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ALMOST WEAKLY CONTINUOUS MULTIFUNCTIONS

1. Introduction

In 1961, Levine [10] introduced the concept of weakly continuous functions in topological spaces. In 1966, Husain [6] introduced the concept of almost continuous functions. In 1978, Smithson [29] and Vopa [17, 18] extended independently these concepts to multifunctions and defined upper (lower) weakly continuous multifunctions and upper (lower) almost continuous multifunctions. Recently, Janković [7] has defined almost weakly continuous functions as a generalization of weakly continuous functions and almost continuous functions.

The purpose of the present paper is to extend the concept of almost weakly continuous functions to multifunctions. In §3, we obtain many characterizations of upper (lower) almost weakly continuous multifunctions. In §4 (resp. §5, §6), we obtain some sufficient conditions for upper (lower) almost weakly continuous multifunctions to be upper (lower) weakly continuous (resp. upper (lower) almost continuous, continuous). It is shown in the final section that the condition "upper almost continuous" in some results established in [26] can be replaced by "upper almost weakly continuous".

2. Preliminaries

Throughout the present paper, X and Y always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) always represents a multivalued (resp. single valued) function. Let A be a subset of the space X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *preopen* [13] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. The family of all preopen sets in X is denoted by $\mathrm{PO}(X)$. For a point x of X, we set

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$\mathrm{PO}(X,x) = \{U \in \mathrm{PO}(X) | \ x \in U\}$. The complement of a preopen set is said to be *preclosed*. The intersection of preclosed sets containing A is called the *preclosure* [5] of A and is denoted by $p\,\mathrm{Cl}(A)$. The union of preopen sets contained in A is called the *preinterior* of A and is denoted by $p\,\mathrm{Int}(A)$. It is obvious that $X - p\,\mathrm{Cl}(A) = p\,\mathrm{Int}(X - A)$.

LEMMA 2.1. *Let x be a point of a space X and A a subset of X. Then the following hold for the preclosure:*

(a) $x \in p\,\mathrm{Cl}(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \mathrm{PO}(X,x)$.
(b) A *is preclosed if and only if* $A = p\,\mathrm{Cl}(A)$.
(c) $p\,\mathrm{Cl}(A) = A \bigcup \mathrm{Cl}(\mathrm{Int}(A))$.

Proof. This follows from Lemmas 2.2 and 2.3 of [5] and ([1], Theorem 1.5).

For a multifunction $F : X \to Y$, following [2], we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively:

$$F^+(B) = \{x \in X \mid F(x) \subset B\} \quad \text{and} \quad F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

DEFINITION 2.2. A multifunction $F : X \to Y$ is said to be

(a) *upper semi-continuous* (*u.s.c.*) [2] if for each $x \in X$ and each open set V containing $F(x)$, there exists an open neighborhood U of x such that $F(U) \subset V$;
(b) *lower semi-continuous* (*l.s.c.*) if for each $x \in X$ and each open set V such that $F(X) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;
(c) *continuous* if it is u.s.c. and l.s.c.

DEFINITION 2.3. A multifunction $F : X \in Y$ is said to be

(a) *upper almost continuous* (*u.a.c.S.*) [22] if for each $x \in X$ and each open set V containing $F(x)$, there exists an open neighborhood U of x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$;
(b) *lower almost continuous* (*l.a.c.S.*) if for each $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for every $u \in U$.

DEFINITION 2.4. A multifunction $F : X \in Y$ is said to be

(a) *upper weakly continuous* (*u.w.c.*) [17, 29] if for each $x \in X$ and each open set V containing $F(x)$, there exists an open neighborhood U of x such that $F(U) \subset (\mathrm{Cl}(V)$;
(b) *lower weakly continuous* (*l.w.c.*) if for each $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$.

DEFINITION 2.5. A multifunction $F : X \to Y$ is said to be

(a) *upper quasi continuous* (*u.q.c.*) [16] if for each $x \in X$, each open set U containing x and each open set V containing $F(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset V$;

(b) *lower quasi continuous* (*l.q.c.*) if for each $x \in X$, each open set U containing x and each open set V such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set G of X such that $G \subset U$ and $F(g) \cap V \neq \emptyset$ for every $g \in G$.

DEFINITION 2.6. A multifunction $F : X \to Y$ is said to be

(a) *upper almost continuous* (*u.a.c.H.*) [18, 29] if for each $x \in X$ and each open set V of Y containing $F(x)$, $x \in \mathrm{Int}(\mathrm{Cl}(F^+(V)))$;

(b) *lower almost continuous* (*l.a.c.H.*) if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, $x \in \mathrm{Int}(\mathrm{Cl}(F^-(V)))$.

The following lemma is useful and will be utilized in the sequel.

LEMMA 2.7. *The following are equivalent for a multifunction* $F : X \to Y$:

(a) F *is u.a.c.H.* (*resp. l.a.c.H.*).

(b) $F^+(V) \in \mathrm{PO}(X)$ (*resp.* $F^-(V) \in \mathrm{PO}(X)$) *for every open set* V *of* Y.

(c) *For each* $x \in X$ *and each open set* V *such that* $F(x) \subset V$ (*resp.* $F(x) \cap V \neq \emptyset$), *there exists* $U \in \mathrm{PO}(X, x)$ *such that* $F(U) \subset V$ (*resp.* $F(u) \cap V \neq \emptyset$ *for every* $u \in U$).

Proof. This is shown in Theorems 2.3 and 2.4 of [26].

DEFINITION 2.8. A multifunction $F : X \to Y$ is said to be

(a) *upper almost weakly continuous* (*u.a.w.c.*) if for each $x \in X$ and each open set V containing $F(x)$, $x \in \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$;

(b) *lower almost weakly continuous* (*l.a.w.c.*) if for each $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, $x \in \mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Cl}(V))))$.

It is shown in Theorems 4 and 6 of [17] that a multifunction $F : X \to Y$ is u.w.c. (resp. l.w.c.) if and only if $F^+(V) \subset \mathrm{Int}(F^+(\mathrm{Cl}(V)))$ (resp. $F^-(V) \subset \mathrm{Int}(F^-(\mathrm{Cl}(V)))$) for every open set V of Y. Therefore, we obtain the following diagram:

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┌──────────────────────────────→ u.q.c.(l.q.c.)
u.s.c.(l.s.c) → u.a.c.S.(l.a.c.S.) → u.w.c.(l.w.c.) → u.a.w.c.(l.a.w.c.)
└──────────────────────────────→ u.a.c.H.(l.a.c.H.) ──────┘↑
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3. Characterizations

In this section we obtain many characterizations of u.w.a.c. (l.a.w.c.) multifunctions.

