DEMONSTRATIO MATHEMATICA
Vol. XXVI No 2 1993

Jan Kurek

ON THE FIRST ORDER NATURAL OPERATORS TRANSFORMING 1-FORMS ON MANIFOLD TO LINEAR FRAME BUNDLE

In this paper we determine all first order natural operators transforming 1-forms on manifold to linear frame bundle.

We deduce that the fundamental operators here are complete lifts and a vertical lift of a 1-form.

In the paper, we use tensor evaluation theorem developed by I. Kolař.

The author is grateful to Professor I. Kolař for suggesting the problem, valuable remarks and useful discussions.

1. Let M be a smooth n dimensional manifold. A section $\omega : M \to T^*M$ of a cotangent bundle $q_M : T^*M \to M$ define a classical field of 1-forms on the manifold M and define a linear map $\omega : TM \to \mathbb{R}$ with respect to a vector bundle structure $p_M : TM \to M$. If a 1-form ω on M has in a local chart (U, x^i) the local expression $\omega = b_i(x)\,dx^i$, then the linear map $\omega : TM \to \mathbb{R}$ in a local induced chart $(p_M^{-1}(U), x^i, X^i)$ on TM is of the form $\omega = b_i(x)X^i$.

Let $p : LM \to M$ be a linear frame bundle. A linear frame u at the point x on M is an linear isomorphism $u : \mathbb{R}^n \to T_xM$. In a local chart (U, x^i) on M the linear frame u is of the form $u(e_k) = x^i{}_k \frac{\partial}{\partial x^i}|_x$. We identify the linear frame u with $u = (X_1, \ldots, X_n)$, where $X_k = u(e_k)$. A local induced chart on LM is of the form $(p^{-1}(U), x^i, x^i{}_k)$.

We define n maps $W^k : LM \to \mathbb{R}$ for $k = 1, \ldots, n$ by formulas

$$W^k : u = (X_1, \ldots, X_n) \mapsto W^k(u) = \omega(X_k), \tag{1.1}$$

where $\omega : TM \to \mathbb{R}$ is a field of 1-forms on M.

If 1-form ω on M in a local chart (U, x^i) is of the form $\omega = b_i(x)\,dx^i$ then the maps $W^k : LM \to \mathbb{R}$ in a local induced chart $(p^{-1}(U), x^i, x^i{}_k)$ are of the form $W^k = b_i(x)x^i{}_k$.

Consider the tangent map $TW^k : T(LM) \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$. The second component of the tangent map defines a linear map $pr_2 \circ TW^k : T(LM) \to \mathbb{R}$, where $pr_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a projection on the second factor.

DEFINITION 1. A field of 1-form ω^k for $k = 1, \dots n$ on LM defined as the second component of the tangent map TW^k by formula

$$\omega^k = pr_2 \circ TW^k : T(LM) \to \mathbb{R} \tag{1.2}$$

is called a k-complete lift of a field of 1-forms ω on M.

DEFINITION 2. A field of 1-forms ω^0 on LM defined as image of a field of 1-forms ω on M by a dual map of the projection $p : LM \to M$ by formula

$$\omega^0 = p^*\omega = \omega \circ Tp : T(LM) \to \mathbb{R} \tag{1.3}$$

is called a vertical lift of a field of 1-forms ω on M.

If a field of 1-forms ω on M has a local expression $\omega = b_i(x)\,dx^i$ in a local chart (U, x^i), then the k-complete lift ω^k for $k = 1, \dots, n$ and the vertical lift ω^0 on LM in a local induced chart $(p^{-1}(U), x^i, x^i{}_k)$ are of the form

$$\begin{aligned} \omega^k &= b_{ij} x^i{}_k\, dx^j + b_i\, dx^i{}_k, \\ \omega^0 &= b_i\, dx^i. \end{aligned} \tag{1.4}$$

We will use the tensor evaluation theorem developed by I. Kolař in [2]. Consider k copies of a vector space V and a finite number of the tensor products $\bigotimes^p V^*, \dots, \bigotimes^q V^*$ of a dual vector space V^*. We denote by $a(x_{i_1}, \dots, x_{i_p}), \dots, b(x_{j_1}, \dots, x_{j_q})$ the values of $a \in \bigotimes^p V^*$ on $x_{i_1}, \dots, x_{i_p} \in V, \dots, b \in \bigotimes^q V^*$ on $x_{j_1}, \dots, x_{j_q} \in V$.

