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ON HERMITE AND HERMITE-FEJÉR INTERPOLATION OF HIGHER ORDER

1. Introduction

There are many different methods of constructing algebraic or trigonometric polynomials which approximate a given continuous function $f(x)$. The process of finding a polynomial which coincides with the function $f(x)$ at certain pre-assigned points, called the nodes of interpolation, and its successive derivatives coinciding with arbirarily chosen numbers is referred to Hermite interpolation [10]. However, the expilicit forms of fundamental polynomials are too complicated and, for that reason, very little is known about the convergence behaviour of this kind of polynomials.

Let there be

$$-1 \le x_{nn} < x_{n-1,n} < \ldots < x_{2n} x_{1n} \le 1 \tag{1.1}$$

the nodes of interpolation,

$$x_{kn} = \cos \frac{2k-1}{2n}\pi, \qquad k = 1, 2, \ldots, n, \tag{1.2}$$

the Tchebysheff nodes and $f(x)$ a function continuous on $[-1, 1]$.

As far as convergence is concerned, Fejér [8] proved the following theorems.

THEOREM A. *If $H_{2n-1}(f, x)$ is a polynomial of degree at most $2n-1$ which satisfies the conditions*

$$H_{2n-1}(f, x_{kn}) = f(x_{kn}), \qquad H'_{2n-1}(f, x_{kn}) = 0, \tag{1.3}$$

then $\lim\limits_{n\to\infty} H_{2n-1}(f, x) = f(x)$ uniformly on $[-1, 1]$.

THEOREM B. *If the polynomial $H_{2n-1}(f, x)$ satisfies the conditions*

$$H_{2n-1}(f, x_{kn}) = f(x_{kn}), \qquad |H'_{2n-1}(f, x_{kn})| < \frac{\varepsilon_n}{\sqrt{1-x_{kn}^2}} \frac{n}{\log n}, \tag{1.4}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$, *then* $\lim_{n\to\infty} H_{2n-1}(f,x) = f(x)$ *uniformly on* $[-1,1]$.

For the order of convergence of $H_{2n-1}(f,x)$ satisfying the conditions

$$(1.5)\qquad H_{2n-1}(f,x_{kn}) = f(x_{kn}), \quad H'_{2n-1}(f,x_{kn}) = f'(x_{kn}), \quad k = 1,2,\ldots,n,$$

we have the following result (cf. [12]).

THEOREM C. *Let* $f \in C^1[-1,1]$ *and* $H_{2n-1}(f,x)$ *be defined by* (1.5). *Then, for* $-1 \le x \le 1$,

$$|H_{2n-1}(f,x) - f(x)| \le c_1 n^{-1} \log n \; E_{2n-2}(f'),$$

where c_1 *(later on* $c_2, c_3, \ldots$*) is an absolute constant and* $E_{2n-2}(f')$ *is the best approximation of* $f'(x)$ *by polynomials of degree at most* $2n-2$.

Let $\{x_{kn}\}_1^n$ be the n distinct zeros of $(1-x^2)P_{n-2}(x)$, where $P_n(x)$ is the Legendre polynomial of degree n with the normalization $P_n(1) = 1$. Let $G_n(f,x)$ be the unique polynomial of degree at most $4n-7$ such that

$$(1.6)\qquad \begin{cases} G_n(f,-1) = f(-1), \quad G_n(f,1) = f(1), \\ G_n(f,x_{kn}) = f(x_{kn}), \quad G_n^{(r)}(f,x_{kn}) = 0, \quad r = 1,2,3; \\ \qquad\qquad k = 2,3,\ldots,n-1. \end{cases}$$