THEOREM 3.1. *The following are equivalent for a multifunction $F : X \to Y$:*

(a) *F is u.a.w.c.*

(b) *$F^+(V) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$ for every open set V of Y.*

(c) *$\mathrm{Cl}(\mathrm{Int}(F^-(V))) \subset F^-(\mathrm{Cl}(V))$ for every open set V of Y.*

(d) *$p\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every open set V of Y.*

(e) *$F^+(V) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ for every open set V of Y.*

(f) *For each $x \in X$ and each open set V containing $F(x)$, there exists $U \in \mathrm{PO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$.*

Proof. (a)⇒(b): Let V be any open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$ and hence $x \in \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$. Therefore, $F^+(V) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$.

(b)⇒(c): Let V be any open set of Y. Since $Y - \mathrm{Cl}(V)$ is open,

$$X - F^-(\mathrm{Cl}(V)) = F^+(Y - \mathrm{Cl}(V)) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(Y - \mathrm{Cl}(V)))))$$
$$\subset \mathrm{Int}(\mathrm{Cl}(F^+(Y - V))) = \mathrm{Int}(\mathrm{Cl}(X - F^-(V))) = X - \mathrm{Cl}(\mathrm{Int}(F^-(V))).$$

Therefore, we obtain $\mathrm{Cl}(\mathrm{Int}(F^-(V))) \subset F^-(\mathrm{Cl}(V))$.

(c)⇒(d): Let V be any open set of Y. By Lemma 2.1, we have

$$p\,\mathrm{Cl}(F^-(V)) = F^-(V) \cup \mathrm{Cl}(\mathrm{Int}(F^-(V))) \subset F^-(\mathrm{Cl}(V)).$$

(d)⇒(e): Let V be any open set of Y. Since $Y - \mathrm{Cl}(V)$ is open, we have

$$X - p\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) = p\,\mathrm{Cl}(X - F^+(\mathrm{Cl}(V))) = p\,\mathrm{Cl}(F^-(Y - \mathrm{Cl}(V)))$$
$$\subset F^-(\mathrm{Cl}(Y - \mathrm{Cl}(V))) \subset F^-(Y - V) = X - F^+(V).$$

Therefore, we obtain $F^+(V) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.

(e)⇒(f): Let $x \in X$ and V be any open set containing $F(x)$. Then $x \in F^+(V) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Therefore, there exists $U \in \mathrm{PO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$.

(f)⇒(a): Let $x \in X$ and V be any open set containing $F(x)$. There exists $U \in \mathrm{PO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$; hence $U \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $x \in U \subset \mathrm{Int}(\mathrm{Cl}(U)) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$.

THEOREM 3.2. *The following are equivalent for a multifunction $F : X \to Y$:*

(a) *F is l.a.w.c.*

(b) *$F^-(V) \subset \mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Cl}(V))))$ for every open set V of Y.*

(c) *$\mathrm{Cl}(\mathrm{Int}(F^+(V))) \subset F^+(\mathrm{Cl}(V))$ for every open set V of Y.*

(d) *$p\,\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ for every open set V of Y.*

(e) *$F^-(V) \subset p\,\mathrm{Int}(F^-(\mathrm{Cl}(V)))$ for every open set V of Y.*

(f) *For each $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \mathrm{PO}(X,x)$, such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.*

Proof. The proof is similar to that of Theorem 3.1 and is thus omitted.

Remark 3.3. For any $U \in PO(X)$, $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(U))) = \mathrm{Cl}(U)$ and hence "open set" in each statement of Theorems 3.1 and 3.2 can be replaced by "preopen set".

THEOREM 3.4. *The following are equivalent for a multifunction* $F : X \to Y$:

(a) F *is u.a.w.c.*
(b) $\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{Int}(W)))) \subset F^-(W)$ *for every closed set* W *of* Y.
(c) $p\,\mathrm{Cl}(F^-(\mathrm{Int}(W))) \subset F^-(W)$ *for every closed set* W *of* Y.
(d) $p\,\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}(B))$ *for every subset* B *of* Y.
(e) $F^+(\mathrm{Int}(B)) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(\mathrm{Int}(B))))$ *for every subset* B *of* Y.

Proof. (a)⇒(b): Let W be any closed set of Y. Since $Y - W$ is open in Y, by Theorem 3.1 we have

$$\begin{aligned} X - F^-(W) = F^+(Y - W) &\subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(Y - W)))) \\ &= \mathrm{Int}(\mathrm{Cl}(F^+(Y - \mathrm{Int}(W)))) \\ &= \mathrm{Int}(\mathrm{Cl}(X - F^-(\mathrm{Int}(W)))) = X - \mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{Int}(W)))). \end{aligned}$$

Therefore, we obtain $\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{Int}(W)))) \subset F^-(W)$.

(b)⇒(c): Let W be any closed set of Y. By Lemma 2.1, we have

$$p\,\mathrm{Cl}(F^-(\mathrm{Int}(W))) = F^-(\mathrm{Int}(W)) \cup \mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{Int}(W)))) \subset F^-(W).$$

(c)⇒(d): This is obvious.

(d)⇒(e): Let B be any subset of Y. Then we have

$$\begin{aligned} X - p\,\mathrm{Int}(F^+(\mathrm{Cl}(\mathrm{Int}(B)))) &= p\,\mathrm{Cl}(X - F^+(\mathrm{Cl}(\mathrm{Int}(B)))) \\ &= p\,\mathrm{Cl}(F^-(Y - \mathrm{Cl}(\mathrm{Int}(B)))) = p\,\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(Y - B)))) \\ &\subset F^-(\mathrm{Cl}(Y - B)) = X - F^+(\mathrm{Int}(B)). \end{aligned}$$

Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(\mathrm{Int}(B))))$.

(e)⇒(a): Let V be any open set of Y. Then $F^+(V) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ and hence F is u.a.w.c. by Theorem 3.1.

