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THE DISCRETE MAXIMUM PRINCIPLE FOR SOME OPTIMIZATION PROBLEM OF MULTIDIMENSIONAL SYSTEMS

In monograph [2] and paper [3] the author discusses the method of metric approximations in the problems of control theory and applies it, among other things, to the derivation of a necessary condition in the optimization of a one-dimensional discrete system with constraints upon the trajectories. In the present paper, by making use of this method, we derive a necessary condition in a multidimensional discrete system linear with respect to the trajectory.

1. Multidimensional discrete systems

Let $\mathbb{Z}^m$ stand for subset of the space $\mathbb{R}^m$, consisting of points $k = (k_1, k_2, \ldots, k_m)$ with integer-valued coordinates, ordered by the relation $k^1 \leq k^2$ meaning that, for each $i \in \{1, 2, \ldots, m\}$, $k_i^1 \leq k_i^2$. For a fixed $k^0 \in \mathbb{Z}^m$, denote $\mathbb{Z}_+^m(k^0) := \{k \mid k \in \mathbb{Z}^m \wedge k^0 \leq k\}$. For an arbitrary set $\mathbb{E}$, let $\mathcal{B}(\mathbb{Z}^m, \mathbb{E})$ denote the set of mapings of $\mathbb{Z}^m$ into $\mathbb{E}$. Define a mapping $\Delta_i : \mathcal{B}(\mathbb{Z}^m, \mathbb{E}) \to \mathcal{B}(\mathbb{Z}^m, \mathbb{E})$ in the following way

$$\Delta_i\phi(k) := \phi(k_1, k_2, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_m), \qquad k \in \mathbb{Z}^m.$$

The mapping $(\Delta_1\phi, \Delta_2\phi, \ldots, \Delta_m\phi)$ is a mapping of the set $\mathcal{B}(\mathbb{Z}^m, \mathbb{E})$ into $\mathcal{B}(\mathbb{Z}^m, \mathbb{E}^m)$.

By a multidimensional discrete system we mean the system

$$(1.1) \qquad \Delta_i x(k) = f_i(k, u(k), x(k)), \qquad k \in D; \qquad i \in I(1, m),$$

where $f_i : \mathbb{Z}^m \times U \times \mathbb{E} \to \mathbb{E}$, $\overline{D} := \{k | k \in \mathbb{Z}_+^m(k^0) \wedge k \leq k^N\}$, $D := \overline{D} \setminus \{k^N\}$, $I(i, j) := \{k | k \in \mathbb{Z} \wedge i \leq k \leq j\}$, $x(\cdot) : D \to \mathbb{E}$, $u(\cdot) : D \to U$.

We shall say that, with a fixed function $u(\cdot) : D \to U$, system (1.1) is fully solvable if there exists a uniquely determined mapping $x(\cdot) : D \to \mathbb{E}$

such that

$$x(k^0) = x^0 \tag{1.2}$$

and system (1.1) is satisfied.

From the general theory of multidimensional discrete systems, [1], it is known that system (1.1) is fully solvable if and only if, for each $k \in \mathbb{Z}^m$, condition

$$\begin{aligned} &f_i(\Delta_j k, u(\Delta_j k), f_j(k, u(k), x(k))) \\ &\quad = f_j(\Delta_i k, u(\Delta_i k), f_i(k, u(k), x(k))), \qquad k \in D;\ i, j \in I(1, m) \end{aligned} \tag{1.3}$$

is satisfied.

We shall consider a special case of system (1.1), namely, the system linear with respect to the trajectory

$$\Delta_i x(k) = A_i(k)x(k) + f_i(k, u(k)), \qquad k \in D;\ i \in I(1, m) \tag{1.4}$$

and the homogeneous linear system, corresponding to it,

$$\Delta_i x(k) = A_i(k)x(k), \qquad k \in D;\ i \in I(1, m), \tag{1.5}$$

where $A(k) := (A_1(k), A_2(k), \ldots, A_m(k))$ with any $k \in D$ is a linear mapping of $\mathbf{E}$ into $\mathbf{E}^m$, $f := (f_1, f_2, \ldots, f_m) : D \times U \to \mathbf{E}^m$.

From condition (1.3) of the full solvability of system (1.1) it follows that systems (1.4) and (1.5) are simultaneously fully solvable when the conditions

$$(\Delta_j A_i(k)) \circ A_j(k) = (\Delta_i A_j(k)) \circ A_i(k), \qquad k \in D,\ i, j \in I(1, m), \tag{1.6}$$

$$\begin{aligned} &(\Delta_j A_i(k)) f_j(k, u(k)) + \Delta_j f_i(k, u(k)) \\ &\quad = (\Delta_i A_j(k)) f_i(k, u(k)) + \Delta_i f_j(k, u(k)), \qquad k \in D;\ i, j \in I(1, m) \end{aligned} \tag{1.7}$$

are satisfied, where $\circ$ stands for the operation of the superposing of mappings.

