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SOME PROPERTIES OF THE PROJECTIVE SPACE

Introduction

In [6], with the use of the notion of vector structure of the set over the field, the definition and some properties were stated of n-dimensional generalized elementary Klein space over arbitrary field. The aim of this paper is to describe the construction of vector structure of projective space, proving that projective space is an example of generalized elementary Klein space.

1. Preliminaries

We shall start from the definitions and some facts from the theory of Klein spaces (cf. [1]–[4]), which will be used in further parts of this paper.

Consider an arbitrary nonempty set M, the group G and mapping $f : M \times G \to M$ satisfying, for each $x \in M$ and $g_1, g_2 \in G$, conditions:

$$f(f(x,g_1),g_2) = f(x,g_2 \cdot g_1), \qquad f(x,e) = x\,,$$

where e is a neutral element of G, and $g_2 \cdot g_1$ denotes a group product. The triplet

$$(M,G,f) \tag{1.1}$$

will be called an *abstract object* supported by the group G. The set M will be called a *fibre* of this object. We will say that the object (1.1) is *transitive* iff the group G acts transitively on M, i.e. for each $x_1, x_2 \in M$ there exists $g \in G$ such that $f(x_1,g) = x_2$. For any $g \in G$ the mapping $f_g : M \to M$, $f_g(x) := f(x,g)$ is a bijection of M onto itself, whereas $\widehat{f} : G \to \mathcal{G}(M)$, $\widehat{f}(g) := f_g$ is a homomorphism of G into the group $\mathcal{G}(M)$ of all transformations of the set M. This homomorphism will be called a *representation* of the object (1.1).

The abstract object (1.1) is a *Klein space* iff its representation $\widehat{f}$ is a monomorphism. Any abstract object

$$(X, G, F) \tag{1.2}$$

will be called a *geometric object* of a given Klein space iff it is supported by the same group G as Klein space (1.1).

For an arbitrary fixed element x of the fibre X of object (1.2), the set

$$S_x^F := \{g \in G : \ F(x,g) = x\}$$

forms a subgroup of G (see [2], p. 23), called a *stability subgroup* of this object in the point x. The following lemma states the basic property of stability subgroups (cf. [2], p. 26).

LEMMA 1.1. *If $F(x_0, g) = x$, then $S_x^F = g \cdot S_{x_0}^F \cdot g^{-1}$.*

It can be also proved (see [3], p. 20), that the following is true:

LEMMA 1.2. *If $F(x_0, g_0) = x$, then the equality $F(x_0, g) = x$ holds iff $g \in g_0 \cdot S_{x_0}^F$.*

According to the definition given in [6], we will call the group of transformations $\mathcal{T}_D(M)$ of the set M the *group of quasi-translations* of this set with quasi-domain D, where $\emptyset \neq D \subset M$, iff it acts straightly transitively on D (i.e. for each $p, q \in D$ there exists unique $\tau \in \mathcal{T}_D(M)$ such that $\tau(p) = q$) and for each $\tau \in \mathcal{T}_D(M)$ $\tau|_{M \setminus D} = \mathrm{id}_{M \setminus D}$.

Let K be an arbitrary field. We will denote zero and unity of this field by 0 and 1. Abelian group of quasi-translations $\mathcal{T}_D(M)$ with outer operation: $K \times \mathcal{T}_D(M) \to \mathcal{T}_D(M)$, satisfying for each $a, b \in K$ and $\tau_1, \tau_2, \tau \in \mathcal{T}_D(M)$ conditions:

$$\begin{cases} a \cdot (\tau_1 \circ \tau_2) = (a \cdot \tau_1) \circ (a \cdot \tau_2) \\ (a + b) \cdot \tau = (a \cdot \tau) \circ (b \cdot \tau) \\ (ab) \cdot \tau = a \cdot (b \cdot \tau) \\ 1 \cdot \tau = \tau \end{cases} \tag{1.3}$$

will be called a *linear space of quasi-translations* of the set M over the field K and denoted by $\mathcal{T}_D(M, K)$.

