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BINORMAL LAW CHARACTERIZED BY POLYNOMIAL CONDITIONAL MOMENTS

1. Introduction

This paper is intended to be a continuation of characterization problems inspired by problem 2 of chapter 6 (p 638 in Russian version) in [4]. That is we study a polynomial regression assumption in characterization of binormal distributions. However contrary to [2] we study mostly regression on r.v. $R=\sqrt{(X^2-2\rho XY+Y^2)}$ where X, Y are standardized and ρ=EXY. The technique used is also different. The results are proved by Hermite expansion of the distribution of (X,Y) and technical lemmas presented in the Appendix under basically the following condition: $\exists\ \lambda>0$ $E\exp(\lambda R^2)<\infty$.

In fact we will assume little bit less (it is shown in the Appendix) namely that either of the following two conditions hold: $\exists\ C>0,\ k>0\ \forall\ n,\ m\geq 0$ either

(A) $$|EH_n(X)H_m((Y-pX)/\sqrt{1-p^2})|\leq Cn^km^k\sqrt{n!m!}$$

or

(A*) $$|EH_n(Y)H_m((X-\rho Y)/\sqrt{1-\rho^2})|\leq Cn^km^k\sqrt{n!m!}$$

here $H_n(x)$ denotes n-th Hermite polynomial i.e.

$$H_n(x)=n!\sum_{m=0}^{[n/2]}(-1)^m x^{n-2m}/(m!2^m(n-2m)!).$$

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We believe that the above mentioned assumption is superfluous. To support this belief it might be of interest to state that Theorems 1.1 & 3.1 of [2] were first proved under this assumption and later generalized.

The main results are in Section 2.

In order to make the formulation of the theorems shorter and more clear let us generalize general polynomial condition considered in [2].

Let (X,Y) be a pair of standardized $\mathbb{L}_2$ random variables. Let ρ=EXY. Finally let $g:\mathbb{R}\times\longrightarrow\mathbb{R}$ be Borel measurable function and let us denote:

(1) $$F(z)=\iint_{g(x,y)\le z} p_\rho(x,y)\,dxdy$$

where $p_\rho(x,y)=\exp(-(x^2-2\rho xy+y^2)/(2(1-\rho^2)))/(2\pi\sqrt{1-\rho^2})$.

Definition 1. We say that (X,Y) belongs to Z-General Polynomial Regression class associated with Normal distribution iff

a) Z=g(X,Y) is distributed according to F

b) $\forall k\ge 1$ $E(X^k|Z)=Q_k^1(Z)$ and $E(Y^k|Z)=Q_k^2(Z)$ a.s.

where Q_k^i i=1,2 are polynomials of degree not exceeding k. We write in this case $(X,Y)\in$Z-GPRN.

The most important cases to be considered are Z=X, Z=Y, Z=aX+bY, $ab\neq 0$, $Z=\sqrt{(X^2-2\rho XY+Y^2)}$.

For the first three cases we introduce the following:

Definition 2. Suppose that the function g in the definition of random variable Z is such that var(Z)=1.Suppose further that $(X,Y)\in$ Z-GPRN and that the polynomials Q_k^i i=1,2 appearing in condition b) of Definition 1 are of the form:

$$Q_k^1(z)=\rho_{X,Z}^k\, z^k+\hat{Q}_{k-1}^1(z),\quad Q_k^2(z)=\rho_{Y,Z}^k\, z^k+\hat{Q}_{k-1}^2(z)$$

where $\rho_{X,Z}$=EXZ, $\rho_{Y,Z}$=EYZ and $\hat{Q}_{k-1}^i$ i=1,2 are polynomials of degree not exceeding k-1. We say in this case that (X,Y) belongs to Polynomial Regression class associated with Normal distribution and write (X,Y)$\in$ Z-PRN.

2. Main Results

We start with the following result which is closely related to Theorem 3.1 of [2]. This time however we will assume that (X,Y) are simultaneously X-GPRN and Y-GPRN. It turns out that in this case one more condition is required. Theorem 1 below gives few examples of such conditions. Namely we have.

Theorem 1. Suppose (X,Y) satisfy (A) or (A*) and are both X-GPRN and Y-GPRN. Moreover if $0<|\rho|<1$ and any of the following condition holds:

1^o $R^2=X^2-2\rho XY+Y^2$ has exponential distribution

2^o $E(Y^2|R)=R^2/2$ a.s.

3^o $E(XY|R)=\rho E(Y^2|R)$ a.s.

then (X,Y) has binormal distribution.

