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A CHARACTERIZATION OF UNIFORMLY DISTRIBUTED RANDOM VARIABLE

Let X be a real random variable defined on some fixed probability space $(\Omega, \mathcal{A}, P)$ and let μ be its probability distribution, i.e. $\mu(B) = P(X^{-1}(B))$ for any Borel subset of reals R. By $S(\mu)$ we denote the support of μ, i.e. the least closed subset of R such that $\mu(S(\mu)) = 1$; see [1], ch. II, §2, Th. 2.1 and Def. 2.1 for the existence and uniqueness of $S(\mu)$.

We say that a random variable X is *uniformly distributed* iff either

(i) $S(\mu) = \{a_1,\dots,a_n\}$ with $a_i = a_1 + (i-1)r$ and $\mu(\{a_i\}) = \frac{1}{n}$ for $i = 1,\dots,n$, $r \neq 0$ or

(ii) $S(\mu) = [a, b]$ for some $-\infty < a < b < +\infty$ and $\mu(A) = \frac{|A|}{(b-a)}$ for any Borel subset A of $S(\mu)$, where $|A|$ stands for Lebesque measure of A.

We shall prove the following characterisation theorem.

Theorem. A real random variable X is uniformly distributed if and only if it satisfies the following condition:

(*) $E[\,X|a \le X \le b] = \frac{1}{2}(a+b)$ for any $a, b, \in S(\mu)$ with $P(a \le X \le b)>0$.

Necessity of the condition (*) is checked by direct calculations in both cases (i) and (ii).

Proof of sufficiency of the condition (*) will be carried in several lemmas. Let us note first the following equivalent formulation of the condition (*) in terms of μ:

(**) $\int_{[a,b]} t\,\mu(dt) = \frac{1}{2}(a+b)\,\mu([a, b])$ for any $a, b \in S(\mu)$ with $\mu([a, b]) > 0$.

In each Lemma we suppose that μ satisfies (**) and by $A(\mu)$

we denote the set of atoms of μ (possibly empty), i.e. $A(\mu) = \{a \in R : \mu(\{a\})>0\}$.

Lemma 1. A point $a \in S(\mu)$ is an isolated point of $S(\mu)$ if and olny if $a \in S(\mu)$.

Sufficiency. Let $a \in A(\mu)$ and suppose to the contrary that a is not an isolated point of $S(\mu)$, i.e. there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset S(\mu)$ such that $\lim_n a_n = a$. We can assume that $a< a_{n+1}< a_n$ for $n = 1, 2,\dots$ (or $a_n< a_{n+1}< a$ for $n = 1, 2,\dots$, which is the case treated analogically) and thus $\mu([a,a_n])> 0$ for $n = 1, 2,\dots$. By applying condition (**) we get

(1) $$\int_{[a,a_1]} t\,\mu(dt) = \frac{1}{2}(a+a_1)\ \mu([a,a_1])$$

and

(2) $$\int_{[a,a_1]} t\,\mu(dt) = \mu(\{a\})a + \int_{(a,a_1]} t\,\mu(dt) = \mu(\{a\})a +$$
$$+ \lim_n \int_{[a_n,a]} t\,\mu(dt) = \mu(\{a\})a + \lim_n \frac{1}{2}(a_n+a_1)\ \mu([a_n,a_1]) =$$
$$= \mu(\{a\})a + \frac{1}{2}(a+a_1)\ \mu((a,a_1]) =$$
$$= \mu(\{a\})a + \frac{1}{2}(a+a_1)\ [\mu([a,a_1]) - \mu(\{a\})] =$$
$$= \frac{1}{2}(a+a_1)\ \mu([a,a_1]) + \frac{1}{2}\ \mu(\{a\})\ (a-a_1).$$

Comparing (1) and (2) we obtain that $a = a_1$, which is the contradiction proving that an atom a of μ is an isolated point of $S(\mu)$.

Necessity. If $a \in S(\mu)$ is an isolated point of $S(\mu)$, then $S(\mu) - \{a\}$ is a proper closed subset of $S(\mu)$ and thus $\mu(S(\mu) - \{a\})< 1$ and in consequence $a \in A(\mu)$.

Lemma 2. If $a,b \in S(\mu)$, $a<b$ and $S(\mu) \cap (a,b) = \phi$, then $\mu(\{a\}) = \mu(\{b\})$.