DEFINITION 1.1. Let $\{\mathcal{T}_D(M, K)\}_{D \in \Lambda}$ will be a system of linear spaces of quasi-translations of M over K, where Λ is a family of quasi-domains of these spaces, and let $\{\mathcal{A}_p(M)\}_{p \in M}$ be a system of groups of transformations of M, for each $p \in M$ and $\alpha \in \mathcal{A}_p(M)$ satisfying condition $\alpha(p) = p$. The pair

$$(\{\mathcal{T}_D(M, K)\}_{D \in \Lambda}, \{\mathcal{A}_p(M)\}_{p \in M}) \tag{1.4}$$

will be called a *vector structure* of the set M over the field K, iff the following axioms are satisfied:

V1. For every distinct $D, D' \in \Lambda$ and each $\tau \in \mathcal{T}_D(M,K)$, $\tau' \in \mathcal{T}_{D'}(M,K)$, $a \in K$, the following conditions hold: $\tau(D') \in \Lambda$ and

$$\tau \circ \mathcal{T}_{D'}(M,K) \circ \tau^{-1} = \mathcal{T}_{\tau(D')}(M,K),$$
$$\tau \circ (a \cdot \tau') \circ \tau^{-1} = a \cdot (\tau \circ \tau' \circ \tau^{-1}).$$

V2. For each p, $q \in M$ there exists a quasi-domain $D \in \Lambda$ such that $p, q \in D$.

V3. For each $p \in M$, $D \in \Lambda$ and $\tau \in \mathcal{T}_D(M,K)$

$$\tau \circ \mathcal{A}_p(M) \circ \tau^{-1} = \mathcal{A}_{\tau(p)}(M).$$

V4. There exists a $p \in M$ such that for every quasi-domains D', D'' belongs to the family

$$\Lambda_p := \{D \in \Lambda : \; p \in D\}$$

there exists a unique transformation $\alpha \in \mathcal{A}_p(M)$ satisfying:

(a) $\alpha(D') = D''$ and $\alpha \circ \mathcal{T}_{D'}(M,K) \circ \alpha^{-1} = \mathcal{T}_{D''}(M,K)$,

(b) for each $a \in K$ and $\tau' \in \mathcal{T}_{D'}(M,K)$

$$\alpha \circ (a \cdot \tau') \circ \alpha^{-1} = a \cdot (\alpha \circ \tau' \circ \alpha^{-1}).$$

Now, let us consider the Klein space (1.1) and the vector structure (1.4) of the fibre of this space over K.

DEFINITION 1.2. The vector structure (1.4) of the fibre of Klein space (1.1) will be called *compatible* iff the following compatibility conditions are satisfied:

(i) for each $g \in G$, $D \in \Lambda$, $\tau \in \mathcal{T}_D(M,K)$ and $a \in K$

$$f_g \circ \mathcal{T}_D(M,K) \circ f_g^{-1} = \mathcal{T}_{f_g(D)}(M,K),$$
$$f_g \circ (a \cdot \tau) \circ f_g^{-1} = a \cdot (f_g \circ \tau \circ f_g^{-1});$$

(ii) for each $g \in G$ and $p \in M$

$$f_g \circ \mathcal{A}_p(M) \circ f_g^{-1} = \mathcal{A}_{f_g(p)}(M).$$

DEFINITION 1.3. The Klein space (1.1) will be called a *generalized elementary Klein space* over K, iff there exists a vector structure (1.4) of the fibre M over K, compatible with this space.

It is possible, using vector structure, to define a tangent bundle for generalized elementary Klein space (see [6]).

2. Groups of quasi-translations of projective space

Consider the linear space K^{n+1} over K and the multiplicative group $GL(n+1, K)$ of non-singular matrixes of the degree $n+1$ with elements $a_{ij} \in K$. Let $\sim$ be a proportionality relation in $K^{n+1}_* = K^{n+1} \setminus \{0\}$ (i.e.