Corollary 1.1 Suppose (X,Y)$\in$Y-GPRN, satisfies (A) or (A*),

$$\forall\ n\geq 1\quad E(X^n|R)=E(Y^n|R)\ \text{a.s.}$$

where as usually $R^2= X^2- 2\rho XY+Y^2$, ρ=EXY and moreover that any of the condition 1^o,2^o or 3^o of Theorem 1 is satisfied. If $0<|\rho|<1$ then (X,Y) has binormal distribution.

Remark 1. If among the assumptions of Corollary 1.1 we assumed (X,Y)$\in$Y-PRN instead of (X,Y)$\in$Y-GPRN then one would not have to assume any of the conditions 1^o, 2^o or 3^o of Theorem 1 to deduce binormality of (X,Y). To show this we argue first as in the proof of Corollary 1.1 and then use Theorem 3.1 of [2].

In the next Theorem we analyze condition R-GPRN (where $R^2=X^2-2\rho XY+Y^2$). Recall that the condition R-GPRN implies that R has exponential distribution (compare the definition of F given by (1)).

Theorem 2. Suppose that (X,Y) are such that (A) or (A*) is satisfied and that $(X,Y)\in$R-GPRN. If $|\rho|<1$, then (X,Y) has binormal distribution.

In the next two results we deduce normality of a random variable by comparing its conditional moments (first two are enough in fact) with the same conditional moments of some normal random variable conditioned upon the same random variable. More precisely our next result is as follows:

Theorem 3. Suppose X and Y are independent and $Y \sim N(0,1)$, $EX=0$, $EX^2=1$, and $\exists\ k>0, C>0\ \forall\ n\in\mathbb{N}\ |EH_n(X)| \le Cn^k\sqrt{n!}$ (B). Suppose further that:

$$E(X|R)=E(Y|R) \text{ a.s.},\ E(X^2|R)=E(Y^2|R) \text{ a.s.} \tag{2}$$

where $R^2=X^2+Y^2$, then $X \sim N(0,1)$.

As a corollary we get the following result in which we deduce binormality of a pair of random variables (X,Y) assuming only some moment conditions, polynomial regressions of one upon the other and comparing conditional moments of every coefficient of the pair with similar conditional moments of an independent (of the pair) Gaussian random variable D conditioned upon sums of squares of X and D and Y and D. More precisely we have:

Corollary 1.3. Let (X,Y) be a pair of standardized random variables satisfying condition (B) and

$$\forall\ k\ge1\quad E(X^k|Y)=\rho^kY^k+W_{k-1}(Y) \text{ a.s.},\ E(Y^k|X)=\tilde{W}_k(X) \text{ a.s.}$$

where $\rho=EXY$ and W_k and $\tilde{W}_k$ $k\ge1$ are polynomials of the degree not exceeding k. Moreover let $D\sim N(0,1)$ be independent of (X,Y). If $E(D|R)=E(X|R)$ a.s. , $E(D|\tilde{R})=E(Y|\tilde{R})$ a.s.

$$E(D^2|R)=E(X^2|R) \text{ a.s.}\quad E(D^2|\tilde{R})=E(Y^2|\tilde{R}) \text{ a.s.}$$

where $R^2=X^2+D^2$, $\tilde{R}^2=Y^2+D^2$, then (X,Y) has normal distribution.

3. Proofs of the results and auxiliary lemmas

The proofs of the results presented in section 2 are based on the following technical lemmas. First two of these lemmas are concerned with the change of variables $(X,Y)\longrightarrow(R,\Phi)$ defined by $Y=R\cos\Phi$, $(X-\rho Y)/\sqrt{(1-\rho^2)} = R\sin\Phi$ and finding the distribution of (R,Φ) given the the expansion of the distribution of (X,Y) in the form of the series (1). The distribution of (R,Φ) will be expanded in the series of Laguerre polynomials $L_n^{(a)}(R)$ $a\in\mathbb{N}\cup\{0\}$ and $\sin(n\varphi)$ and $\cos(n\varphi)$ $n\geq 0$. The proofs of these lemmas and of the others are highly technical, hence will be presented in the Appendix. In this section we will only present the lemmas and the proofs of our results.

Before we will present these lemmas let us recall the definition of Laguerre polynomials:

$$L_s^{(a)}(x)=\sum_{m=0}^{s}(-1)^m\binom{s+\alpha}{s-m}x^m/m!,\ L_s(x)\overset{df}{=}L_s^{(0)}(x),$$

$x\geq 0, a\geq 0, s=0,1,\ldots$.