THEOREM 3.5. *The following are equivalent for a multifunction* $F : X \to Y$:

(a) F *is l.a.w.c.*
(b) $\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{Int}(W)))) \subset F^+(W)$ *for every closed set* W *of* Y.
(c) $p\,\mathrm{Cl}(F^+(\mathrm{Int}(W))) \subset F^+(W)$ *for every closed set* W *of* Y.
(d) $p\,\mathrm{Cl}(F^+(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^+(\mathrm{Cl}(B))$ *for every subset* B *of* Y.
(e) $F^-(\mathrm{Int}(B)) \subset p\,\mathrm{Int}(F^-(\mathrm{Cl}(\mathrm{Int}(B))))$ *for every subset* B *of* Y.

Proof. The proof is similar to that of Theorem 3.4.

A function $f : X \to Y$ is said to be *almost weakly continuous* [7] if $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ for each open set V of Y. In the following corollary "open set" in each statement can be replaced by "preopen set".

COROLLARY 3.6. *The following are equivalent for a function $F : X \to Y$:*

(a) *f is almost weakly continuous.*
(b) $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set V of Y.*
(c) $p\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set V of Y.*
(d) $f^{-1}(V) \subset p\,\mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ *for every open set V of Y.*
(e) *For each $x \in X$ and each open set V containing $f(x)$, there exists $U \in \mathrm{PO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$.*

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

LEMMA 3.7. *The following hold for a multifunction $F : X \to Y$:*

$$(a)\ G_F^+(A \times B) = A \cap F^+(B) \quad \text{and} \quad (b)\ G_F^-(A \times B) = A \cap F^-(B)$$

for every subset $A \subset X$ and $B \subset Y$.

Proof. We shall prove only (b). Let A and B be any subsets of X and Y, respectively. Let $x \in G_F^-(A \times B)$. Then

$$\emptyset \neq G_F(x) \cap (A \times B) = (\{x\} \times F(x)) \cap (A \times B) = (\{x\} \cap A) \times (F(x) \cap B).$$

Therefore, we have $x \in A$ and $F(x) \cap B \neq \emptyset$ and hence $x \in A \cap F^-(B)$. Conversely, let $x \in A \cap F^-(B)$. Then $x \in A$ and $F(x) \cap B \neq \emptyset$ and hence $G_F(x) \cap (A \times B) \neq \emptyset$. Therefore, $x \in G_F^-(A \times B)$. This completes the proof.

THEOREM 3.8. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then $F : X \to Y$ is u.a.w.c. if and only if $G_F : X \to X \times Y$ is u.a.w.c.*

Proof. **Necessity.** Suppose that $F : X \to Y$ is u.a.w.c. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x,y) \in U(y) \times V(y) \subset W$. The family $\{V(y) \mid y \in F(x)\}$ is an open cover of $F(x)$ and there exists a finite number of points, says, $y_1, y_2, \dots, y_n$ in $F(x)$ such that $F(x) \subset \bigcup\{V(y_i) \mid i = 1,2,\dots,n\}$. Set $U = \bigcap\{U(y_i) \mid i = 1,2,\dots,n\}$ and $V = \bigcup\{V(y_i) \mid i = 1,2,\dots,n\}$. Then U and V are open in X and Y, respectively and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is u.a.w.c., by Theorem 3.1 there exists $U_0 \in \mathrm{PO}(X,x)$ such that $F(U_0) \subset \mathrm{Cl}(V)$. By Lemma 3.7, we have

$$U \cap U_0 \subset U \cap F^+(\mathrm{Cl}(V)) = G_F^+(U \times \mathrm{Cl}(V)) \subset G_F^+(\mathrm{Cl}(W)).$$

Therefore, we obtain $U \cap U_0 \in \mathrm{PO}(X,x)$ and $G_F(U \cap U_0) \subset \mathrm{Cl}(W)$. This shows that G_F is u.a.w.c.

Sufficiency. Suppose that $G_F : X \to X \times Y$ is u.a.w.c. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, by Theorem 3.1 there exists $U \in \mathrm{PO}(X,x)$ such that $G_F(U) \subset X \times \mathrm{Cl}(V) = \mathrm{Cl}(X \times V)$. Therefore, by Lemma 3.7 $U \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$ and hence $F(U) \subset \mathrm{Cl}(V)$. This shows that F is u.a.w.c.

THEOREM 3.9. *A multifunction $F : X \to Y$ is l.a.w.c. if and only if $G_F : X \to X \times Y$ is l.a.w.c.*

Proof. **Necessity.** Suppose that F is l.a.w.c. Let $x \in X$ and W be any open set of Y such that $G_F(x) \cap W \neq \emptyset$. There exists $y \in F(x)$ such that $(x,y) \in W$ and hence $(x,y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since F is l.a.w.c. and $y \in F(x) \cap V$, there exists $U_0 \in \mathrm{PO}(X,x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U_0$; hence $U_0 \subset F^-(\mathrm{Cl}(V))$. By Lemma 3.7, $U \cap U_0 \subset U \cap F^-(\mathrm{Cl}(V)) = G_F^-(U \times \mathrm{Cl}(V)) \subset G_F^-(\mathrm{Cl}(W))$. Moreover, $U \cap U_0 \in \mathrm{PO}(X,x)$ and hence G_F is l.a.w.c.

Sufficiency. Suppose that G_F is l.a.w.c. Let $x \in X$ and V be an open set in Y such that $F(x) \cap V \neq \emptyset$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. There exists $U \in \mathrm{PO}(X,x)$ such that $G_F(u) \cap \mathrm{Cl}(X \times V) \neq \emptyset$ for each $u \in U$. By Lemma 3.7, we obtain $U \subset G_F^-(\mathrm{Cl}(X \times V)) = F^-(\mathrm{Cl}(V))$. This shows that F is l.a.w.c.

A subset A of a space X is said to be *semi-open* [11] if there exists an open set U of X such that $U \subset A \subset \mathrm{Cl}(U)$. Yhe family of all semi-open sets in X is denoted by $\mathrm{SO}(X)$. The complement of a semi-open set is said to be *semi-closed*. The intersection of all semi-closed sets containing A is called the *semi-closure* [4] of A and is denoted by $s\,\mathrm{Cl}(A)$. For a multifunction $F : X \to Y$, a multifunction $s\,\mathrm{Cl}\,F : X \to Y$ is defined in [20] as follows: $(s\,\mathrm{Cl}\,F)(x) = s\,\mathrm{Cl}(F(x))$ for each $x \in X$.