$\xi \sim \eta$ iff there exists such $a \in K$ that $\eta = a\xi$ for ξ, $\eta \in K_*^{n+1}$). It is easily observed that $\sim$ is an equivalence relation. Let

$$P^n(K) := K_*^{n+1}/\sim \quad \text{and} \quad GP(n,K) := GL(n+1,K)/C(n+1,K),$$

where $C(n+1,K)$ denotes the centre of the group $GL(n+1,K)$. It can be shown (cf. [2], p. 32) that the mapping

$$f : P^n(K) \times GP(n,K) \to P^n(K), \quad f([\xi],\langle A\rangle) := [A\xi]$$

is an effective action of $GP(n,K)$ on the set $P^n(K)$. The abstract object

$$(2.1) \qquad (P^n(K), GP(n,K), f)$$

will be called an *n-dimensional projective Klein space* over K.

Let X be a family of all $(n-1)$-dimensional projective hyperplanes. Let us consider the set

$$(2.2) \qquad \Lambda := \{D : D \subset P^n(K),\ D = P^n(K) \setminus H,\ H \in X\}$$

and construct the object of subsets of the space (2.1) (see [4], p. 16)

$$(2.3) \qquad (\Lambda, GP(n,K), f^*), \quad \text{where} \quad f^*(D,g) := \{f(p,g) : p \in D\}.$$

Let $D_0 = P^n(K)\setminus H_0$, where H_0 is an $(n-1)$-dimensional hyperplane defined by equation $\xi^1 = 0$. Consider also two partial subobjects (see [4], p. 16) of Klein space (2.1)

$$(2.4) \qquad (D_0, S_{D_0}^{f^*}, f_{D_0}), \quad f_{D_0} = f|_{D_0 \times S_{D_0}^{f^*}},$$

and

$$(2.5) \qquad (D_0, S_D^{f^*}, f_D), \quad f_D = f|_{D \times S_D^{f^*}}, \quad D \in \Lambda,$$

where $S_{D_0}^{f^*}$ and $S_D^{f^*}$ are stability subgroups of object (2.3) in D_0 and D, respectively. Since (2.3) is a transitive object, there exists $\overline{g} \in GP(n,K)$ such that $f^*(D_0,\overline{g}) = D$. By Lemma 1.1

$$S_D^{f^*} = \overline{g} \cdot S_{D_0}^{f^*} \cdot \overline{g}^{-1}.$$

It follows that the mapping

$$\varphi_{\overline{g}} : S_{D_0}^{f^*} \to S_D^{f^*}, \quad \varphi_{\overline{g}}(g) = \overline{g} \cdot g \cdot \overline{g}^{-1}$$

is a group homomorphism, whereas the mapping

$$\psi_{\overline{g}} : D_0 \to D, \quad \psi_{\overline{g}} = f_{\overline{g}}|_{D_0}$$

is a bijection. It is easily seen that these mapping satisfy the condtition

$$f_D(\psi_{\overline{g}}(p), \varphi_{\overline{g}}(g)) = \psi_{\overline{g}}(f_{D_0}(p,g)) \quad \text{for } p \in D_0 \quad \text{and } g \in S_{D_0}^{f^*},$$

which means that objects (2.4) and (2.5) are abstractively equivalent (cf. [4], p. 12). It is easily noted that the set

$$G_{D_0} = \left\{ g \in GP(n,K) : \ g = \langle A \rangle, \ A = \left(\begin{array}{c|c} 1 & 0 \\ \hline (a_{i1}) & (\delta_{ij}) \end{array} \right) \right\}$$

(where δ_{ij}, $i,j = 2,3,\ldots,n+1$ is Kronecker's symbol) is a normal subgroup of the group

$$S_{D_0}^{f^*} = \left\{ g \in GP(n,K) : \ g = \langle B \rangle, \ B = \left(\begin{array}{c|c} 1 & 0 \\ \hline (b_{i1}) & (b_{ij}) \end{array} \right), \ (b_{ij}) \in GL(n,K) \right\}$$

acting straightly transitively on D_0. It follows (cf. [3], Lemma I.4.10, p. 22) that $\varphi_{\overline{g}}(G_{D_0})$ is a straightly transitively acting on D normal subgroup of the group $S_D^{f^*}$. Moreover, as is easily seen, the following equality holds:

$$\varphi_g(G_{D_0}) = \varphi_{\overline{g}}(G_{D_0}) \quad \text{for } g \in \overline{g} \cdot S_{D_0}^{f^*} .$$

In virtue of Lemma 1.2 we can observe that the definition of the group

$$G_D := \overline{g} \cdot G_{D_0} \cdot \overline{g}^{-1} \tag{2.6}$$

does not depend on the choice of $\overline{g}$, satisfying condition $f^*(D_0, \overline{g}) = D$.