2. Formulation of the problem of controlling a multidimensional discrete system linear with respect to the trajectory

Let the multidimensional linear discrete system

$$\Delta_i x(k) = A_i(k)x(k) + f_i(k, u(k)), \qquad k \in D;\ i \in I(1, m), \tag{2.1}$$

with the constraints

$$u(k) = (u_1(k), u_2(k), \ldots, u_r(k))^T \in U \subset \mathbf{R}^r, \qquad k \in D, \tag{2.2}$$

$$x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T \in \Omega(k) \subset \mathbf{R}^n, \qquad k \in \overline{D};\ \Omega(k^0) = \{x^0\}, \tag{2.3}$$

$$\phi_j(x(k), k) \le 0, \qquad k \in \overline{D};\ j \in I(1, l), \tag{2.4}$$

$$\phi_j(x(k), k) = 0, \qquad k \in \overline{D};\ j \in I(l+1, l+p) \tag{2.5}$$

and the functional

$$\mathcal{J}(u(\cdot)) := \max\{\phi_0(x(k),k)|k \in \overline{D}\} \tag{2.6}$$

be given, where $\phi_j : \mathbb{R}^n \times \overline{D} \to \mathbb{R}$, while T denotes the transposition operation. we shall consider the following problem:

$$\mathcal{J}(u(\cdot)) \to \inf \tag{2.7}$$

under constraints (2.1)–(2.5) and (1.6)–(1.7), which will be called the basic problem.

A control $u : D \to U \subset \mathbb{R}^r$ will be called an admissible control if the set U is bounded and, with that function, conditions (1.6)–(1.7) are satisfied.

3. Approximating problem

For any non-empty subset A of a finite dimensional space X, let us introduce the following notations:

$$\rho(a|A) := \inf\{\|a-b\| | b \in A\},$$
$$W(a|A) := \{b|b \in \mathrm{cl}A \wedge \|a-b\| = \rho(a|A)\}$$

where cl denotes the closure operation.

Let a pair $\{x^0(\cdot), u^0(\cdot)\}$ be an optimal process in basic problem (2.7). Let us introduce the following notations:

$$\gamma^0 := \max\{\phi_0(x^0(k),k)|k \in \overline{D}\},$$
$$c^0 := (\gamma^0, 0, \ldots, 0) \in \mathbb{R}^{1+l+p},$$
$$c := (\gamma, 0, \ldots, 0) \in \mathbb{R}^{1+l+p} \quad \text{for } \gamma \in \mathbb{R},$$
$$\mathbf{E}^k_\Omega(\phi) := \{(x, \mu_0, \mu_1, \ldots, \mu_{l+p}) \in \mathbb{R}^{n+1+l+p} | x \in \Omega(k);$$
$$\phi_j(x,k) \le \mu_j \quad \text{for } j \in I(0,1);$$
$$\phi_j(x,k) = \mu_j \quad \text{for } j \in I(l+1, l+p)\},$$
$$\Phi(x,k,c) := \rho((x,c)|\mathbf{E}^k_\Omega).$$

Consider now the problem of the form

$$\Delta_i x(k) = A_i(k)x(k) + f_i(k, u(k)), \quad k \in D;\ i \in I(1,m), \tag{3.1}$$

$$\mathcal{J}_c(u(\cdot)) := \Big[\sum_{k\in\overline{D}} \Phi^2(x(k),k,c)\Big]^{1/2} + \sum_{k\in\overline{D}} \|x(k) - x^0(k)\|^2 \to \inf, \tag{3.2}$$

$$u(k) \in U \subset \mathbb{R}^r, \quad k \in D, \tag{3.3}$$

called further the problem approximating problem (2.7).

For the approximating problems, the following theorems are true:

THEOREM 1 ([5]). *If the pair $\{x^0(\cdot), u^0(\cdot)\}$ is an optimal process in basic problem* (2.7) *and the set $f(k,U)$ is, with each $k \in D$, a closed set,* *then, in each approximating problem* (3.1)–(3.3), *there exist optimal process* $\{x_c^0(\cdot), u_c^0(\cdot)\}$ *such that, for any* $k \in \overline{D}$, $x_c^0(k) \to x^0(k)$ *as* $c \to c^0$.