Then one can easily see that

$$(1.7)\qquad G_n(f,x) = f(-1)\tfrac{1}{2}(1-x)P_{n-2}^4(x) + f(1)\tfrac{1}{2}(1+x)P_{n-2}^4(x) + \sum_{k=2}^{n-1} f(x_{kn})A_{kn}(x),$$

where, for $k = 2,3,\ldots,n-1$,

$$(1.8)\qquad A_{kn}(x) = h_{kn}^2 + \tfrac{2}{3}[(n-2)(n-1)(1-xx_{kn})+1] \times \frac{(1-x^2)(x-x_{kn})^2}{(1-x_{kn}^2)^3} l_{kn}^4(x),$$

$$(1.9)\qquad h_{kn}(x) = \frac{1-x^2}{1-x_{kn}^2} l_{kn}^2(x),$$

$$(1.10)\qquad l_{kn}(x) = \frac{P_{n-2}(x)}{(x-x_{kn})P'_{n-2}(x_{kn})}.$$

For the polynomial $G_n(f,x)$ we shall prove the following theorem.

THEOREM 1. *Let* $f \in C[-1,1]$ *and* $G_n(f,x)$ *be defined by* (1.6). *Then, for* $-1 \le x \le 1$, *we have*

$$(1.11)\qquad |G_n(f,x) - f(x)| \le c_2 n^{-1} \sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k} + \frac{1}{k^2}\right),$$

where $w_f(\delta)$ *is the modulus of continuity of* $f(x)$ *on* $[-1,1]$.

This type of estimates for Hermite-Fejér interpolation has been given in [1]-[4], [9], [11], [13], [14], [16]-[18], [21]-[23]. One can easily see that (1.11) is best possible for $f(x) \in \operatorname{Lip}\alpha$, $0 < \alpha < \frac{1}{2}$. We also show that the result is precise for $\alpha = 1$, by proving the following theorem.

THEOREM 2. *There exists a function* $f(x) \in \operatorname{Lip} 1$ *and a constant* c_3 *such that*

$$|G_n(f,0) - f(0)| \geq c_3 n^{-1} \log n, \quad n = 6, 8, 10, \ldots.$$

Next, let us denote by $F_n(f,x)$ the unique polynomial of degree at most $4n-7$ satisfying the conditions

$$\begin{gathered} F_n(f,-1) = f(-1), \quad F_n(f,1) = f(1), \quad F_n(f,x_{kn}) = f(x_{kn}), \\ F_n'(f,x_{kn}) = \beta_{kn}, \quad F_n''(f,x_{kn}) = \gamma_{kn}, \quad F_n'''(f,x_{kn}) = \lambda_{kn}, \\ k = 2,3,\ldots,n-1, \end{gathered}$$

where x_{kn} are the zeros of $P_{n-2}(x)$ and $\beta_{kn}, \gamma_{kn}, \lambda_{kn}$ are any given numbers. Clearly,

$$\begin{aligned} F_n(f,x) = {} & \sum_{k=1}^{n} f(x_{kn})A_{kn}(x) + \sum_{k=2}^{n-1} \beta_{kn}B_{kn}(x) + \\ & + \sum_{k=2}^{n-1} \gamma_{kn}C_{kn}(x) + \sum_{k=2}^{n-1} \lambda_{kn}D_{kn}(x), \end{aligned} \tag{1.13}$$

where $x_{1n} = 1$ and $x_{nn} = -1$. The fundamental polynomials $A_{kn}(x)$ for $k = 2,3,\ldots,n-1$ are given by (1.8) and for $k = 1,n$ by

$$A_{1n}(x) = \frac{1+x}{2}P_{n-2}^4(x), \quad A_{nn}(x) = \frac{1-x}{2}P_{n-2}^4(x), \tag{1.14}$$

and B_{kn}, C_{kn}, D_{kn} for $k = 2,3,\ldots,n-1$ by

$$B_{kn}(x) = (x - x_{kn})h_{kn}^2(x) + 4\frac{(n-2)(n-1)(1-x_{kn}^2)+1}{(1-x_{kn}^2)^2}D_{kn}(x), \tag{1.15}$$

$$C_{kn}(x) = \frac{(x-x_{kn})^2(1-x^2)(1+x_{kn}^2-2xx_{kn})}{2(1-x_{kn}^2)^2}l_{kn}^4(x), \tag{1.16}$$

$$D_{kn}(x) = \frac{(1-x^2)(x-x_{kn})^3}{6(1-x_{kn}^2)}l_{kn}^4(x), \tag{1.17}$$

where $h_{kn}(x)$ are given by (1.9) and l_{kn} by (1.10). These are all new forms and, as it turns out, they are very convenient for the proofs.

