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ON CONJUGACY CLASSES IN LINEAR GROUPS

Let G be a group, C a conjugacy class of G and Z the center of G. In this paper we will show that if $C \not\subset Z$, then $|C| > |Z|$ for $\mathrm{SL}(n,K)$. Using this fact we will show that: if the matrices $V = \mathrm{diag}(v_1, \dots, v_n)$, $W = \mathrm{diag}(w_1, \dots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$ belong to $G = \mathrm{SL}(n,K)$ or $G = \mathrm{PSL}(n,K)$, then $G = C \cdot \mathrm{Cl}(V) \cdot \mathrm{Cl}(W)$, where C is any non-central conjugacy class in G. Observe that in [2] there was calculated, on a big computer, that $C_1C_2C_3C_4 = \mathrm{PSL}(3,q)$ and $C_1^3 = \mathrm{PSL}(3,q)$ for $q = 2,3,4,5$, where C_i $(i = 1,2,3,4)$—arbitrary conjugacy class different from $\{1\}$.

First of all we will estimate the number of elements of the centralizer $C(A)$ of the matrix

$$A = K_1 \dot{+} K_2 \dot{+} \cdots \dot{+} K_s, \quad \text{where} \tag{1}$$

$$K_i = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ & & & 0 & \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & 0 & & & 1 \\ a_{t_i} & a_{t_i-1} & & \dots & a_1 \end{bmatrix}, \qquad t_1 + t_2 + \ldots + t_s = n$$

in the groups $\mathrm{GL}(n,q)$, $\mathrm{SL}(n,q)$.

Note that each matrix $B \in \mathrm{GL}(n,K)$ is similar to a matrix of the form (1).

LEMMA 1. *Let $C(A)$ is the centralizer of the matrix A in $\mathrm{GL}(n,q)$ or $\mathrm{SL}(n,q)$. Then $|C(A)| \le q^{ns}$.*

Proof. Let $AX = XA$, and let the matrix X be of the form (1). The equation $AX = XA$ is equivalent to the system of equation

$$K_iX_{ij} = X_{ij}K_j, \quad i,j = 1,\dots,s. \tag{2}$$

Note that for each blocks X_{ij} we have only one equation (2) from which we can find entries of X_{ij}. Let the orders of K_i and K_j be k and m, respectivly.

Then the matrix X_{ij} is of the rank $k \times m$. If we denote the entries of X_{ij} by y_{pq}, $(p = 1, \ldots, k;\ q = 1, \ldots, m)$, then from (2) we have:

$$\begin{bmatrix} 0 & 1 & 0 & & 0 \\ & & & 0 & \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & 0 & & & 1 \\ a_k & a_{k-1} & & \ldots & a_1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ y_{k1} & y_{k2} & \cdots & y_{km} \end{bmatrix}$$

$$= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ y_{k1} & y_{k2} & \cdots & y_{km} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & & 0 \\ & & & 0 & \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & 0 & & & 1 \\ b_m & b_{m-1} & & \ldots & b_1 \end{bmatrix}$$

$$\begin{bmatrix} y_{21} & y_{22} & \cdots & y_{2m} \\ y_{31} & y_{32} & \cdots & y_{3m} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ y_{k1} & y_{k2} & \cdots & y_{km} \\ c_1 & c_2 & \ldots & c_m \end{bmatrix} = \begin{bmatrix} y_{1m}b_m, & y_{11} + y_{1m}b_{m-1}, & \cdots, & y_{1m-1} + b_1 y_{1m} \\ y_{2m}b_m, & y_{21} + y_{2m}b_{m-1}, & \cdots, & y_{2m-1} + b_1 y_{2m} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ y_{km}b_m, & y_{k1} + y_{km}b_{m-1}, & \cdots, & y_{km-1} + b_1 y_{km} \end{bmatrix} \tag{3}$$

where $c_r = a_k y_{1r} + a_{k-1} y_{2r} + \cdots + a_1 y_{kr}$, $r = 1, \ldots, m$.

Comparing entries of rows $1, \ldots, k-1$ in the equation (3) we see that all entries of X_{ij} can be expressed as linear combinations of

$$y_{11}, y_{12}, \ldots, y_{1m}\,. \tag{4}$$

If the homogenous system in the variables y_{ij}, obtained from comparison of the last rows of the equation (3), has zero solution then $X_{ij} = 0$. If we consider the equation (2) for $j = 1, \ldots, s$ in this same way, we see that all the entries of matrices $X_{i1}, X_{i2}, \ldots, X_{is}$ will be expressed by entries of the first rows theose matrices. The entries create one row of the matrix X. If we solve all equations (2), we get s such rows. The case $X = 0$ is imposible because the equations $K_i X_{ii} = X_{ii} K_i$ have at least one nonzero solution, for example E_i $(i = 1, \ldots, s)$. Therefore all entries of X are functions of ns variables. The maximal number of all matrices commuting with A is q^{ns} i.e. $|C(A)| \leq q^{ns}$.

Let $\mathrm{Cl}(A)$ denote the conjugacy class of A in $G = \mathrm{GL}(n, q)$ or $G = \mathrm{SL}(n, q)$, where A is of the form (1).