THEOREM 2 ([5]). *If the process* $\{x_c^0(\cdot), u_c^0(\cdot)\}$, $c = (\gamma, 0, \ldots, 0) \in \mathbb{R}^{1+l+p}$, $\gamma < \gamma^0$, *is an optimal process in approximating problem* (3.1)–(3.3) *and the set* $\mathbb{E}_\Omega^k(\phi)$ *is a closed set in a neighbourhood of the point* $(x_c^0(k), c)$, *then* $\{x_c^0(\cdot), u_c^0(\cdot)\}$ *is an optimal process also in the following problem*:

$$\Delta_i x(k) = A_i(k)x(k) + f_i(k, u(k)), \qquad k \in D;\ i \in I(1,m), \tag{3.4}$$

$$\Delta_i x_{n+1}(k) = \sum_{\tilde{k} \in D_i(k)} F(x(\tilde{k}), \tilde{k}, c), \qquad k \in D;\ i \in I(1,m), \tag{3.5}$$

$$\Delta_i x_{n+2}(k) = \sum_{\tilde{k} \in D_i(k)} \|x(\tilde{k}) - x^0(\tilde{k})\|^2, \qquad k \in D;\ i \in I(1,m), \tag{3.6}$$

$$x(k^0) = x^0, \qquad x_{n+1}(k^0) = 0, \qquad x_{n+2}(k^0) = 0, \tag{3.7}$$

$$u(k) \in U \subset \mathbb{R}^r, \tag{3.8}$$

$$\begin{aligned} J_c(u(\cdot)) := {} & [x_{n+1}(k^N) + \|x(k^N) - y_c(k^N)\|^2 \\ & + \|c - w_c(k^N)\|^2]^{1/2} + x_{n+2}(k^N) + \|x(k^N) - x^0(k^N)\|^2 \to \inf, \end{aligned} \tag{3.9}$$

where $x_{n+1}(\cdot), x_{n+2}(\cdot) : \overline{D} \to \mathbb{R}$, $(y_c(k), w_c(k)) \in W((x_c^0(k), c) | \mathbb{E}_\Omega^k(\phi))$,

$$F(x, k, c) := \|x - y_c(k)\|^2 + \|c - w_c(k)\|^2, \qquad k \in \overline{D},$$
$$D(k) := \{\tilde{k} | \tilde{k} \in \overline{D} \wedge \tilde{k} \le k \wedge \tilde{k} \ne k\}, \qquad D_i(k) := D(\Delta_i k).$$

4. A necessary condition for optimality for an approximating problem

We are given in the set $\overline{D}$ a system of points $k^0, k^1, \ldots, k^N$, satisfying the conditions

$$k^{j+1} \ge k^j \wedge \sum_{i=1}^{m} (k_i^{j+1} - k_i^j) = 1, \qquad j \in I(0, N-1),$$

called a discrete curve joining the points k^0 and k^N and denoted by $L(k^0, k^1, \ldots, k^N)$. Let us introduce the following notation:

$$X_{k^0}(k^N) := A_{i_{N-1}}(k^{N-1}) A_{i_{N-2}}(k^{N-2}) \ldots A_{i_0}(k^0) \tag{4.1}$$

where i_j is such that $\Delta_{i_j} k^j = k^{j+1}$. It is known, [1], that, when conditions (1.6) are satisfied, $X_{k^0}(k^N)$ does not depend on the discrete curve joining the points k^0 and k^N. In the case when $k^0 = k^N$, we accept that $X_{k^0}(k^N)$ is the identity mapping.

Let k^l be a fixed point of the discrete curve $L(k^0, k^1, \ldots, k^N)$, $k^l \ne k^N$. We say that $f(k^l, v)$ belongs to the set $\sigma(f(k^l, u_c^0(k^l)), f(k^l, U))$ when $v = v(k^l)$ for some admissible control $v(\cdot)$ and there exists a sequence (ε_s), $\varepsilon_s \downarrow 0$, such that, for every s,

$$f(k^l, u_c^0(k^l)) + \varepsilon_s(f(k^l, v) - f(k^l, u_c^0(k^l))) \in f(k^l, U). \tag{4.2}$$

Hence it appears that if $f(k^l, v) \in \sigma(f(k^l, u_c^0(k^l)), f(k^l, U))$, then there exists elements $v_s \in U$ such that

$$f_i(k^l, v_s) = f_i(k^l, u_c^0(k^l)) + \varepsilon_s(f_i(k^l, v) - f_i(k^l, u_c^0(k^l))), \quad i \in I(1, m). \tag{4.3}$$