Let us also denote $p(z)=\exp(-z^2/2)/\sqrt{(2\pi)}$. We have:

Lemma 1. (i) Let us denote

$$J_{k,n,m}(y)=\sqrt{(2\pi)}\int p(z)H_m(z)H_n(\alpha z+\beta y)H_k(\gamma z+\delta y)\,dz.$$

For any $\alpha,\beta,\gamma,\delta$ such that $\alpha\beta\gamma\delta<0$ $|\alpha|=|\delta|$, $|\beta|=|\gamma|$, $\alpha^2+\beta^2=1$ we have:

$$J_{k,n,m}(y) =$$

$$= m!H_{k+n-m}(y)\sum_{j=0}^{m}\binom{n}{j}\binom{k}{m-j}|\alpha|^{k-m+2j}|\gamma|^{n+m-2j}(\mathrm{sign}\alpha)^j(\mathrm{sign}\gamma)^{m-j}.$$

(ii) $f_n(t):=\int\exp(itx)p(x)H_n(x)dx=(it)^n\exp(-t^2/2)$ (here $i^2=-1$).

(iii) Let us denote $D_{n,m}(r,\varphi):=H_n(r\cos\varphi)H_m(r\sin\varphi)$. We have:

$$D_{n,m}(r,\varphi)=\begin{cases}\displaystyle\sum_{i=0}^{[n/2]+[m/2]}P_i^{n,m}(r)\sin(n+m-2i)\varphi & \text{if } m \text{ is odd}\\[2ex] \displaystyle\sum_{i=0}^{[n/2]+[m/2]}P_i^{n,m}(r)\cos(n+m-2i)\varphi & \text{if } m \text{ is even,}\end{cases}$$

where

$$(3)\quad \begin{cases} P_i^{n,m}(r)=(-1)^i \dfrac{n!m!}{2^{n+m-i-1}(n+m-1)!} F_i^{n,m} r^{n+m-2i}\, L_i^{(n+m-2i)}(r^2/2) & \text{if } 2i<n+m \\ P_{n+m}^{2n+2m}(r)=(-1)^{n+m} \dfrac{n!m!}{2^{n+m}} L_{n+m}(r^2/2) & \text{if } i=n+m\,, \end{cases}$$

and $F_i^{n,m}$ are real numbers given by:

$$(4)\quad \begin{cases} F_0^{n,m}=(-1)^{[m/2]}\binom{n+m}{m} \\ F_i^{n,m}=(-1)^{[m/2]}\displaystyle\sum_{j=0\vee(i-[n/2])}^{i\wedge[m/2]}(-1)^j\binom{m+n-2i}{m-2j}\binom{i}{j}. \end{cases}$$

Corollary 1.2.

(i) Characteristic function of $\varphi_{X,Y}(t,s)$ of (X,Y) has the following form :

$$\varphi_{X,Y}(t,s)=q(t,s)\sum_{n,m\geq 0} b_{n,m}(it)^n(is)^m(1-\rho^2)^{(n+m)/2}/(n!m!)$$

where $i^2=-1$, $q(t,s)= \exp(-(t^2+2\rho ts+s^2))$ and

$$(5)\qquad b_{n,m}=\sum_{k=0}^{n}\binom{n}{k}\rho^k c_{k+m,n-k} = \sum_{k=0}^{n}\binom{n}{k}\rho^k\, \hat{c}_{k+n,m-k}.$$

(ii) Let R,Φ be new random variables defined by the relationships

$$Y = R\cos\Phi$$

$(X-\rho Y)/\sqrt{(1-\rho^2)}=R\sin\Phi$, with $R\geq 0$ a.s. and $\Phi\in(-\pi,\pi]$.Let $f(r,\varphi)$ be the distribution of (R,Φ), then:

$$(6)\quad f(r,\varphi)=(1/(2\pi))r\exp(-r^2/2)\Bigg\{\sum_{s\geq 0}(-1)^s g_{s,0}L_s(r^2/2)/(s!2^s) + $$

$$+\sum_{i\geq 1}\frac{r^{2i}}{2^{2i}}\cos(2i\varphi)\sum_{s\geq 0}(-1)^s g_{s,2i}L_s^{(2i)}(r^2/2)/(2^{s-1}(s+2i)!)) +$$

