DEMONSTRATIO MATHEMATICA
Vol. XXVI No 1 1993

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ON QUASIDIFFERENTIAL KERNELS

The purpose of this paper is to explore the structure of quasidifferential kernels of a class of quasidifferentiable functions in the sense of Demyanov and Rubinov. The corresponding properties and operations concerning kernels are given.

1. Introduction

Let f be a quasidifferentiable function defined on an open set $S \subset R^n$ in the sense of [3] and $x \in S$. We denote by $\mathcal{D}f(x)$ the class of all equivalent quasidifferentials of f at x, by $\underline{\mathcal{D}}f(x)$ the family of all subdifferentials of f at x, by $\overline{\mathcal{D}}f(x)$ the family of all superdifferentials of f at x, i.e.,

$$\mathcal{D}f(x) := \{[\underline{\partial}f(x),\overline{\partial}f(x)] \mid f'(x;d) = p(d)+q(d) = \max_{v\in\underline{\partial}f(x)} \langle v,d\rangle + \min_{w\in\overline{\partial}f(x)} \langle w,d\rangle,\ \forall d\in R\},$$

$$\underline{\mathcal{D}}f(x) := \{\underline{\partial}f(x) \mid \exists \text{ a convex compact set } \overline{\partial}f(x) : [\underline{\partial}f(x),\overline{\partial}f(x)]\in\mathcal{D}f(x)\},$$

$$\overline{\mathcal{D}}f(x) := \{\overline{\partial}f(x) \mid \exists \text{ a convex compact set } \underline{\partial}f(x) : [\underline{\partial}f(x),\overline{\partial}f(x)]\in\mathcal{D}f(x)\},$$

where p(d) is a sublinear operator and q(d) is a sublinear operator. According to the definition of quasidifferentiable functions, if f is quasidifferentiable at x, then its directional derivative at this point in a direction $d\in R^n$ can be represented as

$$f'(x;d) = \max_{v\in\underline{\partial}f(x)} \langle v,d\rangle + \min_{w\in\overline{\partial}f(x)} \langle w,d\rangle,$$

or equivalently,

AMS subject classfication: 90C.
Key words: Quasidifferentiable functions, quasidifferential calculus, convex analysis.

$$(1.1)\qquad f'(x;d) = p_1(d)-p_2(d) = \max_{v\in\underline{\partial}f(x)} \langle v,d\rangle - \max_{w\in-\overline{\partial}f(x)} \langle w,d\rangle,$$

where both of $p_1(d)$ and $p_2(d)$ are sublinear operators, i.e., as the sum form of a pair of sublinear operator and super-linear operator, or as the difference form of two sublinear operators, [4], [8], [13]. This kind of structure of derivatives of quasidifferentiable function brings on that a quasidifferential of a quasidifferentiable function, called bidifferential also in [7], is not unique, but the quasidifferential class of equivalence of a quasidifferentiable function is unique. However, it was pointed out that there is no automatic way to select a representative of a quasidifferential class, [7]. This is one of the reasons that quasidifferentiable functions are not easy to be used. So far one has not found an automatic way, in general case, to select a representative of a quasidifferential class, and every equivalent class of quasidifferentials is too big since their union could cover whole space. In other words, whether there exists some kind of uniqueness used to describe further characters of quasidifferentiable functions, generally speaking, is still open. A recent result shows that in one--dimensional space this problem has been worked out [6]. The author proposed a definition of kernel of quasidifferentials, but the structure and expressions of kernel are too complicated to be used for quasidifferentiable optimization [11]. Some discussions on the continuity of kernel were given in [12], but likewise, they are too complicated to be used in studying convergence of numerical methods. The concepts concerning kernels given in [11] are presented in [12] in which their quasidifferentials have ⊛-equivalent bounded subfamilies. We now go back to some basic definitions and notations in [11].

Let $[\underline{\partial}f(x),\overline{\partial}f(x)]\in\mathcal{D}f(x)$. Since

$$(\underline{\partial}f(x)+\overline{\partial}f(x))-\overline{\partial}f(x) = \underline{\partial}f(x)-(\overline{\partial}f(x)-\overline{\partial}f(x)),$$

it follows from properties of quasidifferentiable functions that

$$[\underline{\partial}f(x)+\overline{\partial}f(x),\overline{\partial}f(x)-\overline{\partial}f(x)]\in\mathcal{D}f(x).$$

Thus the expression (1.1) can be replaced by

$$(1.2) \qquad f'(x;d) = \max_{v\in\underline{\partial}f(x)+\overline{\partial}f(x)} \langle v,d\rangle - \max_{w\in\overline{\partial}f(x)-\overline{\partial}f(x)} \langle w,d\rangle.$$

It is clear that $0\in\overline{\partial}f(x)-\overline{\partial}f(x)$. Hence, for a $\overline{\partial}f(x)\in\overline{\mathcal{D}}f(x)$ the second term on the right-hand side of (1.2),

$$\max_{w\in\overline{\partial}f(x)-\overline{\partial}f(x)} \langle w,d\rangle,$$

is always nonnegative. One has

$$f'(x;d) \le \max_{v\in\underline{\partial}f(x)+\overline{\partial}f(x)} \langle v,d\rangle, \ \forall[\underline{\partial}f(x),\overline{\partial}f(x)]\in\mathcal{D}f(x).$$

Taking the infimum to the inequality above over $\mathcal{D}f(x)$, we have

$$f'(x;d) \le \inf_{\mathcal{D}f(x)} \max_{v\in\underline{\partial}f(x)+\overline{\partial}f(x)} \langle v,d\rangle.$$

Define

$$\underline{f}'(x;d) := \inf_{\mathcal{D}f(x)} \max_{v\in\underline{\partial}f(x)+\overline{\partial}f(x)} \langle v,d\rangle.$$

The function $\underline{f}'(x;d)$ of $d\in R^n$ is called the directional subderivative of f at x. On the other hand, since

$$\max_{w\in\overline{\partial}f(x)-\overline{\partial}f(x)} \langle w,d\rangle = \max_{v\in\underline{\partial}f(x)+\overline{\partial}f(x)} \langle v,d\rangle - f'(x;d) \ge$$
$$\ge \underline{f}'(x;d)-f'(x;d),$$

the set

$$\left\{\max_{w\in\overline{\partial}f(x)-\overline{\partial}f(x)} \langle w,d\rangle \;\middle|\; \overline{\partial}f(x)\in\overline{\mathcal{D}}f(x)\right\}$$

has a finite infimum for every $d\in R^n$. By $\overline{f}'(x;d)$ we denote it, i.e.,

$$\overline{f}'(x;d) := \inf_{\mathcal{D}f(x)} \max_{w\in\overline{\partial}f(x)-\overline{\partial}f(x)} \langle w,d\rangle.$$

It is called the directional superderivative of f at x. Now the directional derivative of f at x in a direction $d\in R^n$ can be rewritten as

$$f'(x;d)= \inf_{\mathcal{D}f(x)} \max_{v\in\underline{\partial}f(x)+\overline{\partial}f(x)} \langle v,d\rangle - \inf_{\mathcal{D}f(x)} \max_{w\in\overline{\partial}f(x)-\overline{\partial}f(x)} \langle w,d\rangle=$$
$$= \underline{f}'(x;d) - \overline{f}'(x;d).$$

For convenience of simplicity, without confusion subderivative and superderivative will be often used instead of directional subderivative and directional superderivative, respectively, later on.