THEOREM 3. *Let* $f \in C[-1,1]$ *and* $F_n(f,x)$ *be given by* (1.13). *Let, for* $k = 2,3,\dots,n-1$,

$$\beta_{kn} = o\left(\frac{n}{\sqrt{1-x_{kn}^2}\log n}\right), \quad \gamma_{kn} = o\left(\frac{n^2}{1-x_{kn}^2}\right),$$
$$\lambda_{kn} = o\left(\frac{n^3}{(1-x_{kn}^2)^{3/2}\log n}\right).$$

Then $\lim_{n\to\infty} F_n(f,x) = f(x)$ *uniformly on* $[-1,1]$.

Theorem 3 is analogous to Theorem B of Fejér. It can also be compared with a result of Erdös and Turán [7].

2. Preliminaries

In this section we state a few known results which we shall use later on. From [5], [19] we have, for $-1 \le x \le 1$,

$$\sum_{k=2}^{n-1} h_{kn}(x) = 1 - P_{n-2}^2(x) \le 1\,. \tag{2.1}$$

By [20], for $-1 \le x \le 1$, we have

$$|P_{n-2}(x)| \le 1, \tag{2.2}$$

$$(1-x^2)^{1/4}|P_{n-2}(x)| \le \left(\frac{2}{\pi}\right)^{1/2}(n-2)^{-1/2}, \tag{2.3}$$

$$(1-x_{kn}^2) > \left(k-\frac{3}{2}\right)^2 n^{-2}, \quad k = 2,3,\dots,\left[\frac{n-2}{2}\right], \tag{2.4}$$

$$(1-x_{kn}^2) > \left(n-k-\frac{3}{2}\right)^2 n^{-2}, \quad k = \left[\frac{n-2}{2}\right]+1,\dots,n-1, \tag{2.5}$$

$$|P_{n-2}'(x_{kn})| \sim \left(k-\frac{3}{2}\right)^{-3/2} n^2, \quad k = 2,3,\dots,\left[\frac{n-2}{2}\right], \tag{2.6}$$

$$|P_{n-2}'(x_{kn})| \sim \left(n-k-\frac{3}{2}\right)^{-3/2} n^2, \quad k = \left[\frac{n-2}{2}\right]+1,\dots,n-1, \tag{2.7}$$

with $x = \cos\theta$, $x_{kn} = \cos\theta_{kn}$ and

$$|P_{n-2}(0)| = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(n-3)}{2\cdot 4\cdot\ldots\cdot(n-4)(n-2)} > \frac{1}{3}(n-2)^{-1/2} > \frac{1}{3}n^{-1/2} \quad (n \text{ even}), \tag{2.8}$$

$$\left(n-\frac{3}{2}\right)^{-1}\left(k-\frac{3}{2}\right)\pi < \theta_{kn} < \left(n-\frac{3}{2}\right)^{-1}(k-1)\pi, \quad k = 2,3,\dots,n-1\,. \tag{2.9}$$

Further, from [15] for $-1 \le x \le 1$, we also have

$$\frac{(1-x^2)^{1/4}|P_{n-2}(x)|}{(1-x_{kn}^2)^{3/4}|P'_{n-2}(x_{kn})|} \le c_4 n^{-1}, \quad n \ge 4, \tag{2.10}$$

$$\frac{(1-x^2)^{1/4}}{(1-x_{kn}^2)^{1/4}}|l_{kn}(x)| \le c_5, \quad k = 2,3,\ldots,n-1. \tag{2.11}$$

and from [6] there follows $|l_{kn}(x)| \le c_6$, $k = 2,3,\ldots,n-1$.