Then the control

$$\widetilde{u}(k) := \begin{cases} v_s, & k = k^l, \\ u_c^0(k), & k \neq k^l \wedge k \in D, \end{cases} \tag{4.4}$$

is an admissible control for system (3.4). Indeed, condition (1.6) is evidently satisfied and condition (1.7) follows from the fact that, at the points k^l, we have

$$\begin{aligned}(\Delta_j A_i(k^l))&f_j(k^l, \widetilde{u}(k^l)) + \Delta_j f_i(k^l, \widetilde{u}(k^l)) \\ &= (\Delta_j A_i(k^l))f_j(k^l, u_c^0(k^l)) + \varepsilon_s(\Delta_j A_i(k^l))(f_j(k^l, v) \\ &\quad - f_j(k^l, u_c^0(k^l))) + \Delta_j f_i(k^l, u_c^0(k^l)) \\ &\quad + \varepsilon_s \Delta_j(f_i(k^l, v) - f_i(k^l, u_c^0(k^l))) \\ &= [(\Delta_i A_j(k^l))f_i(k^l, u_c^0(k^l)) + \Delta_i f_j(k^l, u_c^0(k^l))] \\ &\quad + \varepsilon_s[(\Delta_i A_j(k^l))f_i(k^l, v) + \Delta_i f_j(k^l, v)] \\ &\quad - \varepsilon_s[(\Delta_i A_j(k^l))f_i(k^l, u_c^0(k^l)) + \Delta_i f_j(k^l, u_c^0(k^l))] \\ &= (\Delta_i A_j(k^l)f_i(k^l, v_s)) + \Delta_i f_j(k^l, v_s) \\ &= (\Delta_i A_j(k^l)f_i(k^l, \widetilde{u}(k^l)) + \Delta_i f_j(k^l, \widetilde{u}(k^l),\end{aligned}$$

whereas, at the remaining points, $\widetilde{u}(k) = u_c^0(k)$, that is, condition (1.7) is also satisfied.

THEOREM 3. *If $\{x_c^0(\cdot), u_c^0(\cdot)\}$ is an optimal pair in approximating problem (3.1)–(3.3) with $c = (\gamma, 0, \ldots, 0)$, $\gamma < \gamma^0$, $\mathbf{E}_\Omega^k(\phi)$ is a closed set in a neighbourhood of the point $(x_c^0(k), c)$, $k \in \overline{D}$, and, for each $k \in \overline{D}$, the mappings $A_i(k)$, $i \in I(1, m)$, are invertible, then there exists vectors $(y_c(k), w_c(k)) \in W((x_c^0(k), c)|\mathbf{E}_\Omega^k(\phi))$, $k \in \overline{D}$, such that*

$$\psi^T(k)A_i^{-1}(k)\Delta_v f_i(k, u_c^0(k)) \leq 0, \quad k \in D,\ i \in I(1, m), \tag{4.5}$$

for $v \in U$, and $f(k, v) \in \sigma(f(k, u_c^0(k)), f(k, U))$, $k \in D$, where

$$\psi(k) = A_i^T(k)\psi(\Delta_i k) - g_i(k), \quad k \in D,\ i \in I(1, m), \tag{4.6}$$

$$\psi(k^N) = -h(k^N), \tag{4.7}$$

$$(4.8)\quad g_i(k) := \sum_{\widetilde{k}\in C(k)} X_k^T(\widetilde{k})h(\widetilde{k}) - \sum_{\widetilde{k}\in C_i(k)} A_i^T(k)X_{\Delta_i k}^T(\widetilde{k})h(\widetilde{k}),$$

$$i \in I(1,m),$$

$$(4.9)\quad h(k) := (x_c^0(k) - y_c(k))/m_c + 2(x_c^0(k) - x^0(k)),$$

$$(4.10)\quad m_c := \Big[\sum_{\widetilde{k}\in\overline{D}} F(x_c^0(\widetilde{k}), \widetilde{k}, c)\Big]^{\frac{1}{2}} > 0,$$

$$(4.11)\quad C(k) := \{\widetilde{k} | \widetilde{k} \in D \wedge \widetilde{k} \geq k \wedge \widetilde{k} \neq k\}, \qquad C_i(k) := C(\Delta_i k),$$

$$(4.12)\quad \Delta_v f_i(k, u_c^0(k)) := f_i(k,v) - f_i(k, u_c^0(k)), \quad i \in I(1,m).$$

Proof. If $\{x_c^0(\cdot), u_c^0(\cdot)\}$ is an optimal process in the approximating problem, then Theorem 2 implies that it is an optimal process in problem (3.4)–(3.9), as well.