Let $\hat{\mathcal{D}}f(x)$ be a subfamily of $\mathcal{D}f(x)$. $\hat{\mathcal{D}}f(x)$ is said to be a $\circledast$-equivalent bounded subfamily if the following conditions are satisfied:

(C1) there exists a positive number M such that

$$\underline{\partial}f(x)\cup\overline{\partial}f(x)\subset B_M(0),\ \forall[\underline{\partial}f(x),\overline{\partial}f(x)]\in\hat{\mathcal{D}}f(x),$$

where $B_M(0)$ denotes the Euclidean ball in R^n with the radius M centered at origin;

(C2) the subfamily $\{\underline{\partial}f(x)+\overline{\partial}f(x) \mid [\underline{\partial}f(x),\overline{\partial}f(x)]\in\hat{\mathcal{D}}f(x)\}$ and the subfamily $\{\overline{\partial}f(x)-\overline{\partial}f(x) \mid \overline{\partial}f(x)\in\bar{\hat{\mathcal{D}}}f(x)\}$, where $\bar{\hat{\mathcal{D}}}f(x):=$ $:=\{\overline{\partial}f(x) \mid \exists\underline{\partial}f(x):\ [\underline{\partial}f(x),\overline{\partial}f(x)]\in\hat{\mathcal{D}}f(x)\}$, form a subexhaustive family and a superexhaustive family of u.c.a.s of f at x, respectively, i.e.,

$$\underline{f}'(x;\cdot)=\inf_{\mathcal{D}f(x)}\ \max_{u\in\underline{\partial}f(x)+\overline{\partial}f(x)}\langle u,\cdot\rangle=\inf_{\hat{\mathcal{D}}f(x)}\ \max_{u\in\underline{\partial}f(x)+\overline{\partial}f(x)}\langle u,\cdot\rangle$$

and

$$\overline{f}'(x;\cdot)=\inf_{\mathcal{D}f(x)}\ \max_{u\in\overline{\partial}f(x)-\overline{\partial}f(x)}\langle u,\cdot\rangle=\inf_{\hat{\mathcal{D}}f(x)}\ \max_{u\in\overline{\partial}f(x)-\overline{\partial}f(x)}\langle u,\cdot\rangle.$$

For convenience of discussion without loss of generality, assume that the subfamily

$$\mathcal{D}_Mf(x) := \{[\underline{\partial}f(x),\overline{\partial}f(x)]\in\mathcal{D}f(x) \mid \forall u\in\underline{\partial}f(x)\cup\overline{\partial}f(x):\ \|u\|\le M\}$$

is a $\circledast$-equivalent bounded subfamily of $\mathcal{D}f(x)$, i.e., let $\hat{\mathcal{D}}f(x)=$ $=\mathcal{D}_Mf(x)$.

Some notations and definitions will be introduced below in order to define kernels for $\mathcal{D}f(x)$. To begin with, define two sets of sequences for any $(u,x)\in R^n\times R^n$ as follows,

$$(1.3)\qquad \underline{U}(u,x) := \left\{ \{u_i\}_1^\infty \;\middle|\; \begin{array}{l} \exists\{[\underline{\partial}_if(x),\overline{\partial}_if(x)]\}_1^\infty\subset\mathcal{D}_Mf(x),\\ \exists\{d_i\}_1^\infty\subset R^n:\\ d_i\longrightarrow d\in R^n,\ \text{as } i\longrightarrow\infty,\\ u_i\longrightarrow u\in R^n,\ \text{as } i\longrightarrow\infty,\\ u_i\in \operatorname{Arg}\max_{u\in\underline{\partial}_if(x)+\overline{\partial}_if(x)}\langle u,d_i\rangle,\\ \langle u,d\rangle=\lim_{i\to\infty}\langle u_i,d_i\rangle=\underline{f}'(x;d) \end{array} \right\}$$

and

$$(1.4)\qquad \bar{U}(u,x) := \left\{ \{u_i\}_1^\infty \;\middle|\; \begin{array}{l} \exists\{[\underline{\partial}_i f(x),\bar{\partial}_i f(x)]\}_1^\infty \subset \mathcal{D}_M f(x), \\ \exists\{d_i\}_1^\infty \subset R^n : \\ d_i \longrightarrow d \in R^n, \text{ as } i\longrightarrow\infty, \\ u_i \longrightarrow u \in R^n, \text{ as } i\longrightarrow\infty, \\ u_i \in \underset{u\in\bar{\partial}_i f(x)-\bar{\partial}_i f(x)}{\operatorname{Arg\,max}} \langle u,d_i\rangle, \\ \langle u,d\rangle = \lim_{i\to\infty} \langle u_i,d_i\rangle = \bar{f}'(x;d) \end{array} \right\}$$

Define

$$\underline{U}_0'(x) := \{u\in R^n \mid \underline{U}(u,x)\neq\emptyset\}$$

and

$$\bar{U}_0'(x) := \{u\in R^n \mid \bar{U}(u,x)\neq\emptyset\}.$$

Let $\underline{U}_0(x)$ and $\bar{U}_0(x)$ be the smallest equivalent subsets of $\underline{U}_0'(x)$ and $\bar{U}_0'(x)$, respectively, where "equivalent" means that, for instance,

$$\forall b\in \mathrm{bd}B_1(0),\ \exists u\in\underline{U}_0(x) : \langle u,b\rangle=\underline{f}'(x;b),$$

and "smallest" means that, for instance, for any equivalent subset $\underline{U}_0''(x)$ of $\underline{U}_0'(x)$ one has

$$\mathrm{co}\underline{U}_0(x)\subset\mathrm{co}U_0''(x).$$

$\underline{U}(x)$ and $\bar{U}(x)$ are defined by convex hulls, of $\underline{U}_0(x)$ and $\bar{U}_0(x)$, respectively. It is obvious that for any $d\in R^n$ there exist vectors $\underline{u}\in\underline{U}(x)$, $\bar{u}\in\bar{U}(x)$ such that

$$\langle\underline{u},d\rangle = \underline{f}'(x;d) \quad\text{and}\quad \langle\bar{u},d\rangle = \bar{f}'(x;d)$$

because of the boundedness of $\mathcal{D}_M f(x)$. Hence, $\underline{U}(x)$ and $\bar{U}(x)$ are nonempty, bounded and convex. They are called a sub- and super-kernel of f at x, denoted by $\partial_* f(x)$ and $\partial^* f(x)$, respectively. We call $[\partial_* f(x),\partial^* f(x)]$ a quasidifferential kernel, or a quasikernel of f at x, briefly kernel, denoted by $D_k f(x)$.

2. Structure on kernel under some hypotheses

A purpose of this section is to attempt to further explore the structure on kernel given in [11]. Under some hypotheses, new results presented here show that the directional derivative of a quasidifferentiable function at a given point may be expressed as a difference of two support functions of kernels and the difference expression is unique. Moreover, relations of quasidifferentiable functions in the sense of [3] and [9], and of generalized gradients in the sense of [1] and quasidifferentials, of generalized directional derivative and directional derivatives of a quasidifferentiable function, are clarified.

To begin with, some basic propositions and lemmas are given.

Proposition 2.1. Suppose S is a nonempty compact convex subset of R^n. Let $u\in S$. Then $u\in \operatorname{Arg}\max_{u'\in S} \langle u',d\rangle$ if and only if $d\in N(u,S)$, where $N(u,S)$ denotes the cone normal to S at $u\in S$.

Proof. Suppose $u\in \operatorname{Arg}\max_{u'\in S} \langle u',d\rangle$ for some $d\in R^n$. Then one has $\langle u,d\rangle=\delta^*(d|S)$, i.e., the relation $\langle u'-u,d\rangle\le 0$ holds for any $u'\in S$. Therefore $d\in N(u,S)$. Conversely, suppose $d\in N(u,S)$. Since $\langle u'-u,d\rangle\le 0$, $\forall u'\in S$ and $u\in S$, it follows that $u\in \operatorname{Arg}\max_{u'\in S} \langle u',d\rangle$.□

The following lemma is very important for our discussion below. It shows that a part of quasidifferentials of a q.d. function f have a nonempty intersection in some sense. Clearly, $\bigcap_{\mathcal{D}f(x)}(\overline{\partial}f(x)-\overline{\partial}f(x))\ne\emptyset$. Is $\bigcap_{\mathcal{D}f(x)}(\underline{\partial}f(x)+\overline{\partial}f(x))$ nonempty?

Lemma 2.2 [5,Th.1]. If f is quasidifferentiable at x, then one has that

$$\bigcap_{\mathcal{D}f(x)}(\underline{\partial}f(x)+\overline{\partial}f(x))\ne\emptyset$$

and this intersection is compact convex. □

Define

$$S:=\bigcap_{\mathcal{D}f(x)}(\underline{\partial}f(x)+\overline{\partial}f(x)).$$

Obviously, for any $u\in S$ there exists at least one sequence $\{u_i\in \underline{\partial}_i f(x)+\overline{\partial}_i f(x)\}_1^\infty$ convergent to u, where $[\underline{\partial}_i f(x),\overline{\partial}_i f(x)]\in \mathcal{D}f(x)$. Especially, if $u\in \operatorname{bd}S$, then there exists a sequence $\{u_i\in \operatorname{bd}(\underline{\partial}_i f(x)+\overline{\partial}_i f(x))\}_1^\infty$ convergent to u.

Lemma 2.3. Suppose $\{u_i\in \underline{\partial}_i f(x)+\overline{\partial}_i f(x)\}_1^\infty\to u\in S$ and assume furthermore that for each i, $d_i\in N(u_i,\underline{\partial}_i f(x)+\overline{\partial}_i f(x))\cap B_1(0)$.

Then the set of clusters of $\{d_i\}_1^\infty$ is included in $N(u,S)$.