LEMMA 3.10. *Let $F : X \to Y$ be a multifunction. Then $(s\,\mathrm{Cl}\,F)^-(V) = F^-(V)$ for every $V \in \mathrm{SO}(Y)$.*

Proof. Let V be any semi-open set of Y. Let $x \in (s\,\mathrm{Cl}\,F)^-(V)$. Then $V \cap s\,\mathrm{Cl}(F(x)) = V \cap (s\,\mathrm{Cl}\,F)(x) \neq \emptyset$. Since $V \in \mathrm{SO}(Y)$, $V \cap F(x) \neq \emptyset$ and hence $x \in F^-(V)$. Therefore, we obtain $(s\,\mathrm{Cl}\,F)^-(V) \subset F^-(V)$. Conversely, let $x \in F^-(V)$. Then $\emptyset \neq F(x) \cap V \subset (s\,\mathrm{Cl}\,F)(x) \cap V$ and hence $x \in (s\,\mathrm{Cl}\,F)^-(V)$. Therefore, we obtain $(s\,\mathrm{Cl}\,F)^-(V) = F^-(V)$.

THEOREM 3.11. *A multifunction $F : X \to Y$ is l.a.w.c. if and only if $s\,\mathrm{Cl}\,F : X \to Y$ is l.a.w.c.*

Proof. **Necessity.** Suppose that F is l.a.w.c. Let $x \in X$ and V be any open set of Y such that $(s\,\mathrm{Cl}\,F)(x) \cap V \neq \emptyset$. By Lemma 3.10, we have $F(x) \cap V \neq \emptyset$. Since F is l.a.w.c., by Theorem 3.2. there exists $U \in \mathrm{PO}(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Since $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$, by Lemma 3.10 we have $u \in F^{-}(\mathrm{Cl}(V)) = (s\,\mathrm{Cl}\,F)^{-}(\mathrm{Cl}(V))$ and hence $(s\,\mathrm{Cl}\,F)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. This shows that $s\,\mathrm{Cl}\,F$ is l.a.w.c.

Sufficiency. Suppose that $s\,\mathrm{Cl}\,F$ is l.a.w.c. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. Then $(s\,\mathrm{Cl}\,F)(x) \cap V \neq \emptyset$ and there exists $U \in \mathrm{PO}(X, x)$ such that $(s\,\mathrm{Cl}\,F)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Since $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$, by Lemma 3.10 $U \subset (s\,\mathrm{Cl}\,F)^{-}(\mathrm{Cl}(V)) = F^{-}(\mathrm{Cl}(V))$ and hence $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Therefore, F is l.a.w.c.

LEMMA 3.12 (Mashhour et al. [13]). *Let U and X_0 be subsets of a space X. The following properties hold:*

(a) *If $U \in \mathrm{PO}(X)$ and $X_0 \in \mathrm{SO}(X)$, then $U \cap X_0 \in \mathrm{PO}(X_0)$.*
(b) *If $U \in \mathrm{PO}(X_0)$ and $X_0 \in \mathrm{PO}(X)$, then $U \in \mathrm{PO}(X)$.*

LEMMA 3.13. *If a multifunction $F : X \to Y$ is u.a.w.c. (resp. l.a.w.c. and $X_0 \in \mathrm{SO}(X)$, then the restriction $F|X_0 : X_0 \to Y$ is u.a.w.c. (resp. l.a.w.c.).*

Proof. We shall prove only the case "u.a.w.c." since another is entirely analogous. Let $x \in X_0$ and V be any open set in Y containing $(F|X_0)(x)$. Since F is u.a.w.c. and $(F|X_0)(x) = F(x)$, there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Let $U_0 = U \cap X_0$, then $U_0 \in \mathrm{PO}(X_0, x)$ by Lemma 3.12 and $(F|X_0)(U_0) = F(U_0) \subset F(U) \subset \mathrm{Cl}(V)$. This shows that $F|X_0$ is u.a.w.c.

LEMMA 3.14. *Let $F : X \to Y$ be a multifunction. If for each $x \in X$ there exists $X_0 \in \mathrm{PO}(X, x)$ such that the restriction $F|X_0 : X_0 \to Y$ is u.a.w.c. (resp. l.a.w.c.), then F is u.a.w.c. (resp. l.a.w.c.).*

Proof. We shall prove only the case "u.a.w.c.". Let $x \in X$ and V be any open set in Y containing $F(x)$. There exists $X_0 \in \mathrm{PO}(X, x)$ such that $F|X_0 : X_0 \to Y$ is u.a.w.c. and hence $(F|X_0)(U_0) \subset \mathrm{Cl}(V)$ for some $U_0 \in \mathrm{PO}(X_0, x)$. By Lemma 3.12, $U_0 \in \mathrm{PO}(X, x)$ and $F(U_0) = (F|X_0)(U_0) \subset \mathrm{Cl}(V)$. Therefore, F is u.a.w.c.

A subset A of a space X is called an *α-set* [14] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. It is shown in [15, Lemma 3.1] that a subset is an α-set if and only if it is semi-open and preopen.

THEOREM 3.15. *Let $\{U_\alpha \mid \alpha \in \nabla\}$ be a cover of X by α-sets of X. A multifunction $F : X \to Y$ is u.a.w.c. (resp. l.a.w.c.) if and only if the restriction $F|U_\alpha : U_\alpha \to Y$ is u.a.w.c. (resp. l.a.w.c.) for every $\alpha \in \nabla$.*

Proof. This is an immediate consequence of Lemmas 3.13 and 3.14.

4. Sufficient conditions for a.w.c. multifunctions to be w.c.

In this section, we obtain some sufficient conditions for u.a.w.c. (resp. l.a.w.c.) multifunctions to be u.w.c. (resp. l.w.c.).

THEOREM 4.1. *If a multifunction* $F : X \to Y$ *is u.a.w.c. and l.a.c.S., then* F *is u.w.c.*

Proof. Let V be any open set of Y. Since F is u.a.w.c., by Theorem 3.1 $F^+(V) \subset \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$. Since $\mathrm{Cl}(V)$ is regular closed, it follows from [22, Theorem 2.2] that $F^+(\mathrm{Cl}(V))$ is closed in X. Therefore, we obtain $F^+(V) \subset \mathrm{Int}(F^+(\mathrm{Cl}(V)))$ and hence it follows from [17, Theorem 6] that F is u.w.c.

