us denote

$$x^k_{j_1\dots j_n} := \begin{cases} \widehat{x}^k_{j_1\dots j_n}, & \text{where } (j_1;\dots,j_n) \in P_A,\\ 1, & \text{where } (j_1,\dots,j_n) = q_k,\\ 0, & \text{where } (j_1,\dots,j_n) \in Q_A \setminus \{q_k\}.\end{cases} \tag{2.7}$$

From the above considerations we infer that $x^k = [x^k_{j_1\dots j_n}] \in L^n(F)$, $(j_1,\dots,j_n) \in \mathbb{N}^n$ is a solution of the system (2.5). The solutions x^k ($k = 1,2,\dots$) in $L^n(F)$ are linearly independent and they form a countable set, so that $\dim L^n(0,\overline{A}) = \aleph_0 = \operatorname{card} Q_A$.

c) $\dim Q_A = M < \aleph_0$ and $Q_A = \{q_1,\dots,q_M\}$, where $q_1 < q_2 < \dots < q_M$; $q_s \in \mathbb{N}^n$ for $s = 1,2,\dots,M$.

Analogously as in b) we obtain solutions of the system (2.5) x^k for $k = 1,2,\dots,M$; in the form (2.7). Of course, each linear combination of elements x^k is a solution of (2.5), thus $\dim L^n(0,\overline{A}) \geq M$. Let $\overline{x} = [\overline{x}_{j_1\dots j_n}]$ for $(j_1,\dots,j_n) \in \mathbb{N}^n$ and $\overline{x} \in L^n(0,\overline{A})$. Then the element $\widehat{x} = \sum_{k=1}^{M} x_{q_k} \cdot x^k \in L^n(F)$, so $\widehat{x} \in \operatorname{lin}(x^1,\dots,x^M)$ and $\widehat{x} \in L^n(0,\overline{A})$.

From definition of $\widehat{x}$ it implies that

$$\widehat{x}_{q_j} = \sum_{k=1}^{M} \overline{x}_{q_k} \cdot \delta_{q_k q_j} = \overline{x}_{q_j} \qquad \text{for } j = 1,2,\dots,M\,,$$

where $\delta_{n,m}$ denotes Kronecker's symbol.

If we fix, as in the case b), the values $\widehat{x}_{q_1},\dots,\widehat{x}_{q_M}$ for $\overline{x}_{q_1},\dots,\overline{x}_{q_M}$ we get the system of equations with unknowns $x_{j_1\dots j_k}$, $(j_1,\dots,j_n) \in P_A$, which in view of Theorem 2.4, has the unique solution. Hence $\overline{x} = \widehat{x}$ and each solution of (2.5) belongs to $\operatorname{lin}(x^1,\dots,x^M)$, so $\dim L^n(0,\overline{A}) \leq M$, which completes the proof.

Let $\overline{A} \in \overline{S}^2_P(m,n)$. The map $P : \mathbb{N}^m \ni (i_1,\dots,i_m) \mapsto (j_1,\dots,j_n) = P(i_1,\dots,i_m) \in P_A$ is onto $\overline{S}^2_p(m,n)$ and one-to-one and so there exists the inverse map $P^{-1} : P_A \mapsto \mathbb{N}^m$.

LEMMA 2.2. *If $\overline{A}$ is a simple matrix, $\overline{B} \in L^m(F)$ and $x = [x_{j_1\dots j_n}]$, $(j_1,\dots,j_n) \in \mathbb{N}^n$ is a solution of the equation* (2.1), *then for any fix $(k_1,\dots,k_n) \in \mathbb{N}^n$ the numbers $x_{j_1\dots j_n}$ for $(j_1,\dots,j_n) \leq (k_1,\dots,k_n)$ are uniquely determined by the system of equations*

$$\sum_{(q_1,\dots,q_n)\leq(j_1,\dots,j_n)} a_{P^{-1}(j_1,\dots,j_n)q_1\dots q_n} \cdot x_{q_1\dots q_n} = b_{P^{-1}(j_1,\dots,j_n)} \tag{2.8}$$

for $(j_1,\dots,j_n) \leq (k_1,\dots,k_n)$.