Proof. Without loss of generality, assume that $\{d_i\}_1^\infty \to d$. From Prop. 2.1 one has $u_i \in \underset{u' \in \underline{\partial}_i f(x)+\overline{\partial}_i f(x)}{\text{Arg max}} \langle u', d_i\rangle$. Therefore

$$\langle u,d\rangle = \lim_{i\to\infty} \delta^*(d_i \mid \underline{\partial}_i f(x)+\overline{\partial}_i f(x)) = \lim_{i\to\infty} \langle u_i, d_i\rangle.$$

If $d=0$, then clearly, $d\in N(u,S)$. Assume that $d\neq 0$. Since

$$S\subset \underline{\partial}_i f(x)+\overline{\partial}_i f(x),\ \forall i$$

and $d\neq 0$, one obtains

$$\langle u'-u_i, d_i\rangle \le 0,\ \forall u'\in S \tag{2.1}$$

and $u_i \in \text{bd}(\underline{\partial}_i f(x)+\overline{\partial}_i f(x))$ for i large enough. From (2.1) we have that

$$\langle u',d\rangle = \lim_{i\to\infty} \langle u',d\rangle \le \lim_{i\to\infty} \langle u_i,d_i\rangle = \langle u,d\rangle,\ \forall u'\in S.$$

Hence $d\in N(u,S)$. □

Corollary 1. If $u\in S$ and $d\in R^n$ such that there exist sequences $\{u_i\in \underline{\partial}_i f(x)+\overline{\partial}_i f(x)\}_1^\infty \to u$ and $\{d_i \in N(u_i, \underline{\partial}_i f(x)+ +\overline{\partial}_i f(x))\}_1^\infty \to d$, then $\delta^*(d|S)=\underline{f}'(x;d)$.

Proof. Since $S\subset \underline{\partial}_i f(x)+\overline{\partial}_i f(x)$, one has

$$\delta^*(d|S) \le \underline{f}'(x;d),\ \forall d\in R^n. \tag{2.2}$$

On the other hand, according to Proposition 2.1 and Lemma 2.3, we have

$$\delta^*(d|S)=\langle u,d\rangle = \lim_{i\to\infty} \langle u_i,d_i\rangle = \tag{2.3}$$

$$= \lim_{i\to\infty} \delta^*(d_i \mid \underline{\partial}_i f(x)+\overline{\partial}_i f(x)) \ge \underline{f}'(x;d).$$

Combining (2.2) and (2.3), one has $\delta^*(d|S)=\underline{f}'(x;d)$. □

Hypotheses:

(H^+) For any $d\in R^n$ such that $\underset{u'\in S}{\text{Arg max}} \langle u',d\rangle=\{u\}$ is a singleton, there exist sequences $\{u_i\in \underline{\partial}_i f(x)+\overline{\partial}_i f(x)\}_1^\infty \to u$ and $\{d_i\in N(u_i, \underline{\partial}_i f(x)+\overline{\partial}_i f(x))\}_1^\infty$ such that d is one of the clusters of $\{d_i\}_1^\infty$.

(H^-) The statement is similar to (H^+), but for $\bigcap_{\overline{\mathcal{D}}f(x)} (\overline{\partial}f(x)-\overline{\partial}f(x))$.

In this paper we only study a class of quasidifferentiable functions satisfying hypotheses (H^+) and (H^-), that is, in the rest of this paper, whenever we say that a function f is quasidifferentiable, it always means that f is a quasidifferentiable function satisfying hypotheses (H^+) and (H^-). In one-dimensional space, every quasidifferentiable function possesses the properties (H^+) and (H^-). But, in general case, whether every quasidifferentiable function possesses the properties (H^+) and (H^-) is still open.

Corollary 2. For each $d \in \text{bd}\, B_1(0)$ such that $\operatorname{Arg}\max_{u' \in S} \langle u', d\rangle$ is a singleton, the expression $\underline{f}'(x;d) = \delta^*(d|S)$ holds.

Proof. Let $\operatorname{Arg}\max_{u' \in S} \langle u', d\rangle = \{u\}$. It is clear that $u \in \text{bd}\, S$, otherwise, $N(u,S)=\{0\}$. According to Prop. 2.2.1 and the lemma given above, there exist sequences $\{u_i \in \underline{\partial}_i f(x) + \overline{\partial}_i f(x)\}_1^\infty \to u$ and

$$\{d_i \in N(u_i, \underline{\partial}_i f(x) + \overline{\partial}_i f(x))\}_1^\infty \to d, \quad \|d_i\| = 1.$$

In consequence of the above corollary, one gets

$$\delta^*(d|S) = \underline{f}'(x;d). \quad \square$$

Lemma 2.4 ([5] Lemma 1, [4] Chapter 13). For almost all $d \in \text{bd}\, B_1(0)$, $\operatorname{Arg}\max_{u' \in S} \langle u', d\rangle$ is a singleton. □

Theorem 2.5. $\delta^*(d|S) = \underline{f}'(x;d)$, $\forall d \in R^n$.

Proof. It is sufficient to prove this theorem under the case where $d \in \text{bd}\, B_1(0)$. Take an arbitrary $\bar{d} \in \text{bd}\, B_1(0)$. In view of the lemma given above, there exist sequences $\{d_i\}_1^\infty$ convergent to $\bar{d}$ and $\{u_i\}_1^\infty$ satisfying

$$\operatorname{Arg}\max_{u' \in S} \langle u', d_i\rangle = \{u_i\}, \quad \forall i.$$

Without loss of generality , assume that the sequence $\{u_i\}_1^\infty$ is convergent to a point $\bar{u} \in S$. It follows from $(H^+)\&(H^-)$ that for each pair of u_i and d_i, one can find a pair of u_i' and d_i' such that

$$\|u_i - u_i'\| < 1/i, \quad \|d_i - d_i'\| < 1/i$$

and

$$d_i' \in N(u_i', \underline{\partial}_i f(x) + \overline{\partial}_i f(x)).$$

Since $u_i \to \bar{u}$ and $d_i \to \bar{d}$ as $i \to \infty$, one has $u_i' \to \bar{u}$ and $d_i' \to \bar{d}$ as $t \to \infty$ because of $\|u_i'-\bar{u}\| \le \|u_i'-u_i\|+\|u_i-\bar{u}\|$ and $\|d_i'-\bar{d}\| \le \|d_i'-d_i\|+ +\|d_i-\bar{d}\|$. Hence, we have

$$\{u_i' \in \underline{\partial}_i f(x)+\bar{\partial}_i f(x)\}_1^\infty \longrightarrow \bar{u} \quad \text{and} \quad \{d_i' \in N(u_i', \underline{\partial}_i f(x)+\bar{\partial}_i f(x))\}_1^\infty \longrightarrow \bar{d}.$$

According to Corol. 1, the relation $\delta^*(\bar{d}|S)=\underline{f}'(x;\bar{d})$ holds. Since $\delta^*(\cdot|S)$ and $\underline{f}'(x;\cdot)$ are positively homogeneous, for each $d \in R^n$ one has $\delta^*(d|S)=\underline{f}'(x;d)$. Obviously, $\delta^*(0|S)=\underline{f}'(x;0)$. The demonstration is completed. □

Following the same lines as above, we may obtain the following theorem for $\bigcap_{\bar{\mathcal{D}}f(x)} (\bar{\partial}f(x)-\bar{\partial}f(x))$ and $\bar{f}'(x;d)$.

Theorem 2.6. $\delta^*\Big(\cdot\,\Big|\bigcap_{\bar{\mathcal{D}}f(x)} (\bar{\partial}f(x)-\bar{\partial}f(x))\Big)=\bar{f}'(x;\cdot)$. □

Does the equality

$$\Big[\bigcap_{\mathcal{D}f(x)} (\underline{\partial}f(x)+\bar{\partial}f(x)), \bigcap_{\bar{\mathcal{D}}f(x)} (\bar{\partial}f(x)-\bar{\partial}f(x))\Big]= [\partial_* f(x)-\partial^* f(x)]$$

hold? The following theorem will give this question a positive answer.