It can be checked by a direct calculation that $f_{g_0}|_{H_0} = \mathrm{id}_{H_0}$ for each $g_0 = G_{D_0}$. It follows that

$$f_g|_H = \mathrm{id}_H \quad \text{for each} \quad g \in G_D, \quad \text{where} \quad D = P^n(K) \setminus H . \tag{2.7}$$

Indeed, for each $q \in H, g \in G_D$ there exists $p \in H_0, \overline{g} \in GP(n,K), g_0 \in G_{D_0}$ such that $q = f(p, \overline{g})$ and $g = \overline{g} \cdot g_0 \cdot \overline{g}^{-1}$. Thus

$$f(q,g) = f(f(p,\overline{g}), \overline{g} \cdot g_0 \cdot \overline{g}^{-1}) = f(p, \overline{g} \cdot g_0) = f(f(p,g_0), \overline{g}) = f(p,\overline{g}) = q ,$$

what proves (2.7).

We have shown that for each $D \in \Lambda$ the group G_D, defined by (2.6), is a normal subgroup of $S_D^{f^*}$, acting straightly transitively on D and satysfying condition (2.7). Thus the following lemma is true.

LEMMA 2.1. *For each $D \in \Lambda$ the group of transformations*

$$\mathcal{T}_D(P^n(K)) := \widehat{f}(G_D) \tag{2.8}$$

of the fibre of Klein space (2.1) *is a group of quasi-translations of the set $P^n(K)$ with quasi-domain D.*

We will prove two more lemmas.

LEMMA 2.2. *For every $g \in GP(n,K)$ and $D \in \Lambda$ the equality*

$$f_g \circ \mathcal{T}_D(P^n(K)) \circ f_g^{-1} = \mathcal{T}_{f_g(D)}(P^n(K)) . \tag{2.9}$$

holds.

Proof. Let $\overline{D} = f_g(D)$. Since (2.3) is a transitive object, there exist $\overline{g} \in GP(n,K)$ such that $f^*(D_0,\overline{g}) = D$. Hence $f^*(D_0, g\cdot\overline{g}) = \overline{D}$. By the definition (2.6) of group G_D we have

$$G_D = \overline{g}\cdot G_{D_0}\cdot \overline{g}^{-1} \quad \text{and } G_{\overline{D}} = (g\cdot\overline{g})\cdot G_{D_0}\cdot(g\cdot\overline{g})^{-1},$$

and, therefore, $G_{\overline{D}} = g\cdot G_D\cdot g^{-1}$. Thus we get

$$\begin{aligned}\mathcal{T}_{f_g(D)}(P^n(K)) &= \mathcal{T}_{\overline{D}}(P^n(K)) = \widehat{f}(G_{\overline{D}}) = \widehat{f}(g\cdot G_D\cdot g^{-1})\\ &= \widehat{f}(g)\circ\widehat{f}(G_D)\circ\widehat{f}(g^{-1}) = f_g\circ\mathcal{T}_D(P^n(K))\circ f_g^{-1},\end{aligned}$$

what ends the proof. ∎

By the definition of the group of quasi-translations (2.8) and Lemma 2.2, we immediately get:

LEMMA 2.3. *For every $D, D' \in \Lambda$ and $\tau \in \mathcal{T}_D(P^n(K))$ the following is true*

$$\tau(D') \in \Lambda \text{ and } \tau\circ\mathcal{T}_{D'}(P^n(K))\circ\tau^{-1} = \mathcal{T}_{\tau(D')}(P^n(K)).$$