3. Some lemmas

First, we assume x_{jn} to be that zero of $P_{n-2}(x)$ which is nearest to x. Then, as in [22], it can be seen that

$$|f(x) - f(x_{kn})| \le \begin{cases} c_7\left[w_f\left(\dfrac{\sin\Theta}{n}\right) + w_f\left(\dfrac{1}{n^2}\right)\right] & \text{if } j = k, \\ c_8\left[w_f\left(\dfrac{i\sin\Theta}{n}\right) + w_f\left(\dfrac{i^2}{n^2}\right)\right] & \text{if } j < k = \\ & = j+i \le n-2 \\ & \text{or } 2 \le k = j-i < j, \end{cases} \tag{3.1}$$

where i is a positive integer. By (2.9), it follows that

$$\frac{1}{\sin\frac{1}{2}|\theta - \theta_{kn}|} \le \frac{2n-1}{2i-1}, \quad k \ne j. \tag{3.2}$$

One can also easily see that

$$\sin\Theta_{kn} \le \sin\Theta + \sin\Theta_{kn} \le 2\sin\tfrac{1}{2}(\Theta + \Theta_{kn}), \tag{3.3}$$

$$\sin\Theta \le \sin\Theta + \sin\Theta_{kn} \le 2\sin\tfrac{1}{2}(\Theta + \Theta_{kn}), \tag{3.4}$$

$$\sin\tfrac{1}{2}|\Theta - \Theta_{kn}| \le \sin\tfrac{1}{2}(\Theta + \Theta_{kn}). \tag{3.5}$$

LEMMA 3.1. *If* $-1 \le x \le 1$, *then*

$$A_{kn}(x) \le \begin{cases} 1, & \text{if } k = j \\ c_9\, i^{-2}, & \text{if } j < k = j+i \le n-2 \text{ or } 2 \le k = j-i < j. \end{cases} \tag{3.6}$$

Proof. From (1.7) there follows

$$\sum_{k=2}^{n-1} A_{kn}(x) = 1 - P_{n-2}^4(x) \le 1. \tag{3.7}$$

Since $A_{kn}(x) \ge 0$ for $k = 2,3,\ldots,n-1$, the inequality (3.7) implies

$$A_{jn}(x) \le 1, \quad -1 \le x \le 1, \tag{3.8}$$

which completes the proof of the first part of (3.6).

By [11], [14], we get

$$h_{kn}(x) \le c_{10}\, i^{-2}, \quad k = j \pm i, \tag{3.9}$$

where $h_{kn}(x)$ is given by (1.9). Further, from (1.8) we have

(3.10) $$A_{kn}(x) = h_{kn}^2(x) + \tfrac{2}{3}[(n-2)(n-1)(1-xx_{kn})+1]\times \\ \times\frac{(1-x^2)(x-x_{kn})^2}{(1-x_{kn}^2)^3} l_{kn}^4(x) = \sigma_1+\sigma_2, \quad k=2,3,\dots,n-1\,.$$

If $k \neq j$, due to (3.9) and (2.1), it iollows that

(3.11) $$\sigma_1 \le c_{10}\, i^{-2}\,.$$

Next, consider

(3.12) $$\sigma_2 \le \frac{n^2(1-x^2)(x-x_{kn})^2(1-xx_{kn})}{(1-x_{kn}^2)^3} l_{kn}^4(x) \\ + \frac{(1-x^2)(x-x_{kn})^2}{(1-x_{kn}^2)^3} l_{kn}^4(x) = \sigma_1^* + \sigma^*\,.$$

Since $x = \cos\Theta$, $x_{kn} = \cos\Theta_{kn}$, on using (2.10), (3.2), (3.4) and (3.5), we obtain for $k \neq j$,

(3.13) $$\begin{aligned}\sigma_1^* &\le \frac{c_{11}}{n^2}\left[\frac{1-x^2}{(x-x_{kn})^2} + \frac{1}{|x-x_{kn}|}\right] \\ &\le \frac{c_{12}}{n^2}\left[\frac{\sin^2\Theta}{\sin^2\frac{1}{2}(\theta+\theta_{kn})\sin^2\frac{1}{2}|\theta-\theta_{kn}|} \right.\\ &\qquad \left. + \frac{1}{\sin\frac{1}{2}(\theta+\theta_{kn})\sin\frac{1}{2}|\theta-\theta_{kn}|}\right] \\ &\le \frac{c_{13}}{n^2}\left[\frac{1}{\sin^2\frac{1}{2}|\theta-\theta_{kn}|}\right] \le c_{14}\, i^{-2}\,.\end{aligned}$$