Proof. From Theorem 2.4 we infer that the equation (2.1), equivalent to (2.2), has exactly one solution. Let us consider the systems $(i_1,\dots,i_m) \in \mathbb{N}^m$ for which $P(i_1,\dots,i_m) \leq (k_1,\dots,k_n)$ and take the system of equations of the form (2.2) for $P(i_1,\dots,i_m) \leq (k_1,\dots,k_n)$.

Since the matrix $\overline{A}$ is simple, so there are only equations of the form

$$\begin{cases} a_{P^{-1}(1,\ldots,1)1\ldots1}\cdot x_{1\ldots1} = b_{P^{-1}(1,\ldots,1)} \\ a_{P^{-1}(1,\ldots,2)1\ldots1}\cdot x_{1\ldots1} + a_{P^{-1}(1,\ldots,2)1\ldots2}\cdot x_{1\ldots2} = b_{P^{-1}(1,\ldots,2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{P^{-1}(k_1,\ldots,k_n)1\ldots1}\cdot x_{1\ldots1}+\ldots+a_{P^{-1}(k_1,\ldots,k_n)k_1\ldots k_n}\cdot x_{k_1\ldots k_n}=b_{P^{-1}(k_1,\ldots,k_n)} \end{cases} \tag{2.9}$$

The number of equations and unknowns in (2.9) is equal and the rank of fundamental matrix is equal to the number of equations. Treating (2.9) as the system of equations with respect to $x_{j_1\ldots j_n}$ for $(j_1,\ldots,j_m)\le(k_1,\ldots,k_n)$ we get that unknown are explicitly determined by (2.9). This completes the proof.

2. Let F be the number field $\mathbb{R}$ or $\mathbb{C}$ and $m,n\in\mathbb{N}$ are fixed numbers.

THEOREM 2.6. *If $\overline{A}\in L^{m+n}(F)$ is a simple matrix, $\|\overline{A}^{m+1,m+n}_{j_1,\ldots j_n}\|_2<+\infty$ for any n-index $(j_1,\ldots,j_n)\in\mathbb{N}^n$ and if the product*

$$\prod_A := \prod_{(j_1,\ldots,j_n)\in\mathbb{N}^n}\frac{\|\overline{A}^{m+1,m+n}_{j_1,\ldots,j_n}\|_2^2}{|a^2_{P^{-1}(j_1,\ldots,j_n)j_1\ldots j_n}|} = \prod_{(j_1,\ldots,j_n)\in\mathbb{N}^n}\left(1+\sum_{(i_1,\ldots,i_m)\ne P^{-1}(j_1,\ldots,j_n)}\left|\frac{a_{i_1\ldots i_m\; j_1\ldots j_n}}{a_{P^{-1}(j_1\ldots j_n)j_1\ldots j_n}}\right|\right) \tag{2.10}$$

is convergent, then for any $\overline{B}\in L^m_2(F)$ the equation (2.1) *has exactly one solution $x=[x_{j_1\ldots j_n}]$ in the space $L^n(F)$ and*

$$|x_{j_1\ldots j_n}|\le\sqrt{\Pi_A}\cdot\|\overline{B}\|_2/\|\overline{A}^{m+1,m+n}_{j_1,\ldots,j_n}\|_2 \quad \text{for } (j_1,\ldots,j_n)\in\mathbb{N}^n. \tag{2.11}$$

Moreover, if $I:=\inf_{(j_1,\ldots,j_n)\in\mathbb{N}^n}\|\overline{A}^{m+1,m+n}_{j_1,\ldots,j_n}\|_2>0$, then $x\in L^n_0(F)$ and $\|x\|_1\le\sqrt{\Pi_A}\cdot\|\overline{B}\|_2/I$.