3. Linear spaces of quasi-translations of projective space

Let us consider the Klein space (2.1), its geometric object (2.3) and the product object (see [4], p. 43)

$$(\Lambda\times P^n(K),\ GP(n,K),\ f^*\times f),$$

where the mapping $f^*\times f$ is defined by the formula

$$(f^*\times f)((D,p),g) = (f^*(D,g), f(p,g)).$$

Since the set $\Delta := \{(D,p)\colon D\in\Lambda, p\in D\}$ is a transitive fibre (cf. [4], p. 15) of this object, we can construct a transitive partial object (cf. [4], p. 15)

$$(3.1)\qquad (\Delta, GP(n,K), \widetilde{f}),\qquad \widetilde{f} = f^*\times f|_{\Delta\times GP(n,K)}.$$

Consider also the set D_0 defined in §2, the point

$$(3.2)\qquad p_0 = [(\xi^i)],\quad \text{where } \xi^1 = 1,\ \xi^i = 0\quad \text{for } i = 2,\ldots,n+1$$

and the stability subgroup

$$(3.3)\quad S^{\widetilde{f}}_{(D_0,p_0)} = \left\{ g\in GP(n,K):\ g = \langle A\rangle,\ A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & (a_{ij})\end{array}\right),\ (a_{ij})\in GL(n,K)\right\}$$

of object (3.1) in the point (D_0,p_0). Since (3.1) is a transitive object, there exists for each $(D,p)\in\Delta$ such $\overline{g}\in GP(n,K)$ that

$$(3.4)\qquad \widetilde{f}((D_0,p_0),\overline{g}) = (D,p).$$

By Lemma 1.1 we get

$$S^{\tilde{f}}_{(D,p)} = \overline{g} \cdot S^{\tilde{f}}_{(D_0,p_0)} \cdot \overline{g}^{-1} . \tag{3.5}$$

Thus the mapping

$$\chi_{\overline{g}} : S^{\tilde{f}}_{(D_0,p_0)} \to S^{\tilde{f}}_{(D,p)}, \ \chi_{\overline{g}}(g) = \overline{g} \cdot g \cdot \overline{g}^{-1}$$

is a group isomorphism. Let K_* denotes the multiplicative group of the field K, while $C_{(D_0,p_0)}$–the centre of group (3.3). Then

$$C_{(D_0,p_0)} = \left\{ g \in GP(n,K) : \ g = \langle B \rangle, \ B = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & (a\delta_{ij}) \end{array} \right), \ a \in K_* \right\} .$$

If $C_{(D,p)}$ denotes the centre of the group (3.5), then, obviously

$$C_{(D,p)} = \chi_{\overline{g}}(C_{(D_0,p_0)}) = \overline{g} \cdot C_{(D_0,p_0)} \cdot \overline{g}^{-1} .$$

Consider the mapping

$$d : K_* \to C_{(D_0,p_0)}, \ d(a) = \langle B \rangle, \quad \text{where } B = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & (a\delta_{ij}) \end{array} \right)$$

and note that

$$d(a) \cdot g_0 \cdot (d(a))^{-1} \in G_{D_0} \quad \text{for each } a \in K_* \quad \text{and } g_0 \in G_{D_0} . \tag{3.6}$$

By Lemma 1.2, for each $g \in GP(n,K)$ such that $\tilde{f}((D_0,p_0),g) = (D,p)$ we have $g \in \overline{g} \cdot S^{\tilde{f}}_{(D_0,p_0)}$. Hence, by (3.3), we obtain

$$g \cdot d(a) \cdot g^{-1} = \overline{g} \cdot d(a) \cdot \overline{g}^{-1} \quad \text{for each } a \in K_* .$$

It follows that the definition of mapping

$$d_{D,p} : \ K_* \to C_{(D,p)}, \ d_{D,p} = \chi_{\overline{g}} \circ d \tag{3.7}$$

does not depend on the choice of $\overline{g}$ satistying condition (3.4). Moreover, we will show that for every $D \in \Lambda$, $p_1, p_2 \in D$, $t \in G_D$, $a \in K_*$ the equality

$$d_{D,p_1}(a) \cdot t \cdot (d_{D,p_1}(a))^{-1} = d_{D,p}(a) \cdot t \cdot (d_{D,p}(a))^{-1} \tag{3.8}$$

holds.