Similarly, on using (2.10), (2.11), (2.4), (2.5), (3.2) and (3.5), we get for $k \neq j$,

(3.14) $$\sigma_2^* = \frac{(1-x^2)(x-x_{kn})^2}{(1-x_{kn}^2)^3} l_{kn}^4(x) \le c_{15} n^{-2}|x-x_{kn}|^{-1} \le \\ \le c_{16}(n\sin\tfrac{1}{2}|\theta-\theta_{kn}|)^{-2} \le c_{17}\, i^{-2}\,.$$

Finally, from (3.10), (3.11) and (3.14), we have the second part of (3.6). This completes the proof of Lemma 3.1.

LEMMA 3.2. *For* $-1 \le x \le 1$ *and* $D_{kn}(x)$ *defined by* (1.17), *we have*

(3.15) $$\sum_{k=2}^{n-1} \frac{|D_{kn}(x)|}{(1-x_{kn}^2)^{3/2}} \le c_{18}\, n^{-3}\log n,$$

(3.16) $$\sum_{k=2}^{n-1} \frac{|D_{kn}(x)|}{(1-x_{kn}^2)^{5/2}} \le c_{19} n^{-1}\,.$$

Proof. Due to (1.17), we have

$$\sum_{k=2}^{n-1} \frac{|D_{kn}(x)|}{(1-x_{kn}^2)^{3/2}} \le \frac{(1-x^2)|P_{n-2}(x)|^3|l_{jn}(x)|}{(1-x_{jn}^2)^{5/2}|P'_{n-2}(x_{jn})|^3} + \sum_{k\neq j} \frac{(1-x^2)|P_{n-2}(x)|^3|l_{kn}(x)|}{(1-x_{kn}^2)^{5/2}|P'_{n-2}(x_{kn})|^3} = I_1 + I_2 . \tag{3.17}$$

On using (2.10) and (2.11), it follows that

$$I_1 = \left[\frac{(1-x^2)^{3/4}|P_{n-2}(x)|^3}{(1-x_{jn}^2)^{9/4}|P'_{n-2}(x_{jn})|^3}\right]\left[\frac{(1-x^2)^{1/4}}{(1-x_{jn}^2)^{1/4}}|l_{jn}(x)|\right] \le c_{20}n^{-3} . \tag{3.18}$$

Further, if $k \neq j$, from an earlier result ([15], p. 249), we have for $-1 \le x \le 1$

$$\frac{(1-x^2)^{1/4}}{(1-x_{kn}^2)^{1/4}}|l_{kn}(x)| \le c_{21}i^{-1}, \qquad k = j \pm i . \tag{3.19}$$

Now, using (2.10) and (3.19), we obtain

$$I_2 = \sum_{k\neq j}\left[\frac{(1-x^2)^{3/4}|P_{n-2}(x)|^3}{(1-x_{kn}^2)^{9/4}|P'_{n-2}(x_{kn})|^3}\right]\left[\frac{(1-x^2)^{1/4}}{(1-x_{kn}^2)^{1/4}}|l_{kn}(x)|\right] \le \le c_{22}\, n^{-3}\sum_{i=1}^{n} i^{-1} \le c_{23}\, n^{-3}\log n . \tag{3.20}$$

Consequently, from (3.17), (3.18), (3.20) we get (3.15), and (3.16) is obtained from (1.17), (2.10), (2.11), (2.4), (2.5).

LEMMA 3.3. *If* $-1 \le x \le 1$, *then*

$$\sum_{k=2}^{n-1} \frac{|B_{kn}(x)|}{(1-x^2)^{1/2}} \le c_{24}n^{-1}\log n , \tag{3.21}$$

where $B_{kn}(x)$ *is given by* (1.15).