Proof. If $\mu([a,b]) = 0$, then $\mu(\{a\}) = \mu(\{b\}) = 0$. If $\mu([a,b])> 0$, then by condition (**) and in view of the fact that $\mu((a,b)) = 0$ we obtain

(3) $$\mu(\{a\})a + \mu(\{b\})b = \frac{1}{2}(a+b)\ [\mu(\{a\}) + \mu(\{b\})],$$

which implies that $\mu(\{a\}) = \mu(\{b\})$.

Lemma 3. If $A(\mu) \neq \emptyset$, then μ is purely atomic and X is uniformly distributed.

Proof. Suppose to the contrary that μ is not purely atomic, i.e. there are points $a_0 \in A(\mu)$ and $b_0 \in S(\mu) - A(\mu)$ such that $a_0 < b_0$ (or $a_0 > b_0$, which is the case treated analogically).

Let $b = \inf \{x \in S(\mu) - A(\mu) : x > a_0\}$. Since atoms of μ are isolated points of $S(\mu)$ (Lemma 1) we infer that $b \in S(\mu) - A(\mu)$ and (hence) $a_0 < b$. Thus we have that

$$S(\mu) \cap [a_0,b) \subset A(\mu)$$

and we shall show that the set $S(\mu) \cap [a_0,b)$ is finite. In fact, if $S(\mu) \cap [a_0,b)$ were infinite we could define inductively $a_1 = \inf (S(\mu) \cap (a_0,b))$, $a_{n+1} = \inf (S(\mu) \cap (a_n,b))$ and thus obtain an infinite sequence $\{a_i\}_{i=0}^{\infty}$ of atoms of μ with

$$a_0 < a_i < a_{i+1} < b \qquad \text{for } i = 1,2,\ldots$$

and such that $S(\mu) \cap (a_i,a_{i+1}) = \emptyset$ for $i = 0,1,2,\ldots$. By Lemma 2 $\mu(\{a_0\}) = \mu(\{a_1\}) = \mu(\{a_2\}) = \ldots$, which is a contradition proving that the set $S(\mu) \cap [a_0,b)$ is finite. Puting $a = \sup (S(\mu) \cap [a_0,b))$ we obtain a point $a \in A(\mu)$ such that $a < b$ and $S(\mu) \cap (a,b) = \emptyset$. Again from Lemma 2 we infer that $\mu(\{a\}) = \mu(\{b\})$, which is impossible since $\mu(\{a\}) > 0$ and $\mu(\{b\}) = 0$. This contradition shows that $S(\mu) = A(\mu)$, i.e. μ is purely atomic. As above, the assumption that the set $S(\mu)$ is infinite leads to the contradiction.
Thus $S(\mu) = A(\mu) = \{a_1,\ldots,a_n\}$ with $a_1 < \ldots < a_n$. Again from Lemma 2 we infer that $\mu(\{a_i\}) = \frac{1}{n}$ for $i = 1,\ldots,n$. By applying the condition (**) to the segments $[a_i,a_{i+2}]$ for $i = 1,\ldots,n-2$ we obtain that $a_{i+1} = \frac{1}{2}(a_i+a_{i+2})$ for $i = 1,\ldots,n-2$ and thus (ii) is satisfied, i.e. X is uniformly distributed.

In the subsequent lemmas we assume that μ is non-atomic, i.e. $A(\mu) = \emptyset$.

Lemma 4. If $a,b \in S(\mu)$, $a < b$ and $[a,b] \subset S(\mu)$, then

$$\mu([a,\tfrac{a+b}{2}]) = \mu([\tfrac{a+b}{2},b]) = \tfrac{1}{2}\,\mu([a,b]).$$

Proof. We suppose that $\mu([a,b]) > 0$ (the case $\mu([a,b]) = 0$ is trivial). By applying (**) we get

(4)
$$\int_{[a,b]} t\,\mu(dt) = \tfrac{1}{2}(a+b)\,\mu([a,b]) = $$
$$= \tfrac{1}{2}(a+b)\,[\mu([a,\tfrac{a+b}{2}]) + \mu([\tfrac{a+b}{2},b])]$$

and

(5)
$$\int_{[a,b]} t\,\mu(dt) = \int_{[a,\frac{a+b}{2}]} t\,\mu(dt) + \int_{[\frac{a+b}{2},b]} t\,\mu(dt) =$$
$$= \tfrac{1}{2}(a + \tfrac{a+b}{2})\,\mu([a,\tfrac{a+b}{2}]) + \tfrac{1}{2}(b + \tfrac{a+b}{2})\,\mu([\tfrac{a+b}{2},b]).$$

Comparing (4) and (5) we obtian that $\mu([a,\frac{a+b}{2}]) = \mu([\frac{a+b}{2},b])$.