Since G_{D_0} acts on D_0 straightly transitively, there exists $g_0 \in G_{D_0}$, such that $f(p_0, g_0) = \overline{p}_1$, where $\overline{p}_1 = f(p_1, \overline{g}^{-1}) \in D_0$. Thus $\tilde{f}((D_0,p_0), \overline{g} \cdot g_0) = (D, p_1)$, and therefore $d_{D,p_1} = \chi_{\overline{g} \cdot g_0} \circ d$. Hence, denoting $\overline{t} := \overline{g}^{-1} \cdot t \cdot \overline{g} \in G_{D_0}$ and using (3.6) we obtain

$$\begin{aligned} d_{D,p_1}(a) \cdot t \cdot (d_{D,p_1}(a))^{-1} &= \overline{g} \cdot g_0 \cdot d(a) \cdot \overline{g}^{-1} \cdot \overline{t} \cdot g_0 \cdot (d(a))^{-1} \cdot g_0^{-1} \cdot \overline{g}^{-1} \\ &= \overline{g} \cdot d(a) \cdot \overline{t} \cdot (d(a))^{-1} \cdot \overline{g}^{-1} = d_{D,p}(a) \cdot t \cdot (d_{D,p}(a))^{-1}, \end{aligned}$$

which proves (3.8).

Since G_D and $C_{(D,p)}$ are subgroups of the group $S_D^{f^*}$, while G_D is normal, we get

$$d_{D,p}(a) \cdot t \cdot (d_{D,p}(a))^{-1} \in G_D \quad \text{for every } a \in K_*,\ t \in G_D\,.$$

Thus, by (2.8), we obtain

$$\widehat{f}(d_{D,p}(a)) \circ \tau \circ \widehat{f}((d_{D,p}(a))^{-1}) \quad \text{for each } a \in K_*,\ \tau \in \mathcal{T}_D(P^n(K))\,.$$

Hence, in virtue of (3.8), the value of mapping

$$: K \times \mathcal{T}_D(P^n(K)) \to \mathcal{T}_D(P^n(K))$$

defined by the formula

$$(3.9) \qquad a \cdot \tau = \begin{cases} \mathrm{id}_{P^n(K)} & \text{if } a = 0 \\ \widehat{f}(d_{D,p}(a)) \circ \tau \circ \widehat{f}((d_{D,p}(a))^{-1}) & \text{if } a \neq 0, \end{cases}$$

does not depend on the choice of the point $p \in D$.

LEMMA 3.1. *For every $g \in GP(n,K)$, $a \in K$, $D \in \Lambda$ and $\tau \in \mathcal{T}_D(P^n(K))$ the equality*

$$(3.10) \qquad f_g \circ (a \cdot \tau) \circ f_g^{-1} = a \cdot (f_g \circ \tau \circ f_g^{-1})\,.$$

holds true. In particular, for each $\widetilde{\tau} \in \mathcal{T}_{\widetilde{D}}(P^n(K))$

$$(3.11) \qquad \widetilde{\tau} \circ (a \cdot \tau) \circ \widetilde{\tau}^{-1} = a \cdot (\widetilde{\tau} \circ \tau \circ \widetilde{\tau}^{-1})\,.$$

Proof. For $a = 0$ the equality (3.10) is obvious. Let $a \neq 0$ and let p be an arbitrary point of the set $D \in \Lambda$. For arbitrarily fixed $g \in GP(n,K)$, let $(\overline{D}, \overline{p}) := \widetilde{f}((D,p),g)$. Since (3.1) is a transitive object, there exists $\overline{g} \in GP(n,K)$ such that

$$\widetilde{f}((D_0,p_0),\overline{g}) = (D,p) \quad \text{and} \quad \widetilde{f}((D_0,p_0), g \cdot \overline{g}) = (\overline{D}, \overline{p})\,.$$

Hence, in virtue of (3.7), we have $d_{D,p}(a) = \overline{g} \cdot d(a) \cdot \overline{g}^{-1}$ and

$$d_{\overline{D},\overline{p}}(a) = (g \cdot \overline{g}) \circ d(a) \circ (g \cdot \overline{g})^{-1} = g \cdot d_{D,p}(a) \cdot g^{-1}\,.$$