Let $L(k^0, k^1, \ldots, k^N)$ be an arbitrary discrete curve and $k^l \neq k^N$ a fixed point of this curve. For $v \in U$ such that $f(k^l, v) \in \sigma(f(k^l, u_c^0(k^l)), f(k^l, U))$, define a control $\widetilde{u}(\cdot)$ of form (4.4) which is admissible control for problem (3.4)–(3.9).

Let $\Delta\widetilde{x}(k) := \widetilde{x}(k) - x_c^0(k)$, $\Delta\widetilde{x}_{n+1}(k) := \widetilde{x}_{n+1}(k) - x_{n+1,c}^0(k)$, $\Delta\widetilde{x}_{n+2}(k) := \widetilde{x}_{n+2}(k) - x_{n+2,c}^0(k)$, $k \in \overline{D}$, where $x_{n+1,c}^0(\cdot)$, $x_{n+2,c}^0(\cdot)$ are the solution of system (3.5)–(3.6), corresponding to the control $u_c^0(\cdot)$, and $\widetilde{x}(\cdot)$, $\widetilde{x}_{n+1}(\cdot)$, $\widetilde{x}_{n+2}(\cdot)$ the solution of system (3.4)–(3.6), corresponding to the control $\widetilde{u}(\cdot)$.

From the optimality of the pair $\{x_c^0(\cdot), u_c^0(\cdot)\}$ and from the definition of the function $F(\cdot,\cdot,\cdot)$ it follows that, with $\gamma < \gamma^0$, $m_c > 0$.

Taking account of form (3.9) of the functional $J_c(\cdot)$, we have

$$(4.13)\quad \begin{aligned} \Delta J_c(\widetilde{u}(\cdot)) &:= J_c(\widetilde{u}(\cdot)) - J_c(\overline{u}_c^0(\cdot)) \\ &= h^T(k^N)\Delta\widetilde{x}(k^N) + \frac{1}{2m_c}\Delta\widetilde{x}_{n+1}(k^N) + \Delta\widetilde{x}_{n+2}(k^N) + o(\|\Delta\widetilde{\widetilde{x}}(k^N)\|) \end{aligned}$$

where $\widetilde{\widetilde{x}} = (\widetilde{x}, \widetilde{x}_{n+1}, \widetilde{x}_{n+2})$.

In view of the fact following from equation (3.5) that, for $k \in D$,

$$\begin{aligned} \Delta_i\Delta\widetilde{x}_{n+1}(k) &= \Delta_i\widetilde{x}_{n+1}(k) - \Delta_i x_{n+1,c}^0(k) \\ &= \Delta\widetilde{x}_{n+1}(k) + \sum_{\widetilde{k}\in G_i(k)} [F(\widetilde{x}(\widetilde{k}), \widetilde{k}, c) - F(x_c^0(\widetilde{k}), \widetilde{k}, c)] \\ &= \Delta\widetilde{x}_{n+1}(k) + \sum_{\widetilde{k}\in G_i(k)} [2(x_c^0(\widetilde{k}) - y_c(\widetilde{k}))^T\Delta\widetilde{x}(\widetilde{k})] \\ &\quad + \sum_{\widetilde{k}\in G_i(k)} o(\|\Delta\widetilde{x}(\widetilde{k})\|) \end{aligned}$$

where $G_i(k) := D_i(k) \setminus D(k)$, we shall get

$$(4.14)\qquad \Delta\widetilde{x}_{n+1}(k^N) = \sum_{j=0}^{N-1}\sum_{i=1}^{m}(\Delta_i\Delta\widetilde{x}_{n+1}(k^j) - \Delta_i\widetilde{x}_{n+1}(k^j))(k_i^{j+1} - k_i^j)$$
$$= \sum_{\widetilde{k}\in D} 2(x_c^0(\widetilde{k}) - y_c(\widetilde{k}))^T\Delta\widetilde{x}(\widetilde{k}) + \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|).$$

Analogously, making use of equation (3.6), we shall obtain

$$(4.15)\qquad \Delta\widetilde{x}_{n+2}(k^N) = \sum_{\widetilde{k}\in D} 2(x_c^0(\widetilde{k}) - x^0(\widetilde{k}))^T\Delta\widetilde{x}(\widetilde{k}) + \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|).$$