Proof. From (1.15), (3.15) and (3.16) one can easily see that

$$\sum_{k=2}^{n-1}\frac{|B_{kn}(x)|}{(1-x_{kn}^2)^{1/2}} \le \sum_{k=2}^{n-1}\frac{|x-x_{kn}|}{(1-x_{kn}^2)^{1/2}}h_{kn}^2(x) + \sum_{k=2}^{n-1}4n^2\frac{|D_{kn}(x)|}{(1-x_{kn}^2)^{3/2}} + 4\sum_{k=2}^{n-1}\frac{|D_{kn}(x)|}{(1-x_{kn}^2)^{5/2}} \le \le \sum_{k=2}^{n-1}\frac{|x-x_{kn}|}{(1-x_{kn}^2)^{1/2}}h_{kn}^2(x) + c_{25}n^{-1}\log n + c_{26}n^{-1} . \tag{3.22}$$

Since x_{jn} is that zero of $P_{n-2}(x)$ which is nearest to x, $-1 \le x \le 1$, it is evident that

$$(3.23)\qquad \frac{\sqrt{1-x^2}}{\sqrt{1-x_{jn}^2}} \le c_{27}\,.$$

Also, from [15], for $k = 2, 3, \ldots, n-1$, we have

$$(3.24)\qquad n|x - x_{kn}|h_{kn}(x) \le c_{28}\sqrt{1-x^2}\,.$$

Now, consider

$$(3.25)\qquad \sum_{k=2}^{n-1} \frac{|x - x_{kn}|}{(1-x_{kn}^2)^{1/2}} h_{kn}^2(x) = \frac{|x - x_{jn}|}{\sqrt{1-x_{jn}^2}} h_{jn}^2(x) + \\ + \sum_{k\neq j} \frac{|x - x_{kn}|}{\sqrt{1-x_{kn}^2}} h_{kn}^2(x) = \mu_1 + \mu_2\,.$$

On using (3.23), (3.24) and (2.1), it follows that

$$(3.26)\qquad \mu_1 \le c_{29} n^{-1}\,.$$

Further, making use of (2.10), (3.2)–(3.4), we obtain

$$(3.27)\qquad \mu_2 = \sum_{k\neq j} \frac{|x - x_{kn}|}{\sqrt{1-x_{kn}^2}} h_{kn}^2(x) \le c_{30} \sum_{k\neq j} \frac{\sqrt{1-x_{kn}^2}(1-x^2)}{n^4|x - x_{kn}|^3} \le \\ \le c_{31} n^{-4} \sum_{k\neq j} \frac{\sin\Theta_{kn} \sin^2\Theta}{\sin^3\frac{1}{2}(\Theta + \Theta_{kn}) \sin^3\frac{1}{2}|\Theta - \Theta_{kn}|} \le \\ \le c_{32} n^{-4} \sum_{k\neq j} \frac{1}{sin^3\frac{1}{2}|\Theta - \Theta_{kn}|} \le c_{33} n^{-1}\,.$$

Now, from (3.25)–(3.27) and (3.22), we obtain (3.21).

LEMMA 3.4. *For $-1 \le x \le 1$ and $c_{kn}(x)$ defined by* (1.16) *we have*

$$(3.28)\qquad c_{kn}(x) \ge 0, \quad k = 2, 3, \ldots, n-1,$$

$$(3.29)\qquad \sum_{k=2}^{n-1} \frac{|c_{kn}(x)|}{1-x_{kn}^2} \le c_{34} n^{-2}\,.$$

Proof. From (1.16) we have, for $k = 2, 3, \ldots, n-1$,

$$(3.30)\qquad c_{kn}(x) = \frac{(x-x_{kn})^2(1-x^2)(1+x_{kn}^2 - 2xx_{kn})}{2(1-x_{kn}^2)^2} l_{kn}^4(x) \ge 0\,.$$