$$+\sum_{i\geq 0}\frac{r^{2i+1}}{2^{2i+1}}\cos((2i+1)\varphi)\ \times$$

$$\times\sum_{i\geq 0}(-1)^s g_{s,2i+1}L_s^{(2i+1)}(r^2/2)/(2^{s-1}(s+2i+1)!)\ +$$

$$+\sum_{i\geq 0}\frac{r^{2i+1}}{2^{2i+1}}\sin((2i+1)\varphi)\ \times$$

$$\times\sum_{s\geq 0}(-1)^s f_{s,2i+1}L_s^{(2i+1)}(r^2/2)/(2^{s-1}(s+2i+1)!)+$$

$$+\sum_{i\geq 1}\frac{r^{2i}}{2^{2i}}\sin(2i\varphi)\sum_{s\geq 0}(-1)^s f_{s,2i}L_s^{(2i)}(r^2/2)/(2^{s-1}(s+2i)!)\Big\}.$$

where we have denoted:

$$g_{S,0}=\sum_{k=0}^{s}\binom{s}{k}(1-\rho^2)^k c_{2k,2s-2k}, \tag{7}$$

$$g_{s,2i}=\sum_{k=0}^{s+i}c_{2k,2s+2i-2k}(1-\rho^2)^k F_s^{2k,2s+2i-2k}, \tag{8}$$

$$g_{s,2i+1}=\sum_{k=0}^{s+i}c_{2k+1,2s+2i-2k}(\sqrt{(1-\rho^2)})^{2k+1}F_s^{2k+1,2s+2i-2k}, \tag{9}$$

$$f_{s,2i+1}=\sum_{k=0}^{s+i}c_{2k,2s+2i-2k-1}(1-\rho^2)^k F_s^{2k,2s+2i-2k-1}, \tag{10}$$

$$f_{s,2i}=\sum_{k=0}^{s+i-1}c_{2k+1,2s+2i-2k-1}(\sqrt{(1-\rho^2)})^{2k+1}F_s^{2k+1,2s+2i-2k-1} \tag{11}$$

and numbers $F_s^{i,k} s,i,k\in\mathbb{N}\cup\{0\}$ are defined by (4).

Lemma 3. Suppose (X,Y) satisfy (A) and (A*). We have then:

(i) $(X,Y)\in$Y-GPRN $\Leftrightarrow c_{k,n}=0$ for $k>n\geq 0$ $\Leftrightarrow b_{n,m}=0$ for $m>n\geq 0$.

(ii) $(X,Y)\in$Y-PRN $\Leftrightarrow c_{k,n}=0$ for $k\geq n\geq 0$, $k+n\geq 1$ $\Leftrightarrow$

$\Leftrightarrow b_{n,m}=0$ for $m\geq n\geq 0$, $m+n\geq 1$.

(iii) $(X,Y)\in X$-GPRN $\Leftrightarrow \hat{c}_{k,n}=0$ for $k>n\geq 0 \Leftrightarrow b_{n,m}=0$ for $n>m\geq 0$.

(iv) $(X,Y)\in X$-PRN $\Leftrightarrow \hat{c}_{k,n}=0$ for $k\geq n\geq 0$, $k+n\geq 1 \Leftrightarrow$

$\Leftrightarrow b_{n,m}=0$ for $n\geq m\geq 0$, $m+n\geq 1$.

Remark 2. As an immediate consequence of Lemma 3 we get weaker version of Theorem 3.1 of [2]. Namely under assumptions of Theorem 3.1 and (A) and (A*) we deduce by Lemma 3 that the conditions $(X,Y)\in Y$-GPRN and $(X,Y)\in X$-PRN imply that $b_{n,m}=0$ for $m>n\geq 0$ and for $n\geq m\geq 0$, $m+n\geq 1$. Thus we deduce that the characteristic function of (X,Y) is equal to $c_{0,0}q(t,s)$ which means that (X,Y) are binormaly distributed.