Let $y_{i_1 \dots i_p} \in \mathbb{R}^{k^p}$ and $z_{j_1 \dots j_q} \in \mathbb{R}^{k^q}$ be a canonical coordinates.

THEOREM 1, [2]. *For every smooth $GL(V)$-invariant function $f : \bigotimes^p V^* \times \dots \times \bigotimes^q V^* \times \underbrace{V \times \dots \times V}_{k\text{-times}} \to \mathbb{R}$ there exists a smooth function $g(y_{i_1 \dots i_p}, \dots, z_{j_1 \dots j_q}) : \mathbb{R}^{k^p} \times \dots \times \mathbb{R}^{k^q} \to \mathbb{R}$ such that*

$$f(a, \dots, b, x_1, \dots, x_k) = g(a(x_{i_1}, \dots, x_{i_p}), \dots, b(x_{j_1}, \dots, x_{j_q})) \tag{1.5}$$

for any $a \in \bigotimes^p V^*, \dots, b \in \bigotimes^q V^*$, $x_1, \dots, x_k \in V$.

We need the following

LEMMA 2. *For every smooth $GL(\mathbb{R}^n)$-invariant function $f : \mathbb{R}^n \times \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}$ there exists a smooth function $g : \mathbb{R}^{n^2+3n+2} \to \mathbb{R}$ such that*

$$(1.6)\qquad f(y^i, x^i{}_k, b_i, b_{ij})_{k=1,\dots,n} = g(y^i b_i, x^i{}_k b_i, y^i y^j b_{ij}, y^i x^j{}_k b_{ij}, x^i{}_k y^j b_{ij}, x^i{}_k x^j{}_l b_{ij})_{\substack{k=1,\dots,n\\ l=1,\dots,n}}$$

P r o o f. This result follows easily from the tensor evaluation theorem [2].

2. In this part we prove the main

THEOREM 3. *All first order natural operators $F : T^*M \to T^*LM$ form $(2n+1)$-parameter family of the form*

$$(2.1)\qquad \begin{aligned} F : b_i\,dx^i \mapsto\ & f(b_l x^l{}_m)_{m=1,\dots,n}[b_i\,dx^i] \\ & + g^k(b_l x^l{}_m)_{m=1,\dots,n}[b_{ij} x^j{}_k\,dx^i + b_i\,dx^i{}_k] \\ & + h^k(b_l x^l{}_m)_{m=1,\dots,n}[b_{ji} x^j{}_k\,dx^i + b_i\,dx^i{}_k], \end{aligned}$$

where f, g^k, h^k for $k = 1,\dots,n$ are arbitrary smooth function of n variables.

P r o o f. Any map $F : T^*M \to T^*LM$ in a local coordinates (U, x^i) on M and in the induced coordinates $(p^{-1}(U), x^i, x^i{}_k)$ on LM is of the form

$$(2.2)\qquad F : b_i\,dx^i \mapsto e_i(x^l, x^l{}_m)\,dx^i + g^k{}_i(x^l, x^l{}_m)\,dx^i{}_k.$$

The first order natural operators $F : T^*M \to T^*LM$ are in bijection with natural transformation $F : J^1T^*M \to T^*LM$ and $L^2{}_n$-equivariant maps of standard fibres $F : (J^1T^*\mathbb{R}^n)_0 \to (T^*L\mathbb{R}^n)_0$.

The group $L^2{}_n$ acts on the standard fibre $S = (J^1T^*\mathbb{R}^n)_0$ in the form

$$(2.3)\qquad \begin{aligned} \bar b_i &= b_j \tilde a^j{}_i \\ \bar b_{ij} &= b_{kl} \tilde a^k{}_i \tilde a^l{}_j + b_k \tilde a^k{}_{ij}. \end{aligned}$$

We denote by $(\tilde a^i{}_j, \tilde a^i{}_{jk})$ the coordinates of the inverse element a^{-1} to an element $a \in L^2{}_n$ with coordinates $(a^i{}_j, a^i{}_{jk})$.