COROLLARY 4.2 (Popa [23]). *If a multifunction* $F : X \to Y$ *is u.a.c.H. and l.s.c., then* F *is u.w.c.*

THEOREM 4.3. *If a multifunction* $F : X \to Y$ *is l.a.w.c. and u.a.c.S., then* F *is l.w.c.*

Proof. The proof is similar to that of Theorem 4.1.

COROLLARY 4.4 (Popa [23]). *If a multifunction* $F : X \to Y$ *is l.a.c.H. and u.s.c., then* F *is l.w.c.*

Smithson [29] and Popa [17] showed independently that if $F : X \to Y$ is u.w.c. then $\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every open set V of Y. Clay and Joseph [3] showed that the converse of the previous statement is also true.

LEMMA 4.5 (Clay and Joseph [3]). *A multifunction* $F : X \to Y$ *is u.w.c.* (*resp. l.w.c.*) *if and only if* $\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ (*resp.* $\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$) *for every open set* V *of* Y.

THEOREM 4.6. *If a multifunction* $F : X \to Y$ *is u.a.w.c. and l.q.c., then* F *is u.w.c.*

Proof. Let V be any open set of Y. Since F is l.q.c., it follows from [24, Theorem 2.4] that $F^-(V) \subset \mathrm{Cl}(\mathrm{Int}(F^-(V)))$. Since F is u.a.w.c., by Theorem 3.1 we have $\mathrm{Cl}(\mathrm{Int}(F^-(V))) \subset F^-(\mathrm{Cl}(V))$ and hence $\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$. It follows from Lemma 4.5 that F is u.w.c.

The following corollary and Corollary 4.2 are immediate consequences of Theorem 4.6.

COROLLARY 4.7 (Popa [24]). *If a multifunction* $F : X \to Y$ *is u.a.c.H. and l.q.c., then* F *is u.w.c.*

THEOREM 4.8. *If a multifunction* $F : X \to Y$ *is l.a.w.c. and u.q.c., then* F *is l.w.c.*

Proof. Let V be any open set of Y. Since F is u.q.c., it follows from [24, Theorem 2.3] that $F^+(V) \subset \mathrm{Cl}(\mathrm{Int}(F^+(V)))$. By Theorem 3.2, we have $\mathrm{Cl}(\mathrm{Int}(F^+(V))) \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ and hence F is l.w.c. by Lemma 4.5.

The following corollary and Corollary 4.4 are immediate consequences of Theorem 4.8.

COROLLARY 4.9 (Popa [24]). *If a multifunction $F : X \to Y$ is l.a.c.H. and u.q.c., then F is l.w.c.*

5. Sufficient conditions for a.w.c. multifunctions to be a.c.H.

In this section, we obtain several sufficient conditions for u.a.w.c. (resp. l.a.w.c.) multifunctions to be u.a.c.H. (resp. l.a.c.H.). Rose [28] defined a function $F : X \to Y$ to be *almost open* if $f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U)))$ for every open set U of X. It was shown in [28, Theorem 11] that a function $F : X \to Y$ is almost open if and only if $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every open set V of Y. We shall obtain an analogous result for multifunctions. A multifunction $F : X \to Y$ is said to be *almost open* if $F(U) \subset \mathrm{Int}(\mathrm{Cl}(F(U)))$ for every open set U of X.

THEOREM 5.1. *A multifunction $F : X \to Y$ is almost open if and only if $F^-(\mathrm{Cl}(V)) \subset \mathrm{Cl}(F^-(V))$ for every open set V of Y.*

Proof. **Necessity.** Let V be any open set of Y and $x \in X - \mathrm{Cl}(F^-(V))$. There exists an open neighborhood U of x such that $U \cap F^-(V) = \emptyset$. Therefore, we have $F(U) \cap V = \emptyset$ and hence $\mathrm{Int}(\mathrm{Cl}(F(U))) \cap \mathrm{Cl}(V) = \emptyset$. Since F is almost open, $F(U) \cap \mathrm{Cl}(V) = \emptyset$ and hence $x \in X - F^-(\mathrm{Cl}(V))$. Therefore, we obtain $F^-(\mathrm{Cl}(V)) \subset \mathrm{Cl}(F^-(V))$.

Sufficiency. Suppose that F is not almost open. Then $F(U) - \mathrm{Int}(\mathrm{Cl}(F(U))) \neq \emptyset$ for some open set U of X. Let $V = Y - \mathrm{Cl}(F(U))$. Then V is open in Y and $F(U) \cap V = \emptyset$. Since $F(U) \cap \mathrm{Cl}(V) = F(U) \cap (Y - \mathrm{Int}(\mathrm{Cl}(F(U)))) \neq \emptyset$, $\emptyset \neq U \cap F^-(\mathrm{Cl}(V)) \subset U \cap \mathrm{Cl}(F^-(V))$ and hence $U \cap F^-(V) \neq \emptyset$. Therefore, we obtain $F(U) \cap \neq \emptyset$. This is contradiction.

A multifunction $F : X \to Y$ is said to be *nearly almost open* if there exists an open basis $\Sigma = \{V_\alpha \mid \alpha \in \nabla\}$ of the topology for Y such that $F^-(\mathrm{Cl}(V_\alpha)) \subset \mathrm{Cl}(F^-(V_\alpha))$ for every $\alpha \in \nabla$.