Next, from (3.30), we get

$$\sum_{k=2}^{n-1}\frac{|c_{kn}(x)|}{1-x_{kn}^2}\le\sum_{k=2}^{n-1}\frac{(1-x^2)(x-x_{kn})^2}{(1-x_{kn}^2)^2}l_{kn}^4(x)+ \\ +\sum_{k=2}^{n-1}\frac{(1-x^2)|x-x_{kn}|^3}{(1-x_{kn}^2)^3}l_{kn}^4(x)=\xi_1+\xi_2 \tag{3.31}$$

and, from (2.10), (2.11), (3.19),

$$\xi_1=\sum_{k=2}^{n-1}\frac{(1-x^2)(x-x_{kn})^2}{(1-x_{kn}^2)^2}l_{kn}^4(x)\le \\ \le c_{35}n^{-2}+c_{36}n^{-2}\sum_{i=1}^{n}i^{-2}c_{37}n^{-2}\,. \tag{3.32}$$

Further, we get the relation

$$\xi_2=\left[\frac{(1-x^2)^{1/4}|l_{jn}(x)|}{(1-x_{jn}^2)^{1/4}}\right]\left[\frac{(1-x^2)^{3/4}|P_{n-2}(x)|^3}{(1-x_{jn}^2)^{9/4}|P'_{n-2}(x)|^3}\right] \\ +\sum_{k\neq j}\frac{(1-x^2)|P_{n-2}(x)|^4}{(1-x_{kn}^2)^3|x-x_{kn}||P'_{n-2}(x_{kn})|^4}$$

which, on using (2.10), (2.11), (3.5) and (3.2), yields

$$\xi_2\le c_{38}n^{-2}\,. \tag{3.33}$$

Hence, from (3.31)–(3.33), we obtain (3.29).

4. Proof of Theorem 1

Due to uniqueness of $G_n(f,x)$ given by (1.6), it follows, by (1.7), that

$$1=\tfrac12(1-x)P_{n-2}^4(x)+\tfrac12(1+x)P_{n-2}^4(x)+\sum_{k=2}^{n-1}A_{kn}(x)\,. \tag{4.1}$$

Hence, we obtain

$$G_n(f,x)-f(x)=[[f(1)-f(x)]\tfrac12(1+x)+ \\ +[f(-1)-f(x)]\tfrac12(1-x)]P_{n-2}^4(x)+\sum_{k=2}^{n-1}[f(x_{kn})-f(x)]A_{kn}(x) \tag{4.2}$$

implying, for $-1\le x\le 1$,

$$(4.3)\qquad \begin{aligned}|G_n(f,x)-f(x)| &\le \\ &\le |f(1)-f(x)|\tfrac{1}{2}(1+x)P_{n-2}^4(x)+ \\ &\quad+|f(-1)-f(x)|\tfrac{1}{2}(1-x)P_{n-2}^4(x)+ \\ &\quad+\sum_{k=2}^{n-1}|f(x_{kn})-f(x)|A_{kn}(x)=J_1+J_2+J_3\,.\end{aligned}$$

On using Lemma 5.1 of [15] and (2.2), one can easily see that

$$(4.4)\qquad J_1=|f(1)-f(x)|\tfrac{1}{2}(1+x)P_{n-2}^4(x)\le c_{39}\,w_f\left(\frac{\sqrt{1-x^2}}{n}\right),$$

$$(4.5)\qquad J_2=|f(-1)-f(x)|\tfrac{1}{2}(1-x)P_{n-2}^4(x)\le c_{40}\,w_f\left(\frac{\sqrt{1-x^2}}{n}\right).$$