The group $L^2{}_n$ acts on the standard fibre $W = (T^*L\mathbb{R}^n)_0$ by formulas

$$(2.4)\qquad \begin{aligned} \bar x^i{}_k &= a^i{}_j x^j{}_k \\ \bar e_i &= e_j \tilde a^j{}_i + g^k{}_j \tilde a^j{}_{li} a^l{}_m x^m{}_k \\ \bar g^k{}_i &= g^k{}_j \tilde a^j{}_i. \end{aligned}$$

Any map $F : (J^1T^*\mathbb{R}^n)_0 \to (T^*L\mathbb{R}^n)_0$ in coordinates (b_i, b_{ij}) and $(x^i{}_k, e_i, g^k{}_i)$ is of the form

$$(2.5)\qquad \begin{aligned} e_i &= F_i(x^i{}_k, b_i, b_{ij}) \\ g^k{}_i &= G^k{}_i(x^i{}_k, b_i, b_{ij}). \end{aligned}$$

Our aim is to find a general form of an $L^2{}_n$-equivariant smooth maps $F_i : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ and $G^k{}_i : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ for every $k = 1,\dots,n$.

Consider $L^1{}_n$-invariant smooth map $F : \mathbb{R}^n \times \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}$ of the form

$$F(y^i, x^i{}_k, b_i, b_{ij}) = F_l(x^i{}_k, b_i, b_{ij})y^l. \tag{2.6}$$

Considering equivariancy with respect to homotheties $y^l \mapsto cy^l$ of $L^1{}_n$-invariant map $F = F_l y^l$ of the form (1.6), we get

$$\begin{aligned}\psi(cy^i b_i, x^i{}_k b_i, c^2 y^i y^j b_{ij}, cy^i x^j{}_k b_{ij}, cx^i{}_k y^j b_{ij}, x^i{}_k x^j{}_l b_{ij} \\ = c \cdot \psi(y^i b_i, x^i{}_k b_i, y^i y^j b_{ij}, y^i x^j{}_k b_{ij}, x^i{}_k y^j b_{ij}, x^i{}_k x^j{}_l b_{ij}).\end{aligned} \tag{2.7}$$

From this relation the map ψ is linear in $y^i \cdot b_i$, $y^i x^j{}_k b_{ij}$, $x^i{}_k y^j b_{ij}$ and independent of $y^i y^j b_{ij}$, where coefficients are $(2n+1)$ arbitrary smooth functions f, g^k, h^k for $k = 1, \ldots, n$ of $n + n^2$ variables depending on $x^i{}_k b_i$ and $x^i{}_k x^j{}_l b_{ij}$.

Thus, every $L^1{}_n$-invariant map $F_i : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ is of the form

$$\begin{aligned}F_i(x^i{}_k, b_i, b_{ij}) = f(x^i{}_k b_i, x^i{}_k x^j{}_l b_{ij}) b_i \\ + g^l(x^i{}_k b_i, x^i{}_k x^j{}_l b_{ij}) b_{ij} x^j{}_l + h^l(x^i{}_k b_i, x^i{}_k x^j{}_l b_{ij}) b_{ji} x^j{}_l,\end{aligned} \tag{2.8}$$

with arbitrary smooth functions f, g^k, h^k for $k = 1, \ldots, n$ of $n + n^2$ variables.

Now, we consider smooth maps $G^k : \mathbb{R}^n \times \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}$ for $k = 1, \ldots, n$ of the form

$$G^k(y^i, x^i{}_k, b_i, b_{ij}) = G^k{}_l(x^i{}_k, b_i, b_{ij})y^l. \tag{2.9}$$

Moreover, we consider a smooth map $\widetilde{G}^k : \mathbb{R}^n \times \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \mathbb{R}^{n*} \times \mathbb{R}^{n*} \to \mathbb{R}$ such that $\widetilde{G}^k = G^k \circ \bigotimes$, i.e.

$$\widetilde{G}^k(y^i, x^i{}_k, b_i, u_i, v_i) = G^k{}_l(x^i{}_k, b_i, u_i \cdot v_j)y^l. \tag{2.10}$$

By tensor evaluation theorem there exists n smooth functions η^k for $k = 1, \ldots, n$ of $3n + 3$ variables such that

$$\widetilde{G}^k(y^i, x^i{}_k, b_i, u_i, v_j) = \eta^k(y^i b_i, y^i u_i, y^i v_i, x^i{}_k b_i, x^i{}_k u_i, x^i{}_k v_i). \tag{2.11}$$

Considering $Gl(1, \mathbb{R})$ invariancy for functions η^k with respect to $u_i \mapsto cu_i$, $v_i \mapsto \frac{1}{c} v_i$ for $c \in \mathbb{R} \setminus \{0\}$, we obtain the relation

$$\eta^k(\alpha, \beta, \gamma, \delta_k, \varepsilon_k, \tau_k) = \eta^k\left(\alpha, c\beta, \frac{1}{c}\gamma, \delta_k, c\varepsilon_k, \frac{1}{c}\tau_k\right). \tag{2.12}$$