Lemma 4. Suppose that (X,Y) satisfy (A) or (A*) and at the same time we have both $(X,Y)\in X$-GPRN and $(X,Y)\in Y$-GPRN. Let $\varphi_{X,Y}(t,s)$ denote characteristic function of (X,Y), then:

(i) $$\varphi_{X,Y}(s,t)=q(s,t)\sum_{n\geq 0}(-1)^n(ts)^n b_{n,n}(1-\rho^2)^n/(n!)^2,$$

(ii) $$c_{k,n}=\begin{cases}\binom{n}{\frac{n-k}{2}}(-\rho)^{(n-k)/2}b_{\frac{n+k}{2},\frac{n+k}{2}} & \text{if } n-k \text{ is even}\\ 0 & \text{if } n-k \text{ is odd or } k>n,\end{cases}$$

(iii) the distribution of random variable $U=(X-\rho Y)/\sqrt{(1-\rho^2)}$ is given by:

$$f(u)=p(u)\sum_{k\geq 0}(-\rho)^k\binom{2k}{k}b_{k,k}H_{2k}(u)/(2k)!.$$

(iv) joint distribution of (U,Y) is given by the following formula:

$$f(u,y) = p(u)p(y)\Big\{b_{0,0}+\sum_{s\geq 2}(-\rho)^s b_{s,s}/s!\times$$

$$\times\Big[\sum_{k=0}^{[s/2]}(1-\rho^2)^k\rho^{-2k}H_{2k}(y)H_{2s-2k}(u)/((2k)!(s-2k)!)-$$

$$- \sum_{k=0}^{[(s-1)/2]} (\sqrt{(1-\rho^2)})^{2k+1}\rho^{-2k-1}H_{2k+1}(Y)\times$$

$$\times H_{2s-2k-1}(u)/((2k+1)!(s-1-2k)!)\Big]\Big\} .$$

We will present now the proofs of the main results.

Proof of Theorem 1:

(a) Clearly we have $R^2=Y^2+U^2$ where $U=(X-\rho Y)/\sqrt{(1-\rho^2)}$. Since by assumption R^2 has exponential distribution we deduce that $g_{s,0}=0$, $s\ge1$ which together with (7) and Lemma 4 implies that $b_{k,k}=0$ $k\ge1$ or consequently $c_{k,n}=0$ for $k+n\ge1$.

(b) Simple calculation shows that $E(U^2|R)=E(Y^2|R)=R^2/2$ a.s. Thus $E(Y^2-U^2|R)=0=R^2\cos2\Phi$ where (R,Φ) are defined by $Y=R\cos\Phi$, $U=R\sin\Phi$. This however means that:

$$\forall\ r\in\mathbb{R}^+ \int_{-\pi}^{\pi}\cos(2\varphi)f(r,\varphi)d\varphi = 0$$

where $f(r,\varphi)$ distribution of (R,Φ).

By (6) we see that this implies that $g_{s,2}=0$ for $s \ge 0$. Inserting assertion (ii) of Lemma 4 into (8) we get for each $s\ge0$:

$$\forall s\ge0 \qquad 0=b_{s+1,s+1}\sum_{k=0}^{[(s+1)/2]}(1-\rho^2)^k(-\rho)^{s-k}F_s^{2k,2s+1-2k}\binom{2s+2-2k}{s+1-2k}$$

which implies that $b_{n,n}= 0$ for $n>0$ and consequently that $c_{k,n}= 0$ for all k,n such that $k+n\ge1$.

(c) In this case we have $E(Y(X-\rho Y)|R)=0$ a.s. which means that:

$$\forall\ r\in\mathbb{R}^+ \int_{-\pi}^{\pi}\sin(2\varphi)f(r,\varphi)d\varphi = 0.$$

Using similar arguments as in the proof of (b) we deduce that this condition leads to the conclusion that $c_{k,n}=0$ for all k,n such that $k+n\ge1$. ∎

Proof of Corollary 1.1. Condition $\forall n\geq 1\ E(X^n|R) = E(Y^n|R)$ a.s. implies that X and Y have the same distribution. Thus X can be exchanged with Y. Hence all the assumptions of Theorem 1 are satisfied. ∎

Proof of Theorem 2. Let random variables U,R,Φ be defined similarly as in the proof of Theorem 1. We will show that the assumptions lead to the conditions $g_{s,i}=0$ for all s and i such that $s+i\geq 1$, and $f_{s,i}=0$ for all s, $i\geq 0$. This fact will be shown by induction with respect to i.

1° $g_{s,0}=0$ for $s\geq 1$ since the density of R is equal to $r\exp(-r^2/2)$, $r\geq 0$.