Using now the easy-to-check identity

$$\sum_{j=0}^{N-1}\sum_{i=1}^{m}[\psi^T(k^{j+1})\Delta_i\Delta\widetilde{x}(k^j) - \psi^T(\delta_i k^{j+1})\Delta\widetilde{x}(k^j)](k_i^{j+1} - k_i^j)$$
$$= \psi^T(k^N)\Delta\widetilde{x}(k^N)$$

where δ_i stands for the operator inverse to the operator Δ_i, and from (4.13), (4.14) and (4.15), adopting $\psi(k^N) = -h(k^N)$, we shall get

$$(4.16)\qquad \Delta J_c(\widetilde{u}(\cdot)) = -\psi^T(k^N)\Delta\widetilde{x}(k^N) + \frac{1}{2m_c}\Delta\widetilde{x}_{n+1}(k^N) + \Delta\widetilde{x}_{n+2}(k^N)$$
$$+ o(\|\Delta\widetilde{x}(k^N)\|) = \sum_{j=0}^{N-1}\sum_{i=1}^{m}[\psi^T(\delta_i k^{j+1})\Delta\widetilde{x}(k^j)$$
$$- \psi^T(k^{j+1})\Delta_i\Delta\widetilde{x}(k^j)](k_i^{j+1} - k_i^j) + \sum_{\widetilde{k}\in D} h^T(\widetilde{k})\Delta\widetilde{x}(\widetilde{k})$$
$$+ \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|) + o(\|\Delta\widetilde{\widetilde{x}}(k^N)\|).$$

The form of the control $\widetilde{u}(\cdot)$ implies that

$$(4.17)\qquad \Delta_i\Delta\widetilde{x}(k) = A_i(k)\Delta\widetilde{x}(k) + f_i(k,\widetilde{u}(k)) - f_i(k,u_c^0(k))$$
$$= \begin{cases} A_i(k)\Delta\widetilde{x}(k), & k \neq k^l \wedge k \in D, \\ A_i(k^l)\Delta\widetilde{x}(k^l) + \varepsilon_s\Delta_v f_i(k^l,u_c^0(k^l)), & k = k^l. \end{cases}$$

So, with $\Delta\widetilde{x}(k^0) = 0$, we have, for $k \in C(k^l)$,

$$(4.18)\qquad \Delta\widetilde{x}(k) = \varepsilon_s X_{k^l}(k)A_{i_k}^{-1}(k^l)\Delta_v f_{i_k}(k^l,u_c^0(k^l)),$$

and, for the remaining $k \in \overline{D} \setminus C(k^l)$, $\Delta\widetilde{x}(k) = 0$, where i_k is an index satisfying the condition $\Delta_{i_k}k^l = \tau^1$, with that τ^1 is a point of the discrete curve $L(k^l,\tau^1,\ldots,k)$ joining the points k^l and k.

Let i_l be an index satisfying the condition $\Delta_{i_l} k^l = k^{l+1}$. By the identity

$$(\Delta_{i_k} A_{i_l}(k^l))\Delta_v f_{i_k}(k^l, u_c^0(k^l)) = (\Delta_{i_l} A i_k(k^l))\Delta_v f_{i_l}(k^l, u_c^0(k^l))$$

resulting from (1.6)–(1.7) and by the invertibility of the mappings $A_i(k)$, equality (4.18) can be written down in the form

$$\Delta\widetilde{x}(k) = \varepsilon_s X_{k^l}(k) A_{i_l}^{-1}(k^l)\Delta_v f_{i_l}(k^l, u_c^0(k^l)), \qquad k \in C(k^l). \tag{4.19}$$