By tensor evaluation theorem for $n = 1$ there exists smooth functions ζ^k for $k = 1, \ldots, n$ depending on $n + 1$ parameters α, δ_k i.e. $\zeta^k : \mathbb{R}^{n^2+2n+1} \to \mathbb{R}$

such that:

$$\eta^k(\alpha,\beta,\gamma,\delta_k,\varepsilon_k,\tau_k) = \zeta^k(\alpha,\delta_k;\beta\cdot\gamma,\beta\cdot\tau_k,\gamma\cdot\varepsilon_k,\varepsilon_k\cdot\tau_l). \tag{2.13}$$

Considering equivariancy with respect to homotheties $y^i \mapsto cy^i$ for every $L^1{}_n$ invariant map $G^k = G^k{}_i y^i$ for $k = 1,\ldots,n$ of the form (2.9), we get relation

$$\begin{aligned}\zeta^k(cy^ib_i, x^i{}_kb_i, c^2y^iy^jb_{ij}, cy^ix^j{}_kb_{ij}, cx^i{}_ky^jb_{ij}, x^i{}_kx^j{}_lb_{ij})\\ = c\zeta^k(y^ib_i, x^i{}_kb_i, y^iy^jb_{ij}, y^ix^j{}_kb_{ij}, x^i{}_ky^jb_{ij}, x^i{}_kx^j{}_lb_{ij}).\end{aligned} \tag{2.14}$$

From this relation the maps $\zeta^k : \mathbb{R}^{n^2+2n+1} \to \mathbb{R}$ for $k = 1,\ldots,n$ are linear in y^ib_i, $y^ix^j{}_kb_{ij}$, $x^i{}_ky^jb_{ij}$ and independent of $y^iy^jb_{ij}$, where coefficients are $n+2$ arbitrary smooth functions p^k, q, r of $n + n^2$ variables depending on $x^i{}_kb_i$ and $x^i{}_kx^j{}_lb_{ij}$.

Thus, every $L^1{}_n$-invariant map $G^k{}_l : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ for $k = 1,\ldots,n$ is of the form

$$\begin{aligned}G^k{}_l = {}& p^k(x^i{}_mb_i, x^i{}_mx^j{}_nb_{ij})b_l + q(x^i{}_mb_i, x^i{}_mx^j{}_nb_{ij})b_{lj}x^j{}_k\\ &+ r(x^i{}_mb_i, x^i{}_mx^j{}_nb_{ij})b_{jl}x^j{}_k\end{aligned} \tag{2.15}$$

with arbitrary smooth functions p^k, q, r for $k = 1,\ldots,n$.

We will consider $L^2{}_n$-equivariancy of the map $F : (J^1T^*\mathbb{R}^n)_0 \to (T^*L\mathbb{R}^n)_0$. If the map F is $L^2{}_n$-equivariant, then for every vector $A = (A^i{}_j, A^i{}_{jk})$ of the Lie algebra $l^2{}_n$ of $L^2{}_n$ the corresponding fundamental vector fields A_s on $S = (J^1T^*\mathbb{R}^n)_0$ and A_W on $W = (T^*L\mathbb{R}^n)_0$ must be F related.

This gives the following system of differential equations for the maps F_i and $G^k{}_i$ with parameters $A^i{}_j$, $A^i{}_{ij}$

$$\begin{aligned}-A^j{}_iG^k{}_j = {}& \frac{\partial p^k}{\partial u_{mn}}(-b_lA^l{}_{hj}x^h{}_mx^j{}_n)b_i - p^kA^j{}_ib_j\\ &+ \frac{\partial q}{\partial u_{mn}}(-b_lA^l{}_{hj}x^h{}_mx^j{}_n)b_{ij}x^j{}_k - qb_{lj}A^l{}_ix^j{}_k - qb_lA^l{}_{ij}x^j{}_k\\ &+ \frac{\partial r}{\partial u_{mn}}(-b_lA^l{}_{hj}x^h{}_mx^j{}_n)b_{ji}x^j{}_k - rb_{jl}A^l{}_ix^j{}_k - rb_lA^l{}_{ji}x^j{}_k,\end{aligned} \tag{2.16}$$