Theorem 2.7. $\bigcap_{\mathcal{D}f(x)} (\underline{\partial}f(x)+\bar{\partial}f(x))=\partial_* f(x)$ and

$$\bigcap_{\bar{\mathcal{D}}f(x)} (\bar{\partial}f(x)-\bar{\partial}f(x))=\partial^* f(x).$$

Proof. For each $u \in \text{bd } S$ there exist sequences

$$\{u_i \in \text{bd}(\underline{\partial}_i f(x)+\bar{\partial}_i f(x))\}_1^\infty \quad \text{and} \quad \{d_i \in N(u_i, \underline{\partial}_i f(x)+\bar{\partial}_i f(x))\}_1^\infty$$

such that $u_i \to u$ and $d_i \to d \in \text{bd } B_1(0)$. Since for any ⊛-equivalent bounded subfamily $\hat{\mathcal{D}}f(x)$ of f at x the inclusion

$$S \subset \bigcap_{\hat{\mathcal{D}}f(x)} (\underline{\partial}f(x)-\bar{\partial}f(x))$$

holds, one has that bd S is an equivalent subset of $\underline{U}_0'(x)$. According to the definition of $\partial_* f(x)$, we have

$$\partial_* f(x) \subset S. \tag{2.4}$$

It will be proved below that the relation of opposite inclusion in (2.4) is also true. For any $d \in R^n$ there exists an element $u \in \text{Arg}\max_{u' \in S} \langle u',d\rangle$ such that

$$\delta^*(d|S) = \langle u,d\rangle = \underline{f}'(x;d). \tag{2.5}$$

On the other hand, we also could find an element $u_* \in \partial_* f(x)$

such that

(2.6) $$\langle u_*, d\rangle = \underline{f}'(x;d)$$

by virtue of the structure of $\partial_* f(x)$. It follows from (2.5) and (2.6) that

(2.7) $$\delta^*(d|S) = \underline{f}'(x;d) \le \max_{u\in\partial_* f(x)} \langle u,d\rangle.$$

One has from (2.7) and [10, Sec. 13] that

(2.8) $$\partial_* f(x) \supset S.$$

Thus the equality $\partial_* f(x)=S$ is obtained by combining (2.4) with (2.8). The same way used in getting $\partial_* f(x)=S$ can be used to obtain

$$\partial^* f(x) = \bigcap_{\overline{\mathcal{D}}f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)). \qquad \square$$

Note that we have $[\partial_* f(x), \partial^* f(x)] \in \mathcal{D}f(x)$ and $\partial^* f(x) = -\partial^* f(x)$, since $\overline{\partial} f(x) - \overline{\partial} f(x)$ is symmetric.

Corollary 1. The directional derivative function of a quasidifferentiable function f at x can be expressed as a difference form of two support functions on the kernel of quasidifferentials at x, i.e.,

$$f'(x;\cdot) = \delta^*(\cdot|\partial_* f(x)) - \delta^*(\cdot|\partial^* f(x)) = \underline{f}'(x;\cdot) - \overline{f}'(x;\cdot). \qquad \square$$

From [11], Lemma 3.2, one has

$\varphi(u \circledast d) \ge \langle u,d\rangle$, $\forall d\in R^n$, $\forall u\in\partial_* f(x)$ or $\forall u\in\partial^* f(x)$, and furthermore,

$$\min_{u\in\partial_* f(x)} \langle u \circledast d\rangle = \max_{u\in\partial_* f(x)} \langle u,d\rangle, \ \forall d\in R^n,$$

$$\min_{u\in\partial^* f(x)} \langle u \circledast d\rangle = \max_{u\in\partial^* f(x)} \langle u,d\rangle, \ \forall d\in R^n.$$

Recently, in one-dimensional space the concrete structure of the kernel of quasidifferentials of a quasidifferentiable function at a point x has been worked out, due to Y. Gao [6], [14]. The kernel can be expressed by directional derivatives as follows.

Theorem 2.8 ([6] Theorem 2, [14] Chapter 3, Sec. 3). In one-dimensional space, for any quasidifferentiable function $f:R^1 \longrightarrow R^1$ the kernel of $\mathcal{D}f$ at a point x possesses following component forms

$$\partial_* f(x) = [\min\{f'(x;1), -f'(x;-1)\}, \max\{f'(x;1), -f'(x;-1)\}],$$

and

$\partial^* f(x) = [-\max\{0,-f'(x;1)-f'(x;-1)\},\max\{0,-f'(x;1)-f'(x;-1)\}]$, and furthermore, there exists a quasidifferential $[\underline{\partial}_0 f(x),\overline{\partial}_0 f(x)]\in\mathcal{D}f(x)$ such that

$$\partial_* f(x) = \underline{\partial}_0 f(x)+\overline{\partial}_0 f(x), \quad \partial^* f(x) = \overline{\partial}_0 f(x)-\overline{\partial}_0 f(x),$$

where $\underline{\partial}_0 f(x)=[-c-\max\{-f'(x;1),f'(x;-1)\},-c+f'(x;1)]$ and $\overline{\partial}_0 f(x)=[c,c+\max\{0,-f'(x;1)-f'(x;-1)\}]$ with arbitrarily fixed real-valued number c. □

Corollary 1 ([6], Corollary 3). The directional differentiability of a function at a point is equivalent to its quasidif- ferentiability in one-dimensional space. □

Corollary 2 ([6] Theorem 4). In one-dimensional space, the map- ping $\partial_* f$ is upper semicontinuous at $x\in R^1$ if and only if the function $\min\{f'(\cdot;1),-f'(\cdot;-1)\}$ is lower semicontinuous and the function $\max\{f'(\cdot;1),-f'(\cdot;-1)\}$ is upper semicontinuous at x. In addition, the mapping $\partial^* f$ is lower (upper) semicontinuous at $x\in R^1$ if and only if the function $\max\{0,-f'(\cdot;1)- -f'(\cdot;-1)\}$ is lower (upper) semicontinuous at x. □

An example is also given in [14], Chapter 3, Sec. 5,

$$f(x) = \begin{cases} x^{3/2}\sin(1/x), & \text{if } x>0 \\ x, & \text{if } x\le 0 \end{cases}, \quad x\in R^1.$$

It is easy to calculate the directional derivatives at $0\in R^1$ in directions 1 and -1,

$$f'(0;1) = \lim_{\lambda\downarrow 0} [\lambda^{3/2}\sin(1/\lambda)]/\lambda=0,$$

$$f'(0;-1) = \lim_{\lambda\downarrow 0}[-\lambda-0]/\lambda = -1.$$

Therefore, $\partial_* f(0)=[0,1]$ and $\partial^* f(0)=[-1,1]$. Note that this function is quasidifferentiable at 0, but not locally Lipschi- tzian. According to the last corollary it is not difficult to construct a function being quasidifferentiable, but not local- ly Lipschitzian, in one-dimensional space.

Evidently, it follows from Theorem 2.7 that

$$\text{epi } \delta^*(\cdot|\partial_* f(x)) = \text{cl} \bigcup_{\mathcal{D}f(x)} \text{epi } \delta^*(\cdot|\underline{\partial} f(x)+\overline{\partial} f(x))$$

and

$$\text{epi } \delta^*(\cdot|\partial^* f(x)) = \text{cl} \bigcup_{\overline{\mathcal{D}}f(x)} \text{epi } \delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x)).$$

Since $\delta^*(\cdot|\partial^* f(x)) \le \delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x))$, $\forall\overline{\partial}f(x)\in\overline{\mathcal{D}}f(x)$ and $\text{epi}(f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x)) \supset \text{epi}(f'(x;\cdot)+\delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x))$, $\forall\overline{\partial}f(x)\in\overline{\mathcal{D}}f(x)$, in addition, $\underline{f}'(x;\cdot)$ and $\overline{f}'(x;\cdot)$ are sublinear, so one has

$$0^+(\text{epi}(f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x)))\supset 0^+(\text{epi}(f'(x;\cdot)+\delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x)))$$

where 0^+C denotes the recession cone of a set C [10, sec. 8]. In consequence of

$$\text{epi}(q0^+) = 0^+(\text{epi } q),$$

where $q0^+$ denotes the recession function of q, we obtain

$$\text{epi}((f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x)))0^+)\supset$$
$$\supset \text{epi}((f'(x;\cdot)+\delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x)))0^+).$$

It follows from this that

$$(f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x)))0^+ \le (f'(x;\cdot)+\delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x)))0^+,$$

$\forall\overline{\partial}f(x)\in\overline{\mathcal{D}}f(x)$. Hence, the function $(f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x))$ with a kernel $\partial^* f(x)$ is the minimum recession function to $\{f'(x;\cdot)+ +\delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x))|\overline{\partial}f(x)\in\overline{\mathcal{D}}f(x)\}$ in a certain sense. Actually, from [10, Corol. 8.5.2] we have

$$(f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x)))0^+)(d)=\lim_{\lambda\downarrow 0} ((f'(x;\cdot)+\delta^*(\cdot|\partial^* f(x)))\lambda)(d)=$$
$$=f'(x;d) + \delta^*(d|\partial^* f(x)),$$

likewise,

$$((f'(x;\cdot)+\delta^*(\cdot|\overline{\partial}f(x)-\overline{\partial}f(x)))0^+)(d)=$$
$$=f'(x;d)+\delta^*(d|\overline{\partial}f(x)-\overline{\partial}f(x)),\ \forall\overline{\partial}f(x)\in\overline{\mathcal{D}}f(x).$$