Moreover, by (2.9), we obtain $\overline{\tau} = f_g \circ \tau \circ f_g^{-1} \in \mathcal{T}_{\overline{D}}(P^n(K))$. Hence

$$\begin{aligned} a \cdot (f_g \circ \tau \circ f_g^{-1}) = a \cdot \tau &= \widehat{f}(d_{\overline{D},\overline{p}}(a)) \circ \overline{\tau} \circ \widehat{f}((d_{\overline{D},\overline{p}}(a))^{-1}) \\ &= f_g \circ \widehat{f}(d_{D,p}(a)) \circ \tau \circ \widehat{f}((d_{D,p}(a))^{-1}) \circ f_g^{-1} \\ &= f_g \circ (a \cdot \tau) \circ f_g^{-1}\,, \end{aligned}$$

which proves (3.10). Since for every $\widetilde{D} \in \Lambda$ the group of quasi-translations $\mathcal{T}_{\widetilde{D}}(P^n(K))$ is a subgroup of the group $\widehat{f}(GP(n,K))$, then, by (3.10), (3.11) is also true. ■

LEMMA 3.2. *For each $D \in \Lambda$ the group of quasi-translations $\mathcal{T}_D(P^n(K))$ with multiplication by elements of the field K, defined by the formula* (3.9), *forms a linear space of quasi-translations $\mathcal{T}_D(P^n(K), K)$ of the set $P^n(K)$ over the field K.*

Proof. It can be easily checked by a direct calculation that for any $a, b \in K$ and $\tau_1, \tau_2, \tau \in \mathcal{T}_{D_0}(P^n(K))$, conditions (1.3) are satisfied. Thus, by lemma 3.1, they are also satisfied for any $\overline{\tau}_1, \overline{\tau}_2, \overline{\tau} \in \mathcal{T}_D(P^n(K))$, where $D \in \Lambda$. Indeed, the transitivity of the object (2.3) implies the existence of such $\overline{g} \in GP(n, K)$ that $f^*(D_0, \overline{g}) = D$ and $\overline{\tau}_i = f_g \circ \tau_i \circ f_g^{-1}$ for $i = 1, 2$. Thus, we have

$$\begin{aligned} a \cdot (\overline{\tau}_1 \circ \overline{\tau}_2) &= a \cdot (f_g \circ \tau_1 \circ f_g^{-1} \circ f_g \circ \tau_2 \circ f_g^{-1}) = f_g \circ (a \cdot (\tau_1 \circ \tau_2)) \circ f_g^{-1} \\ &= f_g \circ ((a \cdot \tau_1) \circ (a \cdot \tau_2)) \circ f_g^{-1} \\ &= f_g \circ (a \cdot \tau_1) \circ f_g^{-1} \circ f_g \circ (a \cdot \tau_2) \circ f_g^{-1} \\ &= (a \cdot (f_g \circ \tau_1 \circ f_g^{-1})) \circ (a \cdot (f_g \circ \tau_2 \circ f_g^{-1})) = (a \cdot \overline{\tau}_1) \circ (a \cdot \overline{\tau}_2). \end{aligned}$$

The remaining conditions (1.3) can be proved in a similar way. ■

4. Vector structure of projective space

Let p be an arbitrary point of the fibre of projective Klein space (2.1), and p_0–point of (3.2). The transivity of the space (2.1) implies the existence of $\overline{g} \in GP(n, K)$ satisfying condition $f(p_0, \overline{g}) = p$. By Lemma 1.1 we obtain

$$S_p^f = \overline{g} \cdot S_{p_0}^f \cdot \overline{g}^{-1},$$

and hence the mapping

$$\Phi_{\overline{g}} : S_{p_0}^f \to S_{p'}^f, \; \Phi_{\overline{g}} = \overline{g} \cdot g \cdot \overline{g}^{-1}$$

is a group isomorphism. It is easily observed that the set

$$G_{p_0} = \left\{ g \in GP(n, K) : \; g = \langle A \rangle, \; A = \left(\begin{array}{c|c} 1 & (a_{1j}) \\ \hline 0 & (\delta_{ij}) \end{array} \right) \right\}$$

is a normal subgroup of the group

$$S_{p_0}^f = \left\{ g \in GP(n, K) : g = \langle B \rangle, \; B = \left(\begin{array}{c|c} 1 & (b_{1j}) \\ \hline 0 & (b_{ij}) \end{array} \right), \; (b_{ij}) \in GL(n, K) \right\}.$$