THEOREM 5.2. *If a multifunction $F : X \to Y$ is l.a.w.c. and nearly almost open, then F is l.a.c.H.*

Proof. Let $\Sigma = \{V_\alpha \mid \alpha \in \nabla\}$ be an open basis of the topology for Y such that $F^-(\mathrm{Cl}(V_\alpha)) \subset \mathrm{Cl}(F^-(V_\alpha))$ for every $\alpha \in \nabla$. For any open set V of Y, there exists a subset ∇_0 of ∇ such that $V = \bigcup\{V_\alpha \mid \alpha \in \nabla_0\}$. Therefore, we obtain

$$\begin{aligned}F^-(V) &= F^-\Big(\bigcup_{\alpha\in\nabla_0} V_\alpha\Big) = \bigcup_{\alpha\in\nabla_0} F^-(V_\alpha)\\ &\subset \bigcup_{\alpha\in\nabla_0} \operatorname{Int}(\operatorname{Cl}(F^-(\operatorname{Cl}(V_\alpha)))) \subset \bigcup_{\alpha\in\nabla_0} \operatorname{Int}(\operatorname{Cl}(F^-(V_\alpha)))\\ &\subset \operatorname{Int}\Big(\operatorname{Cl}\Big(\bigcup_{\alpha\in\nabla_0} F^-(V_\alpha)\Big)\Big) = \operatorname{Int}\Big(\operatorname{Cl}\Big(F^-\Big(\bigcup_{\alpha\in\nabla_0} V_\alpha\Big)\Big)\Big)\\ &= \operatorname{Int}(\operatorname{Cl}(F^-(V))).\end{aligned}$$

This shows that $F^-(V) \in \mathrm{PO}(X)$. It follows from Lemma 2.7 that F is l.a.c.H.

COROLLARY 5.3. *If a multifunction $F : X \to Y$ is l.a.w.c. and almost open, then F is l.a.c.H.*

COROLLARY 5.4 (Popa [23]). *If a multifunction $F : X \to Y$ is l.w.c. and if for any open set V of Y the relation $F^-(\operatorname{Cl}(V)) \subset \operatorname{Cl}(F^-(V))$ holds, then F is l.a.c.H.*

Proof. This is an immediate consequence of Theorem 5.1 and Corollary 5.3.

A subset A of a space X is said to be *α-regular* [9] if for each point $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \operatorname{Cl}(G) \subset U$.

THEOREM 5.5. *If a multifunction $F : X \to Y$ is l.a.w.c. and $F(x)$ is α-regular for each $x \in X$, then F is l.a.c.H.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. There exists a point $y \in F(x) \cap V$ and $y \in W \subset \operatorname{Cl}(W) \subset V$ for some open set W of Y. Since F is l.a.w.c. and $F(x) \cap W \neq \emptyset$, $x \in \operatorname{Int}(\operatorname{Cl}(F^-(\operatorname{Cl}(W)))) \subset \operatorname{Int}(\operatorname{Cl}(F^-(V)))$. Therefore, F is l.a.c.H.

COROLLARY 5.6. *If a multifunction $F : X \to Y$ is l.a.w.c. and Y is regular, then F is l.a.c.H.*

Hereafter, in this section, we shall obtain some sufficient conditions for u.a.w.c. multifunctions to be u.a.c.H.

THEOREM 5.7. *If a multifunction $F : X \to Y$ is u.a.w.c. and satisfies $F^+(\operatorname{Cl}(V)) \subset \operatorname{Cl}(F^+(V))$ for every open set V of Y, then F is u.a.c.H.*

Proof. Let V be any open set V of Y. Since F is u.a.w.c., by Theorem 3.1 $F^+(V) \subset \operatorname{Int}(\operatorname{Cl}(F^+(\operatorname{Cl}(V))))$ and hence $F^+(V) \subset \operatorname{Int}(\operatorname{Cl}(F^+(V)))$. Therefore, $F^+(V) \in \mathrm{PO}(X)$ and it follows from Lemma 2.7 that F is u.a.c.H.

COROLLARY 5.8 (Popa [23]). *If a multifunction $F : X \to Y$ is u.w.c. and if for any open set V of Y the relations $F^+(\mathrm{Cl}(V)) \subset \mathrm{Cl}(F^+(V))$ holds, then F is u.a.c.H.*

A subset A of a space X is said to be *α-paracompact* [30] if every X-open cover of A has an X-open X-locally finite refinement which covers A. The following lemma is very useful in the sequel.

LEMMA 5.9 (Kovačević [9]). *If A is an α-regular α-paracompact subset of a space X and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

THEOREM 5.10. *If a multifunction $F : X \to Y$ is u.a.w.c. and if either*

(a) *$F(x)$ is α-regular α-paracompact for each $x \in X$ or*

(b) *$F(x)$ is closed in Y for each $x \in X$ and Y is normal (not necessarily T_1), then F is u.a.c.H.*

Proof. Let $x \in X$ and V be any open set of Y containing $F(x)$. Under each condition of (a) and (b), there exists an open set W of Y such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is u.a.w.c., by Theorem 3.1 there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(W)$; hence $F(U) \subset V$. Therefore, by Lemma 2.7 F is u.a.c.H.

A multifunction $F : X \to Y$ is said to be *complementary continuous* (w*.c.) [17] if $F^-(Fr(V))$ is closed in X for every open set V of Y, where $Fr(V)$ denotes the frontier of V. It was shown in [17, Theorem 7] that a multifunction $F : X \to Y$ is u.s.c. if and only if it is u.w.c. and w*.c.

THEOREM 5.11. *If a multifunction $F : X \to Y$ is u.a.w.c. and w*.c., then F is u.a.c.H.*

Proof. Let $x \in X$ and V be any open set of Y containing $F(x)$. By Theorem 3.1, there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Since $F(x) \subset V$, $F(x) \cap Fr(V) = \emptyset$ and hence $X - F^-(Fr(V))$ is an open niegh-borhood of x. Set $U_0 = U \cap (X - F^-(Fr(V)))$, then we have $U_0 \in \mathrm{PO}(X, x)$ and $F(U_0) \subset V$. This shows that F is u.a.c.H.

For a multifunction $F : X \to Y$, the subset $\{(x, y) \mid x \in X \text{ and } y \in F(x)\}$ of $X \times Y$ is called the *graph* of F and is denoted by $G(F)$. We say that F has a *closed graph* if $G(F)$ is a closed subset of the product space $X \times Y$. A space Y is said to be *rim-compact* if there exists an open basis Σ for the topology on Y such that $Fr(V)$ is compact for each V in Σ.