Proof. It follows from Theorem 2.4 that the equation (2.1) has exactly one solution $x=[x_{j_1\ldots j_n}]$, $(j_1,\ldots,j_n)\in\mathbb{N}^n$. Let us take a fixed n-indices $(k_1,\ldots,k_n)\in\mathbb{N}^n$. From Lemma 2.2 we infer that $x_{j_1\ldots j_n}$ for $(j_1,\ldots,j_n)\le(k_1,\ldots,k_n)$ are determined explicitly by (2.9). The system (2.9) has the solution determined by Cramer formulas. Let us put:

$$D_{k_1,\ldots,k_n}=\begin{vmatrix} a_{P^{-1}(1,\ldots,1)1\ldots1} & 0 & \ldots & 0 \\ a_{P^{-1}(1,\ldots,2)1\ldots1} & a_{P^{-1}(1,\ldots,2)1\ldots2} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{P^{-1}(k_1,\ldots,k_n)1\ldots1} & a_{P^{-1}(k_1,\ldots,k_n)1\ldots2} & \ldots & a_{P^{-1}(k_1,\ldots,k_n)k_1\ldots k_n}\end{vmatrix} = \prod_{(j_1,\ldots,j_n)\le(k_1,\ldots,k_n)} a_{P^{-1}(j_1,\ldots,j_n)j_1,\ldots,j_n}\ne 0,$$

$$D^{j_1,\dots,j_n}_{k_1,\dots,k_n} := \begin{vmatrix} a_{P^{-1}(1,\dots,1)1\dots1} \cdots b_{P^{-1}(1,\dots,1)} & \cdots & 0 \\ a_{P^{-1}(1,\dots,2)1\dots1} \cdots b_{P^{-1}(1,\dots,2)} & \cdots & 0 \\ \dots\dots\dots & \dots & \dots \\ a_{P^{-1}(k_1,\dots,k_n)1\dots1} \cdots b_{P^{-1}(k_1,\dots,k_n)} & \cdots & a_{P^{-1}(k_1,\dots,k_n)k_1\dots k_n} \end{vmatrix}$$

for $(j_1,\dots,j_n) \le (k_1,\dots,k_n)$. Thus we obtain

$$x_{j_1\dots j_n} = \frac{D^{j_1,\dots,j_n}_{k_1,\dots,k_n}}{D_{k_1,\dots,k_n}} \quad \text{for } (j_1,\dots,j_n) \le (k_1,\dots,k_n). \tag{2.12}$$

Next, applying Hadamard's inequality to $D^{j_1,\dots,j_n}_{k_1,\dots,k_n}$ we get

$$|D^{j,\dots,j_n}_{k_1,\dots,k_n}|^2 \le \prod_{(j_1,\dots,j_n)\le(k_1,\dots,k_n)} \Big(\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |a^2_{i_1\dots i_m\ j_1\dots j_n}| \Big) \times \frac{\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |b_{i_1\dots i_m}|^2}{\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |a_{i_1\dots i_m\ k_1\dots k_n}|^2}.$$

From the above and from (2.12), we have

$$|x_{r_1,\dots,r_n}|^2 \le \prod_{(j_1,\dots,j_n)\le(k_1,\dots,k_n)} \frac{\big(\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |a_{i_1\dots i_m\ j_1\dots j_n}|^2}{|a^2_{P^{-1}(i_1\dots i_n)j_1\dots j_n}|} \times \frac{\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |b_{i_1\dots i_m}|^2}{\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |a_{i_1\dots i_m\ r_1\dots r_n}|^2}.$$

for $(r_1,\dots,r_n) \le (k_1,\dots,k_n)$.

For any $(j_1,\dots,j_n) \le (k_1,\dots,k_n)$ the above inequality and the following relations

$$\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |a^2_{i_1\dots i_m\ j_1\dots j_n}| \le \sum_{(i_1,\dots,i_m)\in\mathbb{N}^m} |a^2_{i_1\dots i_m\ j_1\dots j_n}| = \|\overline{A}^{m+1,m+n}_{j_1,\dots,j_n}\|_2^2 < +\infty,$$

$$\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |b^2_{i_1\dots i_m}| \le \sum_{(i_1,\dots,i_m)\in\mathbb{N}^m} |b^2_{i_1\dots i_m}| = \|\overline{B}\|_2^2 < \infty,$$

yield for any $(j_1,\dots,j_n) \le (k_1,\dots,k_n)$

$$|x_{j_1\dots j_n}|^2 \le \prod_{(r_1,\dots,r_n)\le(k_1,\dots,k_n)} \frac{\|\overline{A}^{m+1,m+n}_{r_1,\dots,r_n}\|_2^2}{|a^2_{P^{-1}(r_1\dots r_n)r_1\dots r_n}|}$$

$$\times \frac{\|\overline{B}\|_2^2}{\sum_{(i_1,\dots,i_m)=P^{-1}(1,\dots,1)}^{P^{-1}(k_1,\dots,k_n)} |a_{i_1\dots i_m\ j_1\dots j_n}|^2} .$$