It is easy to see that a directionally differentiable function f is quasidifferentiable at $x\in R^n$ if and only if there exists a sublinear function φ such that $f'(x;\cdot)+\varphi(\cdot)$ is sublinear, i.e.,

$$f'(x;d_1+d_2) - f'(x;d_1)-f'(x;d_2) \le \varphi(d_1)+\varphi(d_2)-\varphi(d_1+d_2),$$

i.e., $f'(x;\cdot)+\varphi(\cdot)$ is subadditive. If f is directionally differentiable at $x\in R^n$ and there exists a sublinear function φ such that $F(\cdot)$ is subadditive, where

$$F(d) := f(x+d)+\varphi(d)-f(x),$$

then f is quasidifferentiable at x, since

$$F'(0;d_1+d_2) = \lim_{\lambda\downarrow 0} F(\lambda(d_1+d_2)) \le$$
$$\le \lim_{\lambda\downarrow 0} F(\lambda d_1) + \lim_{\lambda\downarrow 0} F(\lambda d_2) = F'(0;d_1) + F'(0;d_2)$$

and $F'(0;h)=f'(x;h)+\varphi(h)$, $\forall h\in R^n$. If f is quasidifferentiable at x, by definition, there exists a sublinear function φ satisfying the following relation

$$(f'(x;\cdot)+\varphi(\cdot)) = \{u\in R^n |\ \langle u,d\rangle\le f'(x;d)+\varphi(d),\ \forall d\in R^n\} = \underline{\partial}f(x).$$

By virtue of this, one has

$$(2.9)\qquad \underline{\partial}f(x) = \{u\in R^n |\ \langle u,\lambda d\rangle+f(x)\le$$
$$\le f(x+\lambda d)+\varphi(\lambda d)+\eta(x+\lambda d),\ \forall d\in R^n,\ \forall\lambda\ge)\},$$

where $-\eta(x+\lambda d):=f(x+\lambda d)-f(x)-\lambda f'(x;d)$, and $\eta(x+\lambda d)/\lambda\longrightarrow 0$ as $\lambda\downarrow 0$, $\forall d\in R^n$, and $\eta(x)=0$. The expression (2.9) can be rewritten as

$$(2.10)\qquad \underline{\partial}f(x) = \{u\in R^n |\ \langle u,z-x\rangle+f(x)\le f(z)+\varphi(z-x)+\eta(z),\ \forall z\in R^n\}.$$

Combining φ with η, and define $\psi(z):=\varphi(z-x)+\eta(z)$, we have

$$(2.11)\qquad \underline{\partial}f(x) = \{u\in R^n |\ \langle u,z-x\rangle+f(x)\le f(z)+\psi(z),\ \forall z\in R^n\}.$$

Clearly, $\psi(x)=0$. The form (2.11) is similar to the one of the subdifferential structure of a convex function. Since η is at least Gâteaux differentiable at x and the derivative equals zero, the function ψ is a quasidifferentiable in the sense of Pschenichny [9]. For convenience of distinction between the definitions in the sense of Demyanov and Rubinov and in the sense of Pschenichny, the latter is called p-quasidifferentiable. Thereby, if a function f is quasidifferentiable at x, then there exists a p-quasidifferentiable function ψ vanishing at x, such that $f+\psi$ is p-quasidifferentiable and

$$\partial(f+\psi)(x) = \underline{\partial}f(x) = \{u\in R^n |\ \langle u,z-x\rangle+f(x)\le f(z)+\psi(z),\ \forall z\in R^n\}.$$

For a convex function f the p-quasidifferentiable function ψ can be taken as $\psi\equiv 0$ and in this case $\partial f(x)=\partial_* f(x)$.

A function defined in R^n, f, is said to be uniformly directionally differentiable at $x\in R^n$ if for any $\varepsilon>0$ there exists an $\alpha_0>0$ such that

$$|f(x+\alpha d)-f(x)-f'(x;\alpha d)|<\alpha\varepsilon,\ \forall\alpha\in(0,\alpha_0],\ \forall d\in B_1(0),$$

[13], [4, Ch. 3]. It is known that if f is Lipschitzian in a neighborhood of x and directionally differentiable at x, then f is uniformly directionally differentiable at x, i.e., f

satisfies the condition given above.

Theorem 2.9. Suppose f is Lipschitzian in a neighborhood of x and quasidifferentiable at x. Then

$$|f^0(x;d)-\underline{f}'(x;d)| \le \overline{f}'(x;d),\ \forall d\in R^n \tag{2.12}$$

and

$$\partial_* f(x)\subset \partial f(x)+\partial^* f(x), \tag{2.13}$$

where $f^0(x;\cdot)$ and $\partial f(x)$ are generalized directional derivative and generalized gradient, respectively, due to Clarke [1], [2].

Proof. Since $f'(x;\cdot)\le f^0(x;\cdot)$, one has

$$\underline{f}'(x;\cdot) - \overline{f}'(x;\cdot) \le f^0(x;\cdot).$$

Therefore,

$$-\overline{f}'(x;\cdot) \le f^0(x;\cdot) - \underline{f}'(x;\cdot). \tag{2.14}$$

We prove the converse inequality below. Since f is Lipschitzian at x, one has that for any $d\in R^n$

$$f^0(x;d) = \limsup_{\substack{x'\to x\\ \lambda\downarrow 0}} \frac{1}{\lambda}[f(x'+\lambda d)-f(x')]=$$

$$= \limsup_{\substack{x'\to x\\ \lambda\downarrow 0}} \frac{1}{\lambda}\left[f\left(x+\|x'-x+\lambda d\| \cdot \frac{x'-x+\lambda d}{\|x'-x+\lambda d\|}\right)-f\left(x+\|x'-x\|\cdot\frac{x'-x}{\|x'-x\|}\right)\right].$$

Since f is uniformly directionally differentiable at x, one has that for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$f^0(x;d)\le\limsup_{\substack{x'\to x\\ \lambda\downarrow 0}} \frac{1}{\lambda}[f'(x;x'-x+\lambda d)-f'(x;x'-x)+ \\ +\varepsilon(\|x'-x+\lambda d\|+\|x'-x\|)], \tag{2.15}$$

for any $x'-x+\lambda d,\ x'-x\in B_\delta(0)$. Considering each item on the right hand side of (2.15), by the definition of quasidifferentiable functions, we have

$$f'(x;x'-x+\lambda d)=\delta^*(x'-x+\lambda d|\ \partial_* f(x))-\delta^*(x'-x+\lambda d|\ \partial^* f(x))\le \\ \le\delta^*(x'-x|\ \partial_* f(x))+\lambda\delta^*(d|\ \partial_* f(x))-\delta^*(x'-x+\lambda d|\ \partial^* f(x)) \tag{2.16}$$

and

$$f'(x;x'-x)=\delta^*(x'-x|\ \partial^* f(x))-\delta^*(x'-x|\ \partial^* f(x))\ge \\ \ge\delta^*(x'-x|\ \partial^* f(x))-\delta^*(x'-x+\lambda d|\ \partial^* f(x))-\delta^*(\lambda d|\ \partial^* f(x)). \tag{2.17}$$

For the last item, we have

$$\|x'-x+\lambda d\|-\|x'-x\|\le\lambda\|d\|. \tag{2.18}$$

In view of (2.15), (2.16), (2.17) and (2.18), one has

$$f^0(x;d) \le \underline{f}'(x;d) + \overline{f}'(x;d) + \varepsilon.$$

Taking $\varepsilon \longrightarrow 0$, finally, one obtains

$$f^0(x;d) - \underline{f}'(x;d) \le \overline{f}'(x;d). \tag{2.19}$$

It follows from (2.14) and (2.19) that the first assertion of this theorem is valid. The second assertion is easy to be derived from the inequality

$$\underline{f}'(x;d) \le f^0(x;d) + \overline{f}'(x;d),\ \forall d \in R^n,$$

i.e.,

$$\delta^*(d|\partial_* f(x)) \le \delta^*(d|\partial f(x)) + \delta^*(d|\partial^* f(x)).$$