THEOREM 5.12. *Let $F : X \to Y$ be an u.a.w.c. multifunction with a closed graph $G(F)$. If Y is rim-compact and $F(x)$ is compact for each $x \in X$, then F is u.a.c.H.*

Proof. Since Y is rim-compact, there exists an open basis Σ for the topology on Y such that $Fr(V)$ is compact for each V in Σ. Let $x \in X$ and W be any open set containing $F(x)$. For each $y \in F(x)$, there exists $V(y)$ in Σ such that $Fr(V(y))$ is compact and $y \in V(y) \subset W$. Since $F(x)$ is compact, there exists a finite subset K of $F(x)$ such that $F(x) \subset \bigcup\{V(y) \mid y \in K\} \subset W$. Set $V = \bigcup\{V(y) \mid y \in K\}$, then V is open in Y, $F(x) \subset V \subset W$ and $Fr(V)$ is compact. Since $G(F)$ is closed, $F^-(Fr(V))$ is closed in X and hence $F^+(Y - Fr(V))$ is open in X. Since F is u.a.w.c., by Theorem 3.1 $F^+(V)) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Moreover, we obtain

$$\begin{aligned} F^+(V) &\subset F^+(Y - Fr(V)) \cap p\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \\ &\subset F^+(Y - Fr(V)) \cap F^+(\mathrm{Cl}(V)) = F^+(V). \end{aligned}$$

Therefore, $F^+(V) = F^+(Y - Fr(V)) \cap p\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \in \mathrm{PO}(X)$ and $x \in F^+(V) \subset F^+(W)$. Consequently, we obtain $x \in \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ and hence F is u.a.c.H.

Remark 5.13. In [8], Joseph defined the concept of subclosed graphs for multifunctions as a generalization of closed graphs and showed that if a multifunction $F : X \to Y$ has a subclosed graph and K is a compact subset of Y then $F^-(K)$ is closed in X [8, Theorem 3.15]. Therefore, the condition "closed" on $G(F)$ in Theorem 5.12 can be replaced by "subclosed".

6. Sufficient conditions for a.w.c multifunctions to be continuous

In this section, we obtain some sufficient conditions for l.a.w.c. (resp. u.a.w.c.) multifunctions to be continuous . Some results established in [17], [19] and [29] will be slightly improved.

LEMMA 6.1. *If a multifunction $F : X \to Y$ is l.w.c. and $F(x)$ is α-regular for each $x \in X$, then F is l.s.c.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. There exists $y \in F(x) \cap V$ and hence $y \in W \subset \mathrm{Cl}(W) \subset V$ for some open set W of Y since $F(x)$ is α-regular. Since F is l.w.c. and $F(x) \cap W \neq \emptyset$, there exists an open neighborhood U of x such that $F(u) \cap \mathrm{Cl}(W) \neq \emptyset$ for every $u \in U$. Therefore, we obtain $F(u) \cap V \neq \emptyset$ for every $u \in U$. This shows that F is l.s.c.

COROLLARY 6.2 (Popa [17]). *Let Y be a regular space. A multifunction $F : X \to Y$ is l.s.c if and only if F is l.w.c.*

THEOREM 6.3. *If a multifunction $F : X \to Y$ is u.a.w.c.,l.w.c. and $F(x)$ is α-regular α-paracompact for each $x \in X$, then F is continuous.*

Proof. It follows from Lemma 6.1 that F is l.s.c. We shall show that F is u.s.c. Let $x \in X$ and V be any open set of Y containing $F(x)$. By Lemma 5.9, there exists an open set W of Y such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is l.w.c., by Lemma 4.5 we have $\mathrm{Cl}(F^+(W)) \subset F^+(\mathrm{Cl}(W)) \subset F^+(V)$. Since F is u.a.w.c. and $F(x)$ is α-regular α-paracompact, by Theorem 5.10 F is u.a.c.H. and hence $x \in F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ by Lemma 2.7. Now, set $U = \mathrm{Int}(\mathrm{Cl}(F^+(W)))$, then U is an open neighborhood of x and $F(U) \subset V$. Therefore, F is u.s.c. and hence continuous.

The following three corollaries are immediate consequences of Lemma 4.5 and Theorem 6.3.

COROLLARY 6.4. *Let $F : X \to Y$ be an u.a.w.c. multifunction and Y be regular. If F satisfies the following*:

(a) *$F(x)$ is compact for each $x \in X$*;

(b) $\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every open set V of Y*,

then F is continuous.

COROLLARY 6.5 (Smithson [29]). *If $F : X \to Y$ is an u.a.c.H., l.s.c. and point compact multifunction into a regular space, then F is u.s.c.*

COROLLARY 6.6 (Popa [19]). *Let $F : X \to Y$ be u.a.c.H. and Y regular. If F satisfies the following*:

(a) *$F(x)$ has a finite number of elements for each $x \in X$*;

(b) $\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for any open set V of Y*,

then F is u.s.c.

THEOREM 6.7. *If a multifunction $F : X \to Y$ is l.a.w.c., u.w.c. and $F(x)$ is α-regular for each $x \in X$, then F is l.s.c.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. There exists $y \in F(x) \cap V$ and hence $y \in W \subset \mathrm{Cl}(W) \subset V$ for some open set W in Y since $F(x)$ is α-regular. By Theorem 5.5, F is l.a.c.H. and $x \in \mathrm{Int}(\mathrm{Cl}(F^-(W)))$. Since F is u.w.c., by Lemma 4.5 we have $\mathrm{Cl}(F^-(W)) \subset F^-(\mathrm{Cl}(W)) \subset F^-(V)$. Therefore, set $U = \mathrm{Int}(\mathrm{Cl}(F^-(W)))$, then U is an open neighborhood of x and $F(u) \cap V \neq \emptyset$ for each $u \in U$. This shows that F is l.s.c.

The following two corrolaries are immediate consequences of Lemma 4.5 and Theorem 6.7.

COROLLARY 6.8 (Popa [19]). *Let $F : X \to Y$ be a l.a.c.H. multifunction and Y a regular space. If F has the property* $\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every open set V of Y, then F is l.s.c.*

COROLLARY 6.9 (Smithson [29]). *If $F : X \to Y$ is a l.a.c.H. and u.s.c. multifunction into a regular space Y, then F is l.s.c.*

LEMMA 6.10. *If a multifunction $F : X \to Y$ is u.w.c. and $F(x)$ is α-regular α-paracompact for each $x \in X$, then F is u.s.c.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. By Lemma 5.9, there exists an open set W such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is u.w.c., there exists an open nieghborhood U of x such that $F(U) \subset \mathrm{Cl}(W)$; hence $F(U) \subset V$. Therefore, F is u.s.c.