Hence, if $r_{k_1,\dots,k_n} \to +\infty$ we have

$$(2.13) \qquad |x_{j_1\dots j_n}|^2 \le \prod_{(r_1,\dots,r_n)\in\mathbb{N}^n} \frac{\|\overline{A}^{m+1,m+n}_{r_1,\dots,r_n}\|_2^2}{|a^2_{P^{-1}(r_1\dots r_n)r_1\dots r_n}|} \times \frac{\|\overline{B}\|_2^2}{\|\overline{A}^{m+1,m+n}_{j_1,\dots,j_n}\|_2^2}$$

for $(j_1,\dots,j_n) \in \mathbb{N}^n$. This completes the proof of the first part of the theorem.

If $I = \inf_{(j_1,\dots,j_n)\in\mathbb{N}^n} \|\overline{A}^{m+1,m+n}_{j_1,\dots,j_n}\|_2^2 > 0$, then by (2.13), the inequality

$$|x_{j_1\dots j_n}|^2 \le \Pi_A \cdot \|\overline{B}\|_2^2/\|\overline{A}^{m+1,m+n}_{j_1,\dots,j_n}\|_2^2 \le \Pi_A \cdot \|\overline{B}\|_2^2/I^2$$

holds. It means that $x \in L_0^n(F)$ and $\|x\|_1 \le \sqrt{\Pi_A} \cdot \|\overline{B}\|_2/I$.

THEOREM 2.7. *If $\overline{A} \in L^{m+n}(F)$ is a simple matrix and for every index $(i_1,\dots,i_m) \in \mathbb{N}^m$*

$$(2.14) \qquad \Big(1 + \sum_{(j_1,\dots,j_n)\neq P(i_1,\dots,i_m)} |a_{i_1\dots i_m\ j_1\dots j_n}|\Big)/|a_{i_1\dots i_m\ P(i_1,\dots,i_m)}| \le 1$$

then for any $\overline{B} \in L_0^m(F)$ the equation (2.1) *has exactly one solution $x = [x_{j_1\dots j_n}]$ and $x \in L_0^n(F)$, $\|x\|_1 \le \|\overline{B}\|_1$.*

Proof. Theorem 2.4 implies that the equation (2.1) has exactly one solution $x = [x_{j_1\dots j_n}]$, $(j_1,\dots,j_n) \in \mathbb{N}^n$, so we have to show that $\|x\|_1 \le \|\overline{B}\|_1$. It is sufficient to prove that $|x_{j_1\dots j_n}| \le \|\overline{B}\|_1$ for $(j_1,\dots,j_n) \in \mathbb{N}^n$. We use the induction here. From Lemma 2.2 we conclude that the system (2.8) uniquely determines $x_{j_1\dots j_n}$ for arbitrary $(j_1,\dots,j_n) \in \mathbb{N}^n$. Hence, we get

$$(2.15) \qquad x_{j_1\dots j_n} = \frac{b_{P^{-1}j_1,\dots,j_n} - \sum_{(r_1,\dots,r_n)<(j_1,\dots,j_n)} a_{P^{-1}(j_1,\dots,j_n)r_1\dots r_n} \cdot x_{r_1\dots r_n}}{a_{P^{-1}(j_1,\dots,j_n)j_1\dots j_n}}$$

for $(j_1,\dots,j_n) \in \mathbb{N}^n$.