Moreover, as is easy to note that the group G_{p_0} acts straightly transitively on the subset $\Lambda_{p_0} = \{D \in \Lambda \colon p_0 \in D\}$ of the fibre of object (2.3).

Let us denote

$$G_p := \overline{g} \cdot G_{p_0} \cdot \overline{g}^{-1}. \tag{4.1}$$

In § 2 we have shown that the subgroup $\varphi_{\overline{g}}(G_{D_0})$ of the group $S_D^{f^*}$ acts straightly transitively on the set D and, moreover, that the equality (2.6) holds for each $\overline{g}$ satisfying condition $f^*(D_0, \overline{g}) = D$. Similarly, it can be shown that

1° the subgroup $\Phi_{\bar{g}}(G_{p_0})$ of the group S^f_p acts straightly transitively on the set $\Lambda_p = \{D \in \Lambda : p \in D\}$,

2° the definition (4.1) of the group G_p does not depend on the choice of $\bar{g}$, satisfying condition $f(p_0, \bar{g}) = p$.

It is easily seen that the following lemma is true.

LEMMA 4.1. *For each* $p \in P^n(K)$ *the group*

$$\mathcal{A}_p(P^n(K)) := \hat{f}(G_p) \tag{4.2}$$

of transformations of the fibre of space (2.1), *has the following properties*:

a) $\alpha(p) = p$ *for each* $\alpha \in \mathcal{A}_p(P^n(K))$,

b) *for every* $D', D'' \in \Lambda_p$ *there exists a unique* $\alpha \in \mathcal{A}_p(P^n(K))$ *such that* $\alpha(D') = D''$,

c) *for each* $g \in GP(n, K)$

$$f_g \circ \mathcal{A}_p(P^n(K)) \circ f_g^{-1} = \mathcal{A}_{f_g(p)}(P^n(K)),$$

d) *for each* $D \in \Lambda$ *and* $\tau \in \mathcal{T}_D(P^n(K))$

$$\tau \circ \mathcal{A}_p(P^n(K)) \circ \tau^{-1} = \mathcal{A}_{\tau(p)}(P^n(K)),$$

e) *for every* $\alpha \in \mathcal{A}_p(P^n(K))$, $D \in \Lambda_p$, $\tau \in \mathcal{T}_D(P^n(K))$ *and* $a \in K$

$$\alpha \circ \mathcal{T}_D(P^n(K)) \circ \alpha^{-1} = \mathcal{T}_{\alpha(D)}(P^n(K)),$$

and

$$\alpha \circ (a \cdot \tau) \circ \alpha^{-1} = a \cdot (\alpha \circ \tau \circ \alpha^{-1}).$$

Proof of property c) is similar to the proof of Lemma 2.2. Since for each $D \in \Lambda$ and $p \in P^n(K)$ the groups of transformations (2.8) and (4.2) are subgroups of the group $\hat{f}(GP(n, K)$, then, in virtue of property c) and lemmas 2.2. and 3.1, properties d) and e) can be easily obtained.

Using lemmas proved in this paper, we can easily check that the pair

$$(\{\mathcal{T}_D(P^n(K), K)\}_{D \in \Lambda}, \{\mathcal{A}_p(P^n(K))\}_{p \in P^n(K)}) \tag{4.3}$$

satisfies axioms V1–V4 and compatibility conditions (i) and (ii). That means that the pair (4.3) is a vector structure of the set $P^n(K)$ over the field K, compatible with Klein space (2.1). Thus we have proved the following theorem:

THEOREM 4.1. *The projective Klein space* (2.1) *is a generalized elementary Klein space over the field* K.

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