Lemma 5. If $a,b \in S(\mu)$, $a < b$, then $S(\mu) \cap (a,b) \neq \emptyset$.

Proof. Suppose to the contrary that there are points $a,b \in S(\mu)$, $a < b$ with $S(\mu) \cap (a,b) = \emptyset$ and thus $\mu((a,b)) = 0$. Since $b \in S(\mu)$ is not an isolated point of $S(\mu)$ there is $b_1 \in S(\mu)$ with $b_1 > b$. Since the segment (a,b_1) is neighbourhood of $b \in S(\mu)$, thus $\mu((a,b_1)) > 0$ and hence $\mu([b,b_1)) > 0$. Applying condition (**) we get

(6)
$$\int_{[a,b_1]} t\,\mu(dt) = \tfrac{1}{2}(a+b_1)\,\mu([a,b_1]) = \tfrac{1}{2}(a+b_1)\,\mu([b,b_1])$$

and

(7)
$$\int_{[b,b_1]} t\,\mu(dt) = \tfrac{1}{2}(b+b_1)\,\mu([b,b_1]).$$

Comparing (6) and (7) and in view that $\mu([a,b)) = 0$ we obtain that $a = b$, which is a contradiction completing the proof of Lemma 5.

Lemma 6. Suppose F is a closed non-empty subset of a real line having the property that for any $a,b \in F$, $a < b$ there is a point $c \in F$ such that $a < c < b$. Then F is a segment (possibly unbounded).

Proof. It is clearly sufficient to prove that $[a,b] \subset F$ for any $a,b \in F$ with $a < b$, since then $F = [\inf F, \sup F]$.

Suppose to the contrary that there are $a,b \in F$, $a < b$ such that for some $\gamma \notin F$ $a < \gamma < b$. Let $\alpha = \inf \{x \in F\colon x > \gamma\}$ and

$\beta = \sup\{x \in F: x < \gamma\}$. Then $\alpha,\beta \in F$, $\alpha < \beta$ and for any $\alpha<c<\beta$, $c \notin F$.

Lemma 7. $S(\mu)$ is a closed bounded segment.

Proof. From Lemmas 5 and 6 we infer that $S(\mu)$ is a closed segment. We have to prove only that the segment $S(\mu)$ is bounded.

Suppose to the contrary that $s(\mu) = [a,\infty]$ (the cases: $S(\mu) = (-\infty,b]$ or $S(\mu) = (-\infty,+\infty)$ are treated analogically). From Lemma 4 applied to the segments $[a+n-1, a+n+1]$ and $[a+n, a+n+2]$ for $n = 1,2,\ldots$ we obtain that

(8) $$\mu([a+n-1, a+n]) = \mu([a+n, a+n+1])$$

and

(9) $$\mu([a+n, a+n+1]) = \mu([a+n+1, a+n+2]).$$

Compring (8) and(9) we obtian an infinite sequence of disjoint segments $[a, a+1]$, $[a+2, a+3]$, $[a+4, a+5],\ldots$ having the same strictly positive measure μ. This contradition completes the proof of Lemma 7.

Proof of the Theorem. In view of Lemma 3 we can assume that μ is non-atomic. Then by Lemma 7 $S(\mu) = [a,b]$, where $-\infty < a < b < \infty$. Let us define the sub-segments A_{nk} of $[a,b]$ as follows

$$A_{nk} = [a + \frac{k-1}{2^n}(b-a),\ a + \frac{k}{2^n}(b-a)] \qquad \text{for } n = 0,1,2,\ldots,$$

$k = 1,2,\ldots, 2^n$.

By Lemma 4 combined with an induction argument we obtain that for $n = 0,1,2,\ldots$, $k = 1,2,\ldots,2^n$ $\quad \mu(A_{nk}) = \frac{1}{2^n}$, i.e.

$$\mu(A_{nk}) = \frac{|A_{nk}|}{(b-a)}.$$

Since the family of segments $\{A_{nk}\}$, $n = 0,1,2,\ldots$, $k = 1,2,\ldots,2^n$ generates the σ-algebra of Borel subsets of $[a,b]$, we conclude finally that $\mu(A) = \frac{|A|}{(b-a)}$ for any Borel subset of $S(\mu)$.

REFERENCE

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