I. For $(j_1,\dots,j_n) = (1,\dots,1)$ by (2.15) and (2.14) we have

$$|x_{1\dots 1}| \le |b_{P^{-1}(1,\dots,1)}|/|a_{P^{-1}(1,\dots,1)1\dots 1} \le \|\overline{B}\|_1 .$$

II. Let us suppose that for two arbitrary sequences of indices

$$(r_1,\dots,r_n) < (j_1,\dots,j_n)$$

the following inequality holds

$$|x_{r_1\dots r_n}| \le \|\overline{B}\|_1 .$$

Hence by (2.15) we get

$$|x_{j_1\dots j_n}| = \frac{|b_{P^{-1}(j_1,\dots,j_n)}| + \sum_{(r_1,\dots,r_n)<(j_1,\dots,j_n)} |a_{P^{-1}(j_1,\dots,j_n)r_1\dots r_n}| \cdot \|\overline{B}\|_1}{|a_{P^{-1}(j_1,\dots,j_n)j_1\dots j_n}|}$$
$$\le \|\overline{B}\|_1 \cdot \frac{1 + \sum_{(r_1,\dots,r_n)\neq(j_1,\dots,j_n)} |a_{P^{-1}(j_1,\dots,j_n)r_1\dots r_n}|}{|a_{P^{-1}(j_1,\dots,j_n)j_1\dots j_n}|}.$$

From the above inequality and (2.14) we get $|x_{j_1\dots j_n}| \le \|\overline{B}\|_1$. Thus we obtain by induction that $|x_{j_1\dots j_n}| \le \|\overline{B}\|_1$ for every $(j_1,\dots,j_n) \in \mathbb{N}^n$. Consequently $x \in L_n^0(F)$ and $\|x\|_1 \le \|\overline{B}\|_1$.

THEOREM 2.8. *Let Z be a subset of the set $\mathbb{N}^m$. Let us assume that $\overline{A} \in L^{m+n}(F)$ is a simple matrix, $\overline{B} \in L_0^m(F)$ and the condition* (2.14) *is satisfied for any sequence of indices $(i_1,\dots,i_m) \in \mathbb{N}^m \setminus Z$. If there exists a constant $M \ge 0$ such that $|x_{P(i_1,\dots,i_m)}| \le M \cdot \|\overline{B}\|_1$ for every $(i_1,\dots,i) \in Z$, then the equation* (2.1) *has exactly one solution x in the space $L_n^0(F)$. Moreover $x \in L_0^n(F)$, and $\|x\|_1 \le \|\overline{B}\|_1 \cdot \max(1,M)$.*

Proof. We observe that if the condition (2.14) is fulfilled for the system $(i_1,\dots,i_m) \in \mathbb{N}^m$, then for any constant $K \ge 0$ the following inequality

$$\frac{1 + K \cdot \sum_{(j_1,\dots,j_n)\neq P(i_1,\dots,i_m)} |a_{i_1,\dots,i_m\ j_1\dots j_n}|}{|a_{i_1\dots i_m P(i_1\dots i_m)}|} \le \max(1,K) \tag{2.16}$$

holds, as the result of the inequality

$$1 + K \cdot \sum_{(j_1,\dots,j_n)\in\mathbb{N}^n} |a_{i_1,\dots,i_m\ j_1\dots j_n}|$$
$$\le \Big(1 + \sum_{(j_1,\dots,j_n)\in\mathbb{N}^n} |a_{i_1\dots i_m\ j_1\dots j_n}|\Big) \cdot \max(1,K).$$

Theorem 2.4 implies that the equation (2.1) has exactly one solution $x = [x_{j_1\dots j_n}] \in L^n(F)$. From Lemma 2.2 we get that $x_{j_1\dots j_n}$ are determined uniquely for $(j_1,\dots,j_n) \in \mathbb{N}^n$ by the system (2.8). So we get the formula (2.15).

Next, using the induction with respect to $(j_1,\dots,j_n) \in \mathbb{N}^n$, we prove that $x \in L_0^n(F)$.

I. Let $(j_1,\dots,j_n) = (1,\dots,1)$. If $P^{-1}(1,\dots,1) \in Z$, then from our assumption we have

$$|x_{1\dots 1}| \le M \cdot \|\overline{B}\|_1 \le \|\overline{B}\|_1 \cdot \max(1,M).$$

If $P^{-1}(1,\dots,1) \in Z$, then in view of (2.15) and (2.14) we get $|x_{1\dots 1}| \le |b_{P^{-1}(1,\dots,1)}|/|a_{P^{-1}(1,\dots,1)1\dots 1}| \le \|\overline{B}\|_1 \le \|\overline{B}\|_1 \cdot \max(1,M)$.