Taking account of (4.16)–(4.19) and equation (4.6), we shall obtain

$$\begin{aligned}
\Delta J_c(\widetilde{u}(\cdot)) = {} & \sum_{j=0}^{N-1}\sum_{i=1}^{m}[\{\psi^T(k^{j+1})A_i(\delta_i k^{j+1}) - g_i^T(\delta_i k^{j+1}) \\
& - \psi^T(k^{j+1})A_i(k^j)\}\Delta\widetilde{x}(k^j) - \psi^T(k^{j+1})(f_i(k^j,\widetilde{u}(k^j)) \\
& - f_i(k^j, u_c^0(k^j)))](k_i^{j+1} - k_i^j) + \sum_{\widetilde{k}\in D} h^T(\widetilde{k})\Delta\widetilde{x}(\widetilde{k}) \\
& + \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|) + o(\|\Delta\widetilde{\widetilde{x}}(k^N)\|) \\
= {} & -\varepsilon_s\psi^T(k^{l+1})\Delta_v f_{i_l}(k^l, u_c^0(k^l)) \\
& - \sum_{j=0}^{N-1}\sum_{i=1}^{m} g_i^T(\delta_i k^{j+1})\Delta\widetilde{x}(k^j)(k_i^{j+1} - k_i^j) + \sum_{\widetilde{k}\in D} h^T(\widetilde{k})\Delta\widetilde{x}(\widetilde{k}) \\
& + \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|) + o(\|\Delta\widetilde{\widetilde{x}}(k^N)\|) \\
= {} & -\varepsilon_s[\psi^T(k^l) + g_{i_l}^T(k^l)]A_{i_l}^{-1}(k^l)\Delta_v f_{i_l}(k^l, u_c^0(k^l)) \\
& - \varepsilon_s\Big[\sum_{j=l+1}^{N-1}\sum_{i=1}^{m} g_i^T(\delta_i k^{j+1})X_{k^l}(k^j)(k_i^{j+1} - k_i^j) \\
& - \sum_{\widetilde{k}\in C(k^l)} h^T(\widetilde{k})X_{k^l}(\widetilde{k})\Big]A_{i_l}^{-1}(k^l)\Delta_v f_{i_l}(k^l, u_c^0(k^l)) \\
& + \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|) + o(\|\Delta\widetilde{\widetilde{x}}(k^N)\|) \\
= {} & -\varepsilon_s\psi^T(k^l)A_{i_l}^{-1}(k^l)\Delta_v f_{i_l}(k^l, u_c^0(k^l)) + \sum_{\widetilde{k}\in D} o(\|\Delta\widetilde{x}(\widetilde{k})\|) \\
& + o(\|\Delta\widetilde{\widetilde{x}}(k^N)\|).
\end{aligned}$$

From the optimality of the pair $\{x_c^0(\cdot), u_c^0(\cdot)\}$ it follows that

$$\Delta J_c(\widetilde{u}(\cdot))/\varepsilon_s \geq 0.$$

Since $\|\Delta\tilde{x}(k)\| = 0(\varepsilon_s)$ and $\|\Delta\tilde{\tilde{x}}(k^N)\| = 0(\varepsilon_s)$, therefore, passing to with $\varepsilon_s \downarrow 0$, we shall get the inequality

$$\psi^T(k^l)A_{i_l}^{-1}(k^l)\Delta_v f_{i_l}(k^l, u_c^0(k^l)) \leq 0.$$

In view of the arbitrariness of the discrete curve $L(k^0, k^1, \ldots, k^N)$ and the point k^l lying on it, we obtain inequality (4.5), which ends the proof of Theorem 3.

5. The discrete maximum principle for the basic problem

In this section, using the limit passing in the approximating problem, we shall derive, in the form of the discrete maximum principle, a necessary condition for optimality for the basic problem.

For any non-empty set A of a finite dimensional space X and for any a^0 belonging to the closure of the set A, let us introduce the following notation

$$K(a^0|A) := \limsup_{a \to a^0}[\mathrm{con}(a - W(a|A))]$$

where $\mathrm{con}\, Z := \{\alpha z | \alpha > 0 \wedge z \in Z\}$, $W(a|A) := \{b | b \in \mathrm{cl}A \wedge \|a - b\| = \varrho(a|A)\}$, and the upper limit of multi-valued mapping $Q : X \to 2^Y$ is understood as

$$\limsup_{a \to a^0} Q(a) := \{q \in Y | (\exists a_n \in X)(a_n \to a^0)$$
$$(\exists q_n \in Q(a_n))(q_n \to q)\}.$$

The set $K(a^0|A)$ is called a cone of generalized normals to the non-empty set A at the point $a^0 \in \mathrm{cl}A$. The properties of such cones are fully detailed in monograph [2], (§1–§4).

THEOREM 4. *If $x^0(\cdot)$ is an optimal trajectory in basic problem* (2.1)–(2.6), (1.6)–(1.7), *and*
1) *the sets $f(k, U)$ are convex for $k \in \overline{D}$,*
2) *U is a compact set, $\Omega(k)$ is a closed set for $k \in \overline{D}$,*
3) *the mappings $A_i(k)$, $i \in I(1, m)$, $k \in \overline{D}$, are invertible,*
4) *the functions $f_i(k, \cdot)$, $i \in I(1, m)$, are continuous,*
5) *the functions $\phi_i(\cdot, k)$, $i \in I(0, l)$, are lower semicontinuous, while the functions $\phi_i(\cdot, k)$, $i \in I(l + 1, l + p)$, continuous in a neighbourhood of the optimal trajectory, then there exists a control $u^0(\cdot)$ satisfying* (2.1)–(2.2) *and functions $x^*(\cdot) : \overline{D} \to \mathbf{R}^n$, $y^*(\cdot) = (\lambda_0(\cdot), \lambda_1(\cdot), \ldots, \lambda_{l+p}(\cdot)) : D \to \mathbf{R}^{1+l+p}$ such that*