LEMMA 2. *If $A \notin Z(G)$, then $|\mathrm{Cl}(A)| > |Z(G)|$.*

Proof. From Lemma 1 we have the following inequalities

$$|\mathrm{Cl}(A)| = \frac{|\mathrm{GL}(n,q)|}{|C(A)|} \geq \frac{|\mathrm{GL}(n,q)|}{q^{ns}}, \quad 1 \leq s \leq n\,, \tag{5}$$

$$|\mathrm{Cl}(A)| = \frac{|\mathrm{SL}(n,q)|}{|C(A)|} \geq \frac{|\mathrm{SL}(n,q)|}{q^{ns}}, \quad 1 \leq s \leq n. \tag{6}$$

Now it is sufficient to show that

$$\frac{|\mathrm{GL}(n,q)|}{q^{ns}} > |Z(\mathrm{GL}(n,q))| \quad \text{and} \tag{7}$$

$$\frac{|\mathrm{SL}(n,q)|}{q^{ns}} > |Z(\mathrm{SL}(n,q))| \quad \text{or} \tag{8}$$

$$\frac{|\mathrm{SL}(n,q)|}{q^{ns}} > |Z(\mathrm{GL}(n,q))|. \tag{9}$$

The inequality (9) is stronger than (7) and (8). Thus it suffices to prove the inequality (9) only for $s = n-1$, because the case $s < n-1$ follows from the $s = n-1$ case. After easy transfomations the inequality (9) will be rewritten to the form

$$(q^n - 1)(q^{n-1} - 1)\cdots(q^3 - 1)(q^2 - 1) > q^{\frac{n(n-1)}{2}}(q - 1). \tag{10}$$

The inequality (10) can be proved easily by induction on n. The case $s = n$ occurs when A is a diagonal matrix. If there are at least two different elements on the diagonal then A is similar to the matrix considered earlier i.e. when $s < n$ (see [3], pp 252–253). If the matrix is a scalar matrix then $A \in Z$.

LEMMA 3. *If K is an infinite field, $G = \mathrm{SL}(n,K)$, and C a non-central conjugacy class in G, then $|C| > |Z|$.*

Proof. Since Z is finite, we need only prove that C is infinite. Suppose C is finite and let A be an element of C; then $C(A)$, the centralizer of A in G, has a finite index in G. It follows that the core T of this centralizer is a normal subgroup of finite index in G (see [4], Theorem 3.3.5, p. 53). Since all normal subgroups of G are central, we have T in Z and the contradiction $|G/Z|$ is finite is obtained.

LEMMA 4. *Let N be a subset of G, Z-center of G, $M = G - Z$ and $|N| > |Z|$. Then $G \subseteq NM$.*

Proof. Note that the conditions

(i) $\forall_{x \in G}\, xM \cap N \neq 0$, (ii) $\forall_{y \in G}\, yN \cap M \neq 0$

are equivalent.

Consider the condition (ii). If $y_0 N \cap M = 0$ for $y_0 \in G$, then $y_0 N \subseteq Z$ and $|N| \leq |Z|$, contrary to $|N| > |Z|$. Therefore $xM \cap N \neq 0$ for each $x \in G$ and $x = n_j m_i^{-1}$ for certain $n_j \in N$, $m_i \in M$ but $M^{-1} = M$, so $x \in NM$ i.e. $G \subseteq NM$.

THEOREM 1. *If* $V, W \in \mathrm{SL}(n,K)$, $V = \mathrm{diag}(v_1, \ldots, v_n)$, $W = \mathrm{diag}(w_1, \ldots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ *for* $i \neq j$ *and for* $N \subseteq \mathrm{SL}(n,K)$, *with* $|N| > |Z|$, *then* $\mathrm{SL}(n,K) \subseteq N \cdot \mathrm{Cl}(V) \cdot \mathrm{Cl}(W)$.

Proof. From ([1], Theorem 2) we have $M = \mathrm{SL}(n,K) - Z(\mathrm{SL}(n,K)) \subseteq \mathrm{Cl}(V) \cdot \mathrm{Cl}(W)$. Since $M^{-1} = M$ it follows from Lemma 4 that $\mathrm{SL}(n,K) \subseteq N \cdot \mathrm{Cl}(V) \cdot \mathrm{Cl}(W)$.

From Lemmas 2, 3 we have $|C| > |Z(\mathrm{SL}(n,K))|$ for $C \not\subset Z(\mathrm{SL}(n,K))$.

Hence we have

COROLLARY 1.1. *If* $V, W \in \mathrm{SL}(n,K)$, $V = \mathrm{diag}(v_1, \ldots, v_n)$, $W = \mathrm{diag}(w_1, \ldots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ *for* $i \neq j$ *and* $C \not\subset Z$ *conjugacy class, then* $\mathrm{SL}(n,K) \subseteq C \cdot \mathrm{Cl}(V) \cdot \mathrm{Cl}(W)$.

Since $|C| > |Z(\mathrm{PSL}(n,K))|$ for $C \neq \{E\}$, then we have the next corollary.

COROLLARY 1.2. *If* $V, W \in \mathrm{PSL}(n,K)$, $V = \mathrm{diag}(v_1, \ldots, v_n)$, $W = \mathrm{diag}(w_1, \ldots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ *for* $i \neq j$ *and* $|C| \neq 1$, *then* $\mathrm{PSL}(n,K) \subseteq C \cdot \mathrm{Cl}(V) \cdot \mathrm{Cl}(W)$.

References

[1] J. Ambrosiewicz, *Products of sets in linear groups*, Demonstratio Math., 22 (1989), 677–693.

[2] A. Dold, B. Eckmann, Z. Arad and M. Herzog, *Products of conjugac classes in groups*, Springer-Verlag Berlin Heidelberg, 1985.

[3] A. Mostowski and M. Stark, *Algebra wyższa*, III, PWN, Warszawa 1966.

[4] W. R. Scott, *Group Theory*, Prentice Hall, INC, Englewood Cliffs, New Jersey.

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