II. Let us suppose that there is the estimation for each $(r_1, \ldots, r_n) < (j_1, \ldots, j_n)$ of the form $|x_{r_1 \ldots r_n}| \le \|\overline{B}\|_1 \cdot \max(1, M)$, we will prove that

$$|x_{j_1 \ldots j_n}| \le \|\overline{B}\|_1 \cdot \max(1, M).$$

If $P^{-1}(j_1, \ldots, j_n) \in Z$, then from the assumption we obtain the above inequality.

If $P^{-1}(j_1, \ldots, j_n) \notin Z$, then in view of (2.14), (2.15) and (2.16), we get

$$\begin{aligned} |x_{j_1 \ldots j_n}| &\le \frac{\|\overline{B}|_1(1 + M \cdot \sum_{(r_1,\ldots,r_n)<(j_1,\ldots,j_n)} |a_{P^{-1}(j_1,\ldots,j_n) r_1 \ldots r_n}|)}{|a_{P^{-1}(j_1,\ldots,j_n) j_1 \ldots j_n}|} \\ &\le \frac{\|\overline{B}|_1(1 + \cdot \sum_{(r_1,\ldots,r_n)\neq(j_1,\ldots,j_n)} |a_{P^{-1}(j_1,\ldots,j_n) r_1 \ldots r_n}| \cdot M)}{|a_{P^{-1}(j_1,\ldots,j_n) j_1 \ldots j_n}|} \\ &\le \|\overline{B}\|_1 \cdot \max(1, M). \end{aligned}$$

Hence we infer by the induction that for every $(j_1, \ldots, j_n) \in \mathbb{N}^n$ $|x_{j_1 \ldots j_n}| \le \|\overline{B}\|_1 \cdot \max(1, M)$, thus $x \in L_0^n(F)$ and $\|x\|_1 \le \|\overline{B}\|_1 \cdot \max(1, M)$, which completes the proof.

THEOREM 2.9. *If $\overline{A} \in L^{m+n}(F)$ is a simple matrix and $\exists (q_1, \ldots, q_m) \in \mathbb{N}^m \forall (i_1, \ldots, i_m) \ge (q_1, \ldots, q_m)$ such that the condition (2.14) is fulfilled, then for any $\overline{B} \in L_0^m(F)$ the equation (2.1) has exactly one solution $x = [x_{j_1 \ldots j_n}] \in L^n(F)$. Moreover $x \in L_0^n(F)$ and there exists a constant $M \ge 1$ such that $\|x\|_1 \le M \cdot \|\overline{B}\|_1$.*

Proof. Let us consider the set $\{P(1, \ldots, 1), P(1, \ldots, 2), \ldots, P(q_1, \ldots \ldots, q_m)\} \subset \mathbb{N}^m$ and let $(k_1^0, \ldots, k_n^0)$ be the maximum element of this set, e.i.

$$\forall (i_1, \ldots, i_m) \le (q_1, \ldots, q_m) \quad P(i_1, \ldots, i_m) \le (k_1^0, \ldots, k_n^0). \tag{2.17}$$

Theorem 2.1 implies that the equation (2.1) has exactly one solution $x = [x_{j_1 \ldots j_n}] \in L^n(F)$. Let us consider a fixed n-indices $(k_1, \ldots, k_n) \in \mathbb{N}^n$ which is larger than $(k_1^0, \ldots, k_n^0)$. From Lemma 2.2 we infer that $x_{j_1 \ldots j_n}$ are determined explicitly for $(j_1, \ldots, j_n) \le (k_1, \ldots, k_n)$ by the system (2.8). Hence we get the formula (2.15).

First we prove that there exists a constant $M > 1$ such that $|x_{j_1 \ldots j_n}| \le M \cdot \|\overline{B}\|_1$, for any set of $(j_1, \ldots, j_n) \le (k_1^0, \ldots, k_n^0)$. We get, in view of (2.15), that

$$|x_{1 \ldots 1}| \le |b_{P^{-1}(1,\ldots,1)} / |a_{P^{-1}(1,\ldots,1) 1 \ldots 1}| \le M_{1 \ldots 1} \cdot \|\overline{B}\|_1 < \infty,$$

where $M_{1 \ldots 1} = 1 |a_{P^{-1}(1,\ldots,1) 1 \ldots 1}| < \infty$.