COROLLARY 6.11 (Popa [21]). *Let Y be a regular space and $F : X \to Y$ a multifunction such that $F(x_0)$ is strictly paracompact (equivalently α-paracompact) for a point x_0 in X. Then F is u.s.c. at x_0 if and only if F is u.w.c. at x_0.*

THEOREM 6.12. *If a multifunction $F : X \to Y$ is l.a.w.c., u.w.c. and $F(x)$ is α-regular α-paracompact for each $x \in X$, then F is continuous.*

Proof. This is an immediate consequence of Theorem 6.7 and Lemma 6.10.

COROLLARY 6.13. *Let $F : X \to Y$ be a l.a.w.c. multifunction and Y regular. If F satisfies the following:*

(a) *$F(x)$ is compact for each $x \in X$;*
(b) $\mathrm{Cl}(F^{-}(V)) \subset F^{-}(\mathrm{Cl}(V))$ *for every open set V of Y,*

then F is continuous.

7. Some applications

In [26], the second author obtained several characterizations and properties of u.a.c.H. (resp. l.a.c.H.) multifunctions. It will be shown that the condition "u.a.c.H." in some theorems established in [26] can be replaced by "u.a.w.c.".

A space X is said to be *strongly compact* [12] if every preopen cover of X admits a finite subcover. A space X is said to be *quasi H-closed* [27] if for every open cover $\{U_\alpha \mid \alpha \in \nabla\}$ of X, there exists a finite subset ∇_0 of ∇ such that $X = \bigcup\{\mathrm{Cl}(U_\alpha) \mid \alpha \in \nabla_0\}$.

THEOREM 7.1. *Let $F : X \to Y$ be an u.a.w.c. surjective multifunction such that $F(x)$ is compact for each $x \in X$. If X is strongly compact, then Y is quasi H-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be an open cover of Y. For each $x \in X$, $F(x)$ is compact and there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subset \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Set $V(x) = \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Since F is u.a.w.c., by

Theorem 3.1 there exists $U(x) \in \mathrm{PO}(X, x)$ such that $F(U(x)) \subset \mathrm{Cl}(V(x))$. The family $\{U(x)| \; x \in X\}$ is preopen cover of X and there exists a finite number of points, says $x-1$, x_2, ..., x_n in X such that $X = \bigcup\{U(x_i)| \; i = 1, 2, \ldots, n\}$. Therefore, we have

$$Y = F(X) = F\Big(\bigcup_{i=1}^{n} U(x_i)\Big) = \bigcup_{i=1}^{n} F(U(x_i)) \subset \bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i))$$
$$= \bigcup_{i=1}^{n} \bigcup_{\alpha \in \nabla(x_i)} \mathrm{Cl}(V_\alpha).$$

This shows that Y is quasi H-closed.

A space X is said to be *preconnected* [25] if X can not be expressed by the union of two nonempty disjoint preopen sets.

THEOREM 7.2. *Let $F : X \to Y$ be a l.a.w.c. (or u.a.w.c.) surjective multifunction. If X is preconnected and $F(x)$ is connected for each $x \in X$, then Y is connected.*

Proof. Suppose that Y is not connected. There exist nonempty open sets U and V of Y such that $U \cup V = Y$ and $U \cap V = \emptyset$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

$$F^+(U) \cup F^+(V) = F^+(U \cup V) = X, \tag{1}$$
$$F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset \quad \text{and} \tag{2}$$
$$F^+(U) \neq \emptyset \quad \text{and} \quad F^+(V) \neq \emptyset. \tag{3}$$

Next, we show that $F^+(U)$ and $F^+(V)$ are preopen in X. (i) Let F be l.a.w.c. By Theorem 3.2, $p\,\mathrm{Cl}(F^+(V)) \subset F^+(V)$ and hence $p\,\mathrm{Cl}(F^+(V)) = F^+(V)$. it follows from Lemma 2.1 that $F^+(V)$ is preclosed. Therefore, $F^+(U)$ is preopen in X. Similarly, we obtain $F^+(V) \in \mathrm{PO}(X)$. (ii) Let F be u.a.w.c. By Theorem 3.1, $F^+(V) \subset p\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) = p\,\mathrm{Int}(F^+(V))$ and hence $F^+(V) = p\,\mathrm{Int}(F^+(V))$. Therefore, $F^+(V)$ is preopen in X. Similarly, we obtain $F^+(U) \in \mathrm{PO}(X)$. Consequently, X is not preconnected.

COROLLARY 7.3 (Popa [26]). *If a multifunction $F : X \to Y$ is a l.a.c.H. (or u.a.c.H.) and punctually connected surjection and if X is preconnected, then Y is connected.*

THEOREM 7.4. *If $F : X \to Y$ is an u.a.w.c. multifunction into a Hausdorff space Y and $F(x)$ is compact for each $x \in X$, then the graph $G(F)$ is preclosed in $X \times Y$.*

Proof. Let $(x,y) \in X \times Y - G(F)$. Then $y \in Y - F(x)$. For each $a \in F(x)$, there exist open sets $V(a)$ and $W(a)$ containing a and y, respectively, such that $V(a) \cap W(a) = \emptyset$; hence $\mathrm{Cl}(V(a)) \cap W(A) = \emptyset$. The family $\{V(a) |\ a \in F(x)\}$ is an open cover of $F(x)$ and there exists a finite number of points in $F(x)$, says, $a_1, a_2, \dots, a_n$ such that $F(x) \subset \bigcup\{V(a_i) |\ i = 1,2,\dots,n\}$. Set $V = \bigcup\{V(a_i) |\ i = 1,2,\dots,n\}$ and $W = \bigcap\{W(a_i) |\ i = 1,2,\dots,n\}$. Since $F(x) \subset V$ and F is u.a.w.c., there exists $U \in \mathrm{PO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Therefore, we obtain $F(U) \cap W = \emptyset$ and hence $(U \times W) \cap G(F) = \emptyset$. Since $(x,y) \in U \times W \in \mathrm{PO}(X \times Y)$, $(x,y) \notin p\,\mathrm{Cl}(G(F))$ and by Lemma 2.1 $G(F))$ is preclosed.

COROLLARY 7.5 (Popa [26]). *If $F : X \to Y$ is a multifunction such that*

(a) *F is punctually compact,*
(b) *F is u.a.c.H. and*
(c) *Y is Hausdorff,*

then $G(F)$ is preclosed in $X \times Y$.

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