The demonstration is completed. □

Remark 1. Suppose f is regular in the Clarke's sense and quasidifferentiable at x. Then we have

$$\partial_* f(x) = \partial f(x) + \partial^* f(x). \quad \square$$

Remark 2. By virtue of (2.19), one has

$$\partial f(x) \subset \partial_* f(x) + \partial^* f(x).$$

On the other hand, under some assumptions [4, Ch. 13], one has

$$\partial f(x) = \overline{co}\{z(d) \in \partial_* f(x) + \partial^* f(x) \mid d \in \tilde{C}(f) \subset \overline{C}(f)\},$$

where

$$\overline{C}(f) := \{d \in \text{bd } B_1(0) \mid \operatorname*{Arg\,max}_{u \in \partial_* f(x)} \langle u,d\rangle \text{ and } \operatorname*{Arg\,max}_{u \in \partial^* f(x)} \langle u,d\rangle \text{ are singletons}\},$$

and $\tilde{C}(f)$ is such a set that $m_*(\overline{C}(f) \backslash \tilde{C}(f)) = 0$. □

For any quasidifferentiable function f at x the directional derivative function $f'(x;\cdot)$ is Lipschitzian in direction. It follows from Theorem 2.9 that

$$|f'^0(x;d,h) - \underline{f}''(x;d,h)| \le \overline{f}''(x;d,h),\ \forall d,h \in R^n,$$

where

$$f'^0(x;d,h) := [f'(x;\cdot)]^0(d;h),$$

$$f'^0(x;d,h) = \max_{u \in \partial f'(x;d)} \langle u,h\rangle,$$

$$\underline{f}''(x;d,h) = \max_{u \in \hat{\partial}\underline{f}'(x;d)} \langle u,h\rangle,$$

$$\hat{\partial}\underline{f}'(x;d) = \{u \in \partial_* f(x) \mid \langle u,d\rangle = \underline{f}'(x;d)\},$$

$$\overline{f}''(x;d,h) = \max_{u \in \hat{\partial}\overline{f}'(x;d)} \langle u,h\rangle,$$

$$\hat{\partial}\bar{f}'(x;d) = \{u\in\partial^* f(x) \mid \langle u,d\rangle=\bar{f}'(x;d)\}.$$

3. Basic calculus on kernels

We now present some rules, partly given in [11], in which one could calculate kernels of f_1+f_2, λf, $\max_{i\in I} f_i$, etc. at a given point where f_1, f_2, f, $f_i (i\in I)$ are quasidifferentiable at the point and λ is a scalar.

For the sake of convenience of description, define

$$\mathcal{D}f(x):=\mathcal{D}f_1(x)+\mathcal{D}f_2(x)$$

if f_1 and f_2 are quasidifferentiable at x and $f:=f_1+f_2$.

Rule 3.1. Suppose f is defined by f_1+f_2, and f_1 and f_2 are quasidifferentiable at x. Then

$$\partial_* f(x) = \partial_* f_1(x) + \partial_* f_2(x)$$

and

$$\partial^* f(x) = \partial^* f_1(x)+\partial^* f_2(x).$$

Proof. Since

$$\partial_* f(x) = \bigcap_{\mathcal{D}f(x)} [(\underline{\partial}f_1(x)+\bar{\partial}f_1(x)) + (\underline{\partial}f_2(x)+\bar{\partial}f_2(x))],$$

one has

$$\underline{f}'(x;\cdot) = \inf_{\mathcal{D}f(x)} \max_{u\in[(\underline{\partial}f_1+\bar{\partial}f_1)+(\underline{\partial}f_2+\bar{\partial}f_2)](x)} \langle u,\cdot\rangle =$$

$$= \inf_{\mathcal{D}f_1(x)} \max_{u\in\underline{\partial}f_1(x)+\bar{\partial}f_1(x)} \langle u,\cdot\rangle + \inf_{\mathcal{D}f_2(x)} \max_{u\in\underline{\partial}f_2(x)+\bar{\partial}f_2(x)} \langle u,\cdot\rangle.$$

In consequence, we obtain

$$\delta^*(\cdot\,|\,\partial_* f(x)) = \delta^*(\cdot\,|\,\partial_* f_1(x)) + \delta^*(\cdot\,|\,\partial_* f_2(x)) =$$

$$= \delta^*(\cdot\,|\,\partial_* f_1(x)+\partial_* f_2(x)).$$

Hence, $\partial_* f(x) = \partial_* f_1(x) + \partial_* f_2(x)$ holds. Likewise, $\partial^* f(x) = $ $= \partial^* f_1(x)+\partial^* f_2(x)$ holds too. □

Rule 3.2. For a quasidifferentiable function f at x and any scalar $\lambda\in R^1$ the following formulae hold

$$\partial_*(\lambda f)(x) = \lambda\partial_* f(x) \tag{3.1}$$

and

$$\partial^*(\lambda f)(x) = \begin{cases} \lambda\partial^* f(x), & \text{if } \lambda\geq 0 \\ |\lambda|\partial^*(-f)(x), & \text{if } \lambda<0 \end{cases}$$

or compactly

(3.2) $$D_k(\lambda f)(x) = |\lambda| D_k((\text{sign } \lambda)f)(x).$$

Proof. Since

$$[\underline{\partial}(\lambda f)(x), \overline{\partial}(\lambda f)(x)] = |\lambda|[\underline{\partial}((\text{sign}\lambda)f)(x), \overline{\partial}((\text{sign}\lambda)f)(x)] =$$

$$= \begin{cases} \lambda[\underline{\partial}f(x), \overline{\partial}f(x), & \text{if } \lambda \geq 0 \\ |\lambda|[-\overline{\partial}f(x), -\underline{\partial}f(x)], & \text{if } \lambda < 0, \end{cases}$$

it follows that

$$\partial_*(\lambda f)(x) = \bigcap_{\mathcal{D}f(x)} \lambda(\underline{\partial}f(x), \overline{\partial}f(x)) = \lambda\partial_* f(x), \quad \forall\lambda\in R^1,$$

and if $\lambda \geq 0$, then one has

$$\partial^*(\lambda f)(x) = \bigcap_{\overline{\mathcal{D}}(\lambda f)(x)} (\overline{\partial}(\lambda f)(x) - \overline{\partial}(\lambda f)(x)) =$$

$$= \bigcap_{\overline{\mathcal{D}}f(x)} \lambda(\overline{\partial}f(x) - \overline{\partial}f(x)) = \lambda\partial^* f(x),$$

and if $\lambda < 0$, then one has

$$\partial^*(\lambda f)(x) = \bigcap_{\overline{\mathcal{D}}(-f)(x)} |\lambda|(\overline{\partial}(-f)(x) - \overline{\partial}(-f)(x)) = |\lambda|\partial^*(-f)(x).$$

By the way, the following formula is also vàlid for $\lambda < 0$,

(3.3) $$\partial^*(\lambda f)(x) = |\lambda| \bigcap_{\underline{\mathcal{D}}f(x)} (\underline{\partial}f(x) - \underline{\partial}f(x)).$$ □

Corollary 1. Suppose f_i, $i\in I$ are quasidifferentiable àt x and λ_i, $i\in I$ are scalars, where I denotes a finite index set. Then we have

$$D_k(\sum \lambda_i f_i)(x) = \sum|\lambda_i| D_k((\text{sign } \lambda_i)f_i(x).$$ □

Rule 3.3. Suppose f_1 and f_2 are quasidifferentiable at x. Then

(3.4) $$D_k(f_1 f_2)(x) = |f_1(x)| D_k((\text{sign } f_1(x))f_2)(x) + $$
$$+ |f_2(x)| D_k((\text{sign } f_2(x))f_1)(x).$$

Proof. Since from the ordinary operation rules of quasidifferentials one has

$$D(f_1 f_2)(x) = f_1(x) Df_2(x) + f_2(x) Df_1(x),$$

it follows that

$$\mathcal{D}(f_1 f_2)(x) = f_1(x)\mathcal{D}f_2(x) + f_2(x)\mathcal{D}f_1(x).$$

Let $k_1 := f_1(x)$ and $k_2 := f_2(x)$. We have

$$\mathcal{D}(f_1 f_2)(x) = \mathcal{D}[k_1 f_2(x) + k_2 f_1(x)].$$

By the Corol. of Rule 3.2, it is easy to get

$$D_k(f_1 f_2)(x) = D_k[k_1 f_2(x) + k_2 f_1(x)] =$$

$$= |k_1|D_k((\text{sign } k_1)f_2)(x) + |k_2|D_k((\text{sign } k_2)f_1)(x). \quad \square$$

Rule 3.4. Suppose f_1 and f_2 are quasidifferentiable at x and $f_2(x)\neq 0$. Then

$$(3.5) \qquad D_k\left(\frac{f_1}{f_2}\right)(x) = [|f_2(x)|D_k((\text{sign } f_2(x))f_1)(x) + $$
$$+ |f_1(x)|D_k((\text{sign}(-f_1(x)))f_2)(x)]/[f_2^2(x)].$$