Similarly in view of (2.15) we have

$$|x_{1\dots 2}| \le \frac{(|b_{P^{-1}(1,\dots,2)}| + |a_{P^{-1}(1,\dots,2)1\dots 1}| \cdot |x_{1\dots 1}|)}{|a_{P^{-1}(1,\dots,2)1\dots 2}|}\,.$$

Hence by the estimate $|x_{1\dots 1}| \le M_{1\dots 1} \cdot \|\overline{B}\|_1$ we infer that

$$|x_{1\dots 2}| \le \frac{\|\overline{B}\|_1 \cdot (1 + M_{1\dots 1} \cdot |a_{P^{-1}(1,\dots,2)1\dots 1}|)}{|a_{P^{-1}(1,\dots,2)1\dots 2}|}\,.$$

Denote the by $M_{1\dots 2}$ coefficient at $\|\overline{B}\|_1$ in the above estimation, we have:

$$|x_{1\dots 2}| \le M_{1\dots 2} \cdot \|\overline{B}\|_1 < +\infty\,.$$

Similarly, by the same reasoning as above, we obtain

$$|x_{k_1^0\dots k_n^0}| \le \frac{|b_{P^{-1}(k_1^0\dots k_n^0)}| + \sum_{(j_1,\dots,j_n)<(k_1^0\dots k_n^0)} |a_{P^{-1}(k_1^0\dots k_n^0)j_1\dots j_n}| \cdot |x_{j_1\dots j_n}|}{|a_{P^{-1}(k_1^0\dots k_n^0)k_1^0\dots k_n^0}|}\,.$$

From the estimate obtained earlier of the form

$$|x_{j_1\dots j_n}| \le M_{j_1\dots j_n} \cdot \|\overline{B}\|_1 < \infty$$

which holds for any $(j_1,\dots,j_n) < (k_1^0,\dots,k_n^0)$ we infer that

$$|x_{k_1^0\dots k_n^0}| \le \frac{\|\overline{B}\|_1(1 + \sum_{(j_1,\dots,j_n)<(k_1^0\dots k_n^0)} M_{j_1\dots j_n} \cdot |a_{P^{-1}(k_1^0\dots k_n^0)j_1\dots j_n}|)}{|a_{P^{-1}(k_1^0\dots k_n^0)k_1^0\dots k_n^0}|}$$

and which, for brevity, can be writen in the form

$$|x_{k_1^0\dots k_n^0}| \le M_{k_1^0\dots k_n^0} \cdot \|\overline{B}\|_1 < \infty\,,$$

where $M_{k_1^0\dots k_n^0}$ is the coefficient of $\|\overline{B}\|_1$.

Let us assume

$$M' := \max_{(j_1,\dots,j_n)\le(k_1^0,\dots,k_n^0)} \{M_{j_1\dots j_n}\} < \infty; \quad M = \max\{1, M'\}\,.$$

From the above we immediately get

$$\forall (j_1,\dots,j_n) \le (k_1^0,\dots,k_n^0) \quad |x_{j_1\dots j_n}| \le M \cdot \|\overline{B}\|_1\,, \tag{2.18}$$

where $M \ge 1$. Next, we assume:

$$Z = \{(i_1,\dots,i_m) \in \mathbb{N}^m : P(i_1,\dots,i_m) \le (k_1^0,\dots,k_n^0)\}\,.$$

We infer, in view of (2.18), that $|x_{P(i_1,\dots,i_m}| \le M\cdot\|\overline{B}\|_1$ for any $(i_1,\dots,i_m) \in Z$. Now we show out that the condition (2.14) is satisfied for any $(i_1,\dots,i_m) \in \mathbb{N}^m \setminus Z$.

Let $(i_1,\dots,i_m) \notin Z$ and let us suppose that $(i_1,\dots,i_m) < (q_1,\dots,q_m)$. Hence, in view of (2.17), we have $P(i_1,\dots,i_m) \le (k_1^0,\dots,k_n^0)$. Thus $(i_1,\dots,i_m) \in Z$ and we get the contradiction. So $(i_1,\dots,i_m) \ge (q_1,\dots,q_m)$

and we conclude that condition (2.14) is satisfied. Since the index $(i_1, \dots, i_m) \in \mathbb{N}^m \setminus Z$ is arbitrary, so we get that the condition (2.14) is fulfilled for every $(i_1, \dots, i_m) \in \mathbb{N}^m \setminus Z$.