Next, (3.1) and Lemma 3.1 yield

$$(4.6)\qquad \begin{aligned}J_3 &= \sum_{k=2}^{n-1}|f(x_{kn})-f(x)|A_{kn}(x)=|f(x_{jn})-f(x)|A_{jn}(x)+ \\ &\quad+\sum_{k\neq j}|f(x_{kn})-f(x)|A_{kn}(x)\le \\ &\le c_7\left[w_f\left(\frac{\sin\Theta}{n}\right)+w_f\left(\frac{1}{n^2}\right)\right]+ \\ &\quad+c_{41}\sum_{k\neq j}\frac{1}{i^2}\left[w_f\left(\frac{i\sin\Theta}{n}\right)+w_f\left(\frac{i^2}{n^2}\right)\right]\le \\ &\le c_{42}\sum_{i=1}^{n}\frac{1}{i^2}\left[w_f\left(\frac{i\sin\Theta}{n}\right)+w_f\left(\frac{i^2}{n^2}\right)\right].\end{aligned}$$

Consequently, from (4.3)–(4.6) it follows that

$$\begin{aligned}|G_n(f,x)-f(x)| &\le c_{43}\,w_f\left(\frac{\sqrt{1-x^2}}{n}\right)+c_{42}\sum_{i=1}^{n}\frac{1}{i^2}\left[w_f\left(\frac{i\sin\Theta}{n}\right)+\right. \\ &\quad\left.+w_f\left(\frac{i^2}{n^2}\right)\right]\le c_{44}\sum_{i=1}^{n}\frac{1}{i^2}\left[w_f\left(\frac{i\sin\Theta}{n}\right)+w_f\left(\frac{i^2}{n^2}\right)\right].\end{aligned}$$

Now, following the same lines as in [16], it can be seen that

$$|G_n(f,x)-f(x)|\le c_{45}\,n^{-1}\sum_{k=1}^{n}w_f\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right).$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2

Let $f(x) = |x|$, $x = \cos\Theta = 0$, $\Theta = \frac{\pi}{2}$, and $n = 6, 8, \ldots$. From (4.2) and (1.8) we obtain

$$\begin{aligned}
G_n(f,0) - f(0) &= P_{n-2}^4(0) + \sum_{k=2}^{n-1} |x_{kn}| A_{kn}(0) = P_{n-2}^4(0) + \\
&+ \sum_{k=2}^{n-1} |x_{kn}| \left\{ h_{kn}^2(0) + \frac{2}{3}[(n-2)(n-1)+1] \frac{x_{kn}^2}{(1-x_{kn}^2)^3} l_{kn}^4(0) \right\} \geq \\
&\geq \frac{2}{3} \sum_{k=2}^{n-1} (n^2 - 3n + 3) \frac{|x_{kn}|^3}{(1-x_{kn}^2)^3} l_{kn}^4(0) \geq \\
&\geq \frac{2}{3} \sum_{k=2}^{n-1} n(n-3) \frac{|x_{kn}|^3}{(1-x_{kn}^2)^3} l_{kn}^4(0) \geq \\
&\geq \frac{2}{3} n(n-3) \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{|x_{kn}|^3}{(1-x_{kn}^2)^3} l_{kn}^4(0) = \\
&= \frac{2}{3} n(n-3) \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{P_{n-2}^4(0)}{(1-x_{kn}^2)^3 x_{kn} [P_{n-2}'(x_{kn})]^4} .
\end{aligned}$$

Hence, on using (2.6), (2.8) and (2.9), it follows that

$$\begin{aligned}
G_n(f,0) - f(0) &\geq c_{46} n^{-2} \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{1}{\cos\Theta_{kn}} \geq \\
&\geq c_{46} n^{-2} \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{1}{\cos\Theta_{kn} - \cos\Theta} = \\
&= c_{47} n^{-2} \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{1}{\sin\frac{1}{2}(\Theta + \Theta_{kn}) \sin\frac{1}{2}(\Theta - \Theta_{kn})} \geq \\
&\geq c_{48} n^{-2} \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{1}{\Theta - \Theta_{kn}} \geq \\
&\geq c_{49} n^{-1} \sum_{k=2}^{\frac{1}{2}(n-2)} \frac{1}{n - 2k + 2} \geq c_{50} n^{-1} \log n .
\end{aligned}$$

This completes the proof of Theorem 2.

Remark. Theorem 3 follows from (1.13), (4.2), (3.15), (3.21) and (3.29). We omit the details.

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