Proof. Let $k_1:=1/f_2(x)$ and $k_2:=-f_1(x)/f_2^2(x)$. It is known that

$$D(f_1/f_2)(x) = k_1Df_1(x) + k_2Df_2(x) = D(k_1f_1+k_2f_2)(x).$$

From the rule given above, one has

$$D_k(f_1/f_2)(x) = |k_1|D_k((\text{sign } k_1)f_1)(x)+|k_2|D_k((\text{sign } k_2)f_2)(x).$$

It follows from this that (3.5) holds. $\square$

Let f be a maximum of quasidifferentiable functions f_i, $i\in I$, at a point x, where I is a finite index set. By [3, Th.1], [4, Th. 10.3], it is known that

$$\underline{\partial}f(x) = \text{co}\{\underline{\partial}f_k(x) - \sum_{i\in R(x)\setminus\{k\}} \overline{\partial}f_i(x) \mid k\in R(x)\},$$

$$\overline{\partial}f(x) = \sum_{k\in R(x)} \overline{\partial}f_k(x),$$

where $R(x):=\{i\in I \mid f_i(x)=f(x)\}$. Define

$$\mathcal{D}f(x):=\{Df(x) \mid Df_i(x)\in\mathcal{D}f_i(x),\ i\in I\}.$$

Rule 3.5. For $f=\max_{i\in I} f_i$, where f_i, $i\in I$ are quasidifferentiable at x, one has

$$(3.6) \qquad \partial_*f(x) = \text{co} \bigcup_{k\in R(x)} [\partial_*f_k(x) + \sum_{i\in R(x)\setminus\{k\}} \partial^*f_i(x)],$$

$$(3.7) \qquad \partial^*f(x) = \sum_{k\in R(x)} \partial^*f_k(x).$$

Proof. To begin with, we calculate $\underline{\partial}f(x)+\overline{\partial}f(x)$ and $\overline{\partial}f(x)-\overline{\partial}f(x)$. Observe

$$\underline{\partial}f(x)+\overline{\partial}f(x) = \text{co}\{\underline{\partial}f_k(x) - \sum_{i\in R(x)\setminus\{k\}} \overline{\partial}f_i(x) \mid k\in R(x)\} +$$

$$+\sum_{k\in R(x)}\bar\partial f_k(x) = \Big\{\sum_{k\in R(x)}\lambda_k[\underline{\partial} f_k(x) - \sum_{i\in R(x)\setminus\{k\}}\bar\partial f_i(x)] \mid \lambda_k\geq 0,$$

$$\Sigma\lambda_k=1\Big\} + \sum_{k\in R(x)}\bar\partial f_k(x).$$

Since for any convex sets C and C_i, $i\in A$, one has

$$\mathrm{co}\bigcup_{i\in A}C_i + C = \mathrm{co}\bigcup_{i\in A}(C_i+C),$$

it follows that

$$\underline{\partial} f(x)+\bar\partial f(x)=\mathrm{co}\bigcup_{k\in R(x)}\Big[\underline{\partial} f_k(x)+\bar\partial f_k(x)+\sum_{i\in R(x)\setminus\{k\}}\big(\bar\partial f_i(x)-\bar\partial f_i(x)\big)\Big]$$

and

$$\bar\partial f(x)-\bar\partial f(x) = \sum_{k\in R(x)}\big(\bar\partial f_k(x)-\bar\partial f_k(x)\big).$$

Since

$$\bar f'(x;\cdot) = \inf_{\bar{\mathcal D}f(x)}\ \max_{u\in\bar\partial f(x)-\bar\partial f(x)} \langle u,\cdot\rangle =$$

$$= \inf_{\substack{\bar{\mathcal D}f_k(x)\\ k\in R(x)}}\ \max_{u\in\sum_{k\in R(x)}\big(\bar\partial f_k(x)-\bar\partial f_k(x)\big)} \langle u,\cdot\rangle =$$

$$= \sum_{k\in R(x)}\ \inf_{\bar{\mathcal D}f_k(x)}\ \delta^*(\cdot\mid \bar\partial f_k(x)-\bar\partial f_k(x)) = \sum_{k\in R(x)}\bar f_k'(x;\cdot),$$

one has

$$\partial^* f(x) = \sum_{k\in R(x)}\partial^* f_k(x).$$

It remains to prove (3.6). Evidently,

$$f'(x;d) = \delta^*(d\mid\partial_* f(x)) - \delta^*(d\mid\partial^* f(x)) =$$

$$= \delta^*(d\mid\partial_* f(x)) - \delta^*(d\mid\sum_{k\in R(x)}\partial^* f_k(x)),\ \forall d\in R^n.$$

On the other hand, we have

$$f'(x;d) = \delta^*\Big(d\mid \mathrm{co}\bigcup_{k\in R(x)}\Big[\partial_* f_k(x) + \sum_{i\in R(x)\setminus\{k\}}\partial^* f_i(x)\Big]\Big) -$$

$$-\delta^*\Big(d\mid \sum_{k\in R(x)}\partial^* f_k(x)\Big) = \delta^*\Big(d\mid \mathrm{co}\bigcup_{k\in R(x)}\Big[\partial_* f_k(x) +$$

$$+ \sum_{i\in R(x)\setminus\{k\}} \partial^* f_i(x)\Big]\Big) - \delta^*\Big(d \Big| \sum_{k\in R(x)} \partial^* f_k(x)\Big), \ \forall d\in R^n,$$

since

$$\Big[\mathrm{co} \bigcup_{k\in R(x)} \Big[\partial_* f_k(x) - \sum_{i\in R(x)\setminus\{k\}} \partial^* f_i(x)\Big], \sum_{k\in R(x)} \partial^* f_k(x)\Big] \in \mathcal{D}f(x).$$

Therefore, one obtains

$$\delta^*(\cdot\,|\,\partial_* f(x)) = \delta^*\Big(\cdot\,\Big|\,\mathrm{co} \bigcup_{k\in R(x)} \Big[\partial_* f_k(x) + \sum_{i\in R(x)\setminus\{k\}} \partial^* f_i(x)\Big]\Big).$$

It follows from this that (3.6) is valid. □

Remark 1. We denote by $\partial^- f(x)$ the intersection

$$\bigcap_{\mathcal{D}f(x)} \big(\underline{\partial} f(x) - \underline{\partial} f(x)\big)$$

and then have $\partial^*(-f)(x) = \partial^- f(x) = -\partial f(x)$ from (3.1). It also can be seen easily from the following expressions

$$(-f)'(x;\cdot) = \max_{u\in -\overline{\partial} f(x)} \langle u,\cdot\rangle + \min_{u\in -\underline{\partial} f(x)} \langle u,\cdot\rangle =$$

$$= \delta^*(\cdot\,|\,-\overline{\partial} f(x) - \underline{\partial} f(x)) - \delta^*(\cdot\,|\,\underline{\partial} f(x) - \underline{\partial} f(x)) =$$

$$= \delta^*(\cdot\,|\,-\partial_* f(x)) - \delta^*(\cdot\,|\,\partial^- f(x)).$$

Actually, in quasidifferential calculus we need three different kinds of kernels: $\partial_* f(x)$, $\partial^* f(x)$ and $\partial^- f(x)$. The first two kernels are basic and the last one is auxiliary in our case. Let $f := \min_{i\in I} f_i$, where I is finite and f_i, $i\in I$ are quasidifferentiable at x. Since $f(x) = -\max\{f_i(x)\,|\, i\in I\}$, one has

$$\partial_* f(x) = -\partial_* \max\{-f_i(x)\,|\, i\in I\}$$

and

$$\partial^* f(x) = \partial^*(-\max\{(-f_i)(x)\,|\, i\in I\}) = \partial^-(\max\{(-f_i)(x)\,|\, i\in I\} =$$

$$= \bigcap_{\mathcal{D}(\max\{(-f_i)(x)\,|\, i\in I\})} [\underline{\partial}(\max\{(-f_i)(x)\,|\, i\in I\}) - \underline{\partial}(\max\{(-f_i)(x)\,|\, i\in I\}) =$$

$$= \bigcap_{\substack{\mathcal{D}f_i(x)\\ i\in R^-(x)}} \Big[\mathrm{co} \bigcup_{k\in R^-(x)} \Big(\overline{\partial} f_k(x) - \sum_{\substack{i\in R^-(x)\\ i\neq k}} \underline{\partial} f_i(x)\Big) -$$

$$- \operatorname{co} \bigcup_{k\in R^{-}(x)} \Big[\bar{\partial} f_k(x) - \sum_{\substack{i\in R^{-}(x)\\ i\neq k}} \underline{\partial} f_i(x)\Big]\Big],$$

where $R^{-}(x):=\{i\in I \mid f_i(x)=f(x)\}$. Therefore, it seems that the definition of kernel $D_k f=[\partial_* f,\partial^* f]$ of quasidifferentials for the quasidifferentiable function f at a given point x needs to be improved further, because the definition of kernel $D_k f=[\partial_* f,\partial^* f]$, proposed in [11] and improved in this paper, is still not very convenient for calculating the kernels of $(-f)$ and $\min\{f_i \mid i\in I\}$.