$$(5.1) \qquad (x^*(k), -y^*(k)) \in K((x^0(k), c^0)|\mathbf{E}_\Omega^k(\phi)) \qquad k \in \overline{D},$$

$$(5.2) \qquad \lambda_i(k) \geq 0, \qquad k \in \overline{D}, \qquad i \in I(0, l),$$

$$(5.3) \qquad \lambda_0(k)(\phi_0(x^0(k), k) - \gamma^0) = 0, \qquad k \in \overline{D},$$

$$(5.4) \qquad \lambda_i(k)(\phi_i(x^0(k),k) = 0, \quad k \in \overline{D}, \quad i \in (I(1,l),$$

$$(5.5) \qquad \sum_{k\in\overline{D}} (\|x^*(k)\|^2 + \|y^*(k)\|^2) = 1,$$

and, for any $v \in U$ and $k \in D$, the condition

$$(5.6) \qquad \psi^T(k)A_i^{-1}(k)\Delta_v f_i(k, u^0(k)) \leq 0, \quad i \in I(1,m),$$

is satisfied, where the function $\psi(\cdot) : \overline{D} \to \mathbb{R}^n$ is a solution of the system of equations

$$(5.7) \qquad \psi(k) = A_i^T(k)\psi(\Delta_i k) - g_i^*(k), \quad k \in D,$$

with the condition

$$(5.8) \qquad \psi(k^N) = -x^*(k^N),$$

where

$$(5.9) \qquad g_i^*(k) := \sum_{\widetilde{k}\in C(k)} X_k^T(\widetilde{k})x^*(\widetilde{k}) - A_i^T(k) \sum_{\widetilde{k}\in C(\Delta_i k)} X_{\Delta_i k}^T(\widetilde{k})x^*(\widetilde{k}).$$

Proof. In view of the continuity of the functions $f_i(k,\cdot)$, $i \in I(1,m)$, and the compactness of U, the set $f(k,U)$ is closed, that is, the assumptions of Theorem 1 are satisfied. Consequently, in each approximating problem (3.1)–(3.3) there are optimal processes $\{x_c^0(\cdot), u_c^0(\cdot)\}$ such that, for each $k \in \overline{D}$, $x_c^0(k) \to x^0(k)$ as $c \to c^0$. From the assumptions concerning the functions $\phi_i(\cdot,k)$, $i \in I(0,l+p)$, it follows that the set $\mathbb{E}_\Omega^k(\phi)$ is, for each $k \in \overline{D}$, a closed set in a neighbourhood of the point $(x_c^0(k), c)$. From Theorem 3, with $c = (\gamma, 0, \ldots, 0)$, $\gamma < \gamma^0$, it follows that there exist vectors $(y_c(k), w_c(k) \in W((x_c^0(k),c)|\mathbb{E}_\Omega^k(\phi))$, $k \in \overline{D}$, such that conditions (4.5)–(4.7) and (4.10) are satisfied. Since the convexity of the set $f(k,U)$ implies that $\sigma(f(k,u_c^0(k)), f(k,U)) = f(k,U)$, therefore condition (4.5) is satisfied for each $v \in U$. In view of the compactness of the set U, one can choose from the set $\{u_c^0(k)\}$, $c \uparrow c^0$,$(\gamma{\uparrow}\gamma^0)$, a subsequence $u_{c_n}^0(k)$ converging to some $u^0(k) \in U, k \in D$, as $n \to \infty$. Let $x_{c_n}^0(\cdot)$ denote the trajectory corresponding to the control $u_{c_n}^0(\cdot)$. Since the sequence

$$(x_{c_n}^0(k) - y_{c_n}(k), c_n - w_{c_n}(k))/m_c \\ \in \operatorname{con}((x_{c_n}^0(k), c_n) - W((x_{c_n}^0(k), c)|\mathbb{E}_\Omega^k(\phi)))$$

is bounded, therefore by choosing a subsequence if necessary, the limit of this sequence belongs to $K((x^0(k),c)|\mathbb{E}_\Omega^k(\phi))$. Denote this limit by $(x^*(k), -y_0^*(k))$. Passing now in (4.5)–(4.10) with $c \uparrow c^0$, we shall obtain (5.1), (5.5)–(5.9).

The properties of cones of generalized normals ([2], th. 3.3) imply conditions (5.2)–(5.4), which completes the proof of Theorem 4.

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