$$\begin{aligned}-A^j{}_iF_j - A^j{}_{li}x^l{}_kG^k{}_j = {}& \frac{\partial f}{\partial u_{mn}}(-b_lA^l{}_{hj}x^h{}_mx^j{}_n)b_i - fA^j{}_ib_j\\ &+ \frac{\partial g^l}{\partial u_{mn}}(-b_kA^k{}_{ij}x^i{}_mx^j{}_n)b_{ij}x^j{}_l - g^lb_{kj}A^k{}_ix^j{}_l - g^lb_kA^k{}_{ij}x^j{}_l\\ &+ \frac{\partial h^l}{\partial u_{mn}}(-b_kA^k{}_{ij}x^i{}_mx^j{}_n)b_{ji}x^j{}_l - h^lb_{jk}A^k{}_ix^j{}_l - h^lb_kA^k{}_{ji}x^j{}_l.\end{aligned} \tag{2.17}$$

Setting $A^i{}_j = 0$ in the first equation (2.16), we get

$$\frac{\partial p^k}{\partial u_{mn}} = 0, \quad \frac{\partial q}{\partial u_{mn}} = 0, \quad \frac{\partial r}{\partial u_{mn}} = 0, \tag{2.18}$$

$$q + r = 0. \tag{2.19}$$

By means of (2.18), we obtain that smooth functions p^k, q, r of $n + n^2$ variables u_m, u_{mn} are independent of the second variables u_{mn}.

Thus, the maps $G^k{}_l : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ for $k = 1, \ldots, n$ are of the form

$$G^k{}_l(x^i{}_j, b_i, b_{ij}) = p^k(x^i{}_m b_i) b_l + q(x^i{}_m b_i) b_{lj} x^j{}_k - q(x^i{}_m b_i) b_{jl} x^j{}_k. \tag{2.20}$$

Now, setting $A^i{}_j = 0$ in the equation (2.17) and using the relation (2.19), we obtain

$$\frac{\partial f}{\partial u_{mn}} = 0, \quad \frac{\partial g^k}{\partial u_{mn}} = 0, \quad \frac{\partial h^k}{\partial u_{mn}} = 0, \tag{2.21}$$

$$p^k = g^k + h^k, \quad q = 0. \tag{2.22}$$

By means of relations (2.21), we get that smooth functions f, g^k, h^k of $n + n^2$ variables u_m, u_{mn} are independent of the variables u_{mn}.

Thus, the maps $F_l : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ and $G^k{}_l : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n\text{-times}} \times \mathbb{R}^{n*} \times \bigotimes^2 \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ for $k = 1, \ldots, n$, are of the form

$$F_l(x^i{}_k, b_i, b_{ij}) = f(x^i{}_k b_i) b_l + g^k(x^i{}_m b_i) b_{lj} x^j{}_k + h^k(x^i{}_m b_i) b_{jl} x^j{}_k \tag{2.23}$$

$$G^k{}_l(x^i{}_m, b_i, b_{ij}) = (g^k(x^i{}_m b_i) + h^k(x^i{}_m b_i)) b_l. \tag{2.24}$$

Finally, using (2.23) and (2.24) in (2.2), we obtain the $(2n+1)$ parameter system of natural operators of the form (2.1). This proves our theorem.

The geometrical interpretation of the $(2n+1)$ parameter system (2.1) of the first order natural operators $F : T^*M \to T^*LM$ is

$$F : \omega \mapsto f(\omega(X_m)) \cdot \omega^0 + g^k(\omega(X_m)) \omega^k + h^k(\omega(X_m)) \omega^k \circ \iota_M \tag{2.25}$$

where ω^0 and ω^k are the vertical lift and k-complete lifts of 1-form ω to LM, respectively, and $\iota_M : TLM \to LTM$ is a canonical diffeomorphism.

References

[1] I. Kolař, *Some natural operators in differential geometry*, Diff. Geom. and its Appl., Proceedings of the Conference, Brno 1986, 91–110, D. Reidel Publishing Company.

[2] I. Kolař, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, to appear.

[3] D. Krupka, J. Janyška, *Lectures on differential invariants*, Masaryk University, Brno, 1990.
[4] J. Kurek, *On the first order natural operators transforming* 1*-forms on manifold to tangent bundle*, Annales UMCS 43 (1989), 79–83.

INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA UNIVERSITY
Plac Marii Curie-Skłodowskiej 1
20-031 LUBLIN, POLAND

Received September 17, 1990.