Let φ be a function $R^m \longrightarrow R$ defined by

$$y \longmapsto \varphi(y) = \max\{f(x,y) \mid x\in X\} = \max\{f_x(y) \mid x\in X\},$$

where $f: X\times R^m \longrightarrow R$ and $X\subset R^n$ is compact.

Theorem 3.6. Suppose that $f(\cdot,y)$ is continuous for every $h\in N(y_0)\subset R^m$, and $f(x,\cdot)$ is uniformly directionally differentiable and uniform in $x\in X$, i.e., for any $h\in R^m$ and $\varepsilon>0$, that there exist scalars $\delta>0$ and $\alpha_0>0$ such that

$$|f(x,y+\alpha h) - f(x,y) - \alpha f_x'(y;d)| < \alpha\varepsilon$$

is satisfied for all $x\in X$ whenever $\|h-d\|<\delta$, $0<\alpha<\alpha_0$, where $N(y_0)$ is neighborhood of y_0. Then φ is uniformly directionally differentiable and

$$\varphi'(y;d) = \max\{f_x'(y;d) \mid x\in X(y)\},$$

where $X(y):=\{x\in X \mid f(x,y)=\varphi(y)\}$ and $f_x'(y;d):=f'(x,\cdot;d)(y)= = \lim_{\lambda\downarrow 0}[f(x,y+\lambda d)-f(x,y)]/\lambda$, due to [13]. Furthermore, assume that for any $x\in X$, $f(x,\cdot)$, i.e., $f_x(\cdot)$ is quasidifferentiable and there exist $Df_x(y)=[\underline{\partial} f_x(y),\bar{\partial} f_x(y)]\in \mathcal{D}f_x(y)$, $x\in X(y)$, compact sets $A_x(y)$ and $B(y)$ such that

$$B(y) = \bar{\partial} f_x(y) + A_x(y),\ \forall x\in X(y),$$

i.e., for any $x\in X(y)$, $B_x(y)=B(y)$, where $B_x(y)=\bar{\partial} f_x(y) + A_x(x)$. Then, φ is quasidifferentiable at y and

(3.7) $$\underline{\partial}\varphi(y) = \operatorname{co} \bigcup_{x\in X(y)} [\underline{\partial} f_x(y)-A_x(y)],\quad \bar{\partial}\varphi(y)=B(y),$$

due to [15]. □

Define, for any $x\in X(y)$,

$\mathcal{D}_s f_x(y):=\{Df_x(y)=[\underline{\partial} f_x(y)-\bar{\partial} f_x(y)]\in \mathcal{D}f_x(y) \mid \exists B(y): \bar{\partial} f_x(y)=B(y)$

and there exists at least one $\bar{\partial} f_z(y)\in \bar{\mathcal{D}} f_z(y)$ such

$$\text{that} \quad \bar{\partial} f_z(y)=B(y) \quad \text{for each} \quad z\in X(y)\backslash\{x\},$$

and also $\underline{\mathcal{D}}_s f_x(y)$ and $\overline{\mathcal{D}}_s f_x(y)$, can be defined correspondingly. Using these notations, we have

$$\overline{\mathcal{D}}_s f_x(y) = \mathcal{F}, \ \forall x\in X(y),$$

where $\mathcal{F}$ is a subfamily of nonempty compact convex sets, included in each $\overline{\mathcal{D}} f_x(y)$. Then (3.7) can be rewritten as

$$(3.8) \qquad \underline{\partial}\varphi(y) = \text{co} \bigcup_{x\in X(y)} \underline{\partial} f_x(y), \ \bar{\partial}\varphi(y)=B(y),$$

where $[\underline{\partial} f_x(y), B(y)]\in\mathcal{D}_s f_x(y)$ and $B(y)\in\mathcal{F}$. Since

$$\bigcap_{\mathcal{D}_s f_x(y)} [\underline{\partial} f_x(y)+B(y)]\neq\emptyset, \quad \bigcap_{\overline{\mathcal{D}}_s f_x(y)} [B(y)-B(y)]\neq\emptyset,$$

$\forall x\in X(y)$, and the fact,

$$f_x'(y;d) = \delta^*(d|\ \partial_{s*} f_x(y)) - \delta^*(d|\ \partial_s^* f_x(y)),$$

i.e., $[\partial_{s*} f_x(y), \partial_s^* f_x(y)]\in\mathcal{D}_s f_x(y)$, $\forall x\in X(y)$, where

$$\partial_{s*} f_x(y)) := \bigcap_{\mathcal{D}_s f_x(y)} [\underline{\partial} f_x(y)+B(y)]$$

and

$$\partial_s^* f_x(y) := \bigcap_{\overline{\mathcal{D}}_s f_x(y)} [B(y)-B(y)],$$

one has from (3.7) that

$$[\text{co} \bigcup_{x\in X(y)} \partial_{s*} f_x(y), \partial_s^* f_x(y)]\in\mathcal{D}\varphi(y),$$

where $\mathcal{D}\varphi(y)$ is defined by the collection of $[\underline{\partial}\varphi(y), \bar{\partial}\varphi(y)]$ given by (3.8) and satisfying all assumptions in the above theorem. In this case, we have

$$\begin{aligned} \bar{\varphi}'(y;\cdot) &= \delta^*(\cdot\,|\,\partial_s^* f_x(y)), \ \forall x\in X(y) \\ &= \delta^*(\cdot\,|\,\partial^*\varphi(y)), \end{aligned}$$

i.e., $\partial^*\varphi(y)=\partial_s^* f_x(y)$, $\forall x\in X(y)$, and then

$$\underline{\varphi}'(y;\cdot) = \delta^*(\cdot\,|\ \text{co} \bigcup_{x\in X(y)} \partial_{s*} f_x(y)) = \delta^*(\cdot\,|\,\partial_*\varphi(y)),$$

i.e., $\partial_*\varphi(y) = \text{co} \bigcup_{x\in X(y)} \partial_{s*} f_x(y)$. Now we have the following rule used to evaluate the kernels of φ at y.

Rule 3.7. Under the assumptions given in the theorem above, one has

$\partial_*\varphi(\cdot) = \text{co} \bigcup_{x\in X(\cdot)} \partial_{s*}f_x(\cdot)$, $\partial_s^*\varphi(\cdot) = \partial_s^* f_x(\cdot)$ for some $x\in X(\cdot)$.□

At the end of this part, we discuss a simple example [8], concerning the addition of kernels. Let $f:=f_1+f_2+f_3:R^2 \to R^1$, where f_1, f_2, f_3 are defined by

$$f_1(x)=|x_1|,\ f_2(x)=-|x_2|$$

and

$$f_3(x) = \begin{cases} \dfrac{|x_1^3 x_2|}{x_1^4+x_2^2}, & x\neq 0 \\ 0, & x=0 \end{cases},$$

respectively, where $x=(x_1,x_2)^T\in R^2$. Thus for any $x\in R^2$,

$$f(x) = \begin{cases} |x_1| - |x_2| + \dfrac{|x_1^2 x_2|}{x_1^4+x_2^2}, & x\neq 0 \\ 0 & , x=0 . \end{cases}$$

It is easy to check that f_1, f_2, f_3 and then f are quasidifferentiable at 0, and that it is easy to find their quasidifferential kernels,

$$\begin{aligned} \partial_* f_1(0) &= [-1,1]\times\{0\}, & \partial^* f_1(0) &= 0\in R^2, \\ \partial_* f_2(0) &= \{0\}\times[-1,1], & \partial^* f_2(0) &= \{0\}\times[-2,2], \\ \partial_* f_3(0) &= 0\in R^2, & \partial^* f_3(0) &= 0\in R^2. \end{aligned}$$

Finally, we have $\partial_* f(0)=[-1,1]\times[-1,1]$ and $\partial^* f(0)=\{0\}\times[-2,2]$.

Acknowledgements: The author is greatly indebted to Professor Emilio Spedicato for his encouragement, advice and help, and acknowledges a financial support from CNR/GNIM Programma Professori Visitatori for his stay in Bergamo.

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Received December 5, 1990.