Hence by Theorem 2.8 we obtain the conclusion of the theorem.

D.2.4. We call the matrix $\overline{A}^*$ the reduced matrix with respect to $\overline{A} = [a_{i_1 \dots i_m\ j_1 \dots j_n}] \in \overline{S}^2(m,n)$, if $\overline{A}^*$ is taken from $\overline{A}^*$ by deleting the sections $\overline{A}^{m+1,m+n}_{j_1,\dots,j_n}$ for $(j_1, \dots, j_n) \in Q_A$.

The operation of reduction explicitly determined $\overline{A}^*$ and $\overline{A}^*$ is a simple matrix after a renumeration with respect to indices $(j_1, \dots, j_n)$.

In the space $\overline{S}^2_P(m,n)$ let us define functionals $\overline{S}^2(m,n) \ni \overline{A} \to \Pi_A \in F \cup \{\infty\}$ and $\overline{S}^2(m,n) \ni \overline{A} \to \sum_A \in F \cup \{\infty\}$, where

$$\text{(2.19)}\ \prod_A := \prod_{(j_1,\dots,j_n) \in P_A} \left(1 + \sum_{(i_1,\dots,i_m) \neq P^{-1}(j_1,\dots,j_n)} \left| \frac{a_{i_1 \dots i_m\ j_1 \dots j_n}}{a_{P^{-1}(j_1,\dots,j_n) j_1 \dots j_n}} \right|^2 \right),$$

$$\text{(2.20)}\ \sum_A := \sum_{(j_1,\dots,j_n) \in P_A} \left(\sum_{(i_1,\dots,i_m) \neq P^{-1}(j_1,\dots,j_n)} \left| \frac{a_{i_1 \dots i_m\ j_1 \dots j_n}}{a_{P^{-1}(j_1,\dots,j_n) j_1 \dots j_n}} \right|^2 \right).$$

LEMMA 2.3. *If $\overline{A} \in \overline{S}^2_P(m,n)$, then $\sum_A = \sum_{A^*}$, and $\Pi_A = \Pi_{A^*}$. The proof of the lemma is very easy, if we observe that in formulas* (2.19) *and* (2.20) *we do not take indices $(j_1, \dots, j_n)$, which belong to Q_A.*

LEMMA 2.4. *If $\overline{A}$ is a simple matrix, then $\sum_A < +\infty$ if and only if $\Pi_A < +\infty$.*

Proof. In view of the inequality $\log(1+x) \le x$ for $x > 0$ and since $P_A = \mathbb{N}^n$ for any simple matrix $\overline{A}$ we infer that

$$\log \Pi_A = \sum_{(j_1,\dots,j_n) \in \mathbb{N}^n} \log \left(1 + \sum_{(i_1,\dots,i_m) \neq P^{-1}(j_1,\dots,j_n)} \left| \frac{a_{i_1 \dots i_m\ j_1 \dots j_n}}{a_{P^{-1}(j_1,\dots,j_n) j_1 \dots j_n}} \right|^2 \right)$$

$$\le \sum_{(j_1,\dots,j_n) \in \mathbb{N}^n} \left(\sum_{(i_1,\dots,i_m) \neq P^{-1}(j_1,\dots,j_n)} \left| \frac{a_{i_1 \dots i_m\ j_1 \dots j_n}}{a_{P^{-1}(j_1,\dots,j_n) j_1 \dots j_n}} \right|^2 \right) = \sum_{A'}$$

Hence $\Pi_A < +\infty$.

D.2.5. A matrix $\overline{A} = [a_{i_1 \dots i_m\ j_1 \dots j_n}] \in \overline{S}^2_P(m,n)$ is said to regular, if the following conditions are satisfied:

$$\text{(i)} \qquad \sum_A < +\infty,$$

$$\text{(ii)} \qquad \forall (j_1, \dots, j_n) \in \mathbb{N}^n \quad \overline{A}^{m+1,m+n}_{j_1,\dots,j_n} \in L^m_2(F).$$

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF ŁÓDŹ
Al. Politechniki 11
90-924 ŁÓDŹ, POLAND

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