2° Suppose that $g_{s,j}=0$ for each $s\geq 0$ and $j<i$. Since the condition $E(Y^k|R)=Q_k(R)$ a.s. $k\geq 1$ where Q_k is a polynomial of degree not exceeding k, means that

$$r^k \int_{-\pi}^{\pi}\cos^k\varphi\ f(r,\varphi)d\varphi = Q_k(r)r\exp(-r^2/2) \quad \text{for } r\in\mathbb{R}^+.$$

Now since: $$\cos^k\varphi = \begin{cases} 2^{-2n}\binom{2n}{n}+2^{1-2n}\sum_{j=1}^{n}\binom{2n}{n+j}\cos(2j\varphi), \text{if } k=2n \\ 2^{-2n}\sum_{j=0}^{n}\binom{2n+1}{n+j+1}\cos((2j+1)\varphi)\ , \quad \text{if } k=2n+1, \end{cases}$$

we deduce from (6) that for $i=2n+1$

$$Q_{2n+1}(r)=(r/2)^{2n+1}2\sum_{j=0}^{n}\binom{2n+1}{n+j+1}(r/2)^{2j+1}\ ,$$

$$\sum_{s\geq 0}\frac{(-1)^s g_{s,2j+1}}{2^{s-1}(2j+s+1)!}\ L_s^{(2j+1)}(r^2/2)\Big]=4(r/2)^{2n+1}(r/2)^{2n+1}\ ,$$

$$\sum_{s\geq 0}(-1)^s g_{s,2n+1}L_s^{(2n+1)}(r^2/2)/((2^{s-1}(2n+s+1)!)\ ,$$

because $g_{s,j}=0$ for $j<2n+1$. Hence we deduce that $g_{s,2n+1}=0$ for $s\geq 0$.

Similarly for i=2n we get:

$$Q_{2n}(r) =$$

$$=(r/2)^{2n}\left[\binom{2n}{n}g_{0,0}+2\sum_{j=0}^{n}\binom{2n}{n+j}(r/2)^{2j}\sum_{s\geq 0}\frac{(-1)^s g_{s,2j}}{2^{s-1}(2j+s)!}L_s^{(2j)}(r^2/2)\right]=$$

$$= (r/2)^{2n}g_{0,0}\binom{2n}{n}+2(r/2)^{2n}(r/2)^{2n}\binom{2n}{2n}\times$$

$$\times\sum_{s\geq 0}\frac{(-1)^s g_{s,2n}}{2^{s-1}(s+2j)!}L_s^{(2n)}(r^2/2)\ ,$$

from which we deduce that $g_{s,2n}=0$ for $s\geq 0$.

Since $g_{s,j}=0$ for $s+j\geq 1$ and since $\sin^k\varphi$ can be expressed as a finite linear combination of $\sin(k\varphi)$ and $\cos(k\varphi)$, $k\geq 0$ we deduce in the similar way that $f_{s,i}=0$ for every s, $i\geq 0$. Now by (7)...(11) we deduce that $c_{k,n}=0$ for k,n such that $k+n\geq 1$. ∎

Proof of Theorem 3. Since condition (B) is satisfied we deduce that the distribution g(x,y) of (X,Y) satisfies (A). Moreover g(x,y) can be expanded as follows:

$$g(x,y)=(1/(2\pi)\exp(-(x^2+y^2)/2)\sum_{n\geq 0}e_nH_n(x)/n!$$

where $e_n=E(H_n(X))$. Thus $c_{m,n}=0$ for $m\neq n$.

Now notice that condition (2) means in fact that $E(Y|R)=E(X|R)$ a.s. and $E(Y^2|R)=E(X^2|R)$ a.s. Using expansions (6) we see that this condition implies that $\forall s\geq 0$ $g_{s,1}= f_{s,1}=0$ which implies $\forall s\geq 0$ $e_{2s+1}=0$ and $\forall s\geq 0$ $g_{s,2}= f_{s,2}=0$ which implies $\forall s\geq 0$ $e_{2s}=0$. This completes the proof. ∎

Proof of Corollary 1.4. We use Theorem 4 and deduce that our assumptions assure that X and Y are normally distributed. Now notice that $(X,Y)\in Y$-PRN and $(X,Y)\in X$-GPRN hence we apply Theorem 3.1 of [2] to deduce that (X,Y) is binormal. ∎

Appendix

We will use elements of the theory of distributions, however contrary to original Schwartz's approach we will rather use sequential one as presented in [1]. Let $\mathcal{S}$ denote the space of all rapidly decreasing functions on $\mathbb{R}^2$ and let $\mathcal{S}'$ be its dual i.e. the space of all tempered distributions on $\mathbb{R}^2$.

Let us denote further $p(x)=\exp(-x^2/2)/\sqrt{2\pi}$, $x\in\mathbb{R}$.

$\forall\ \rho\in(0,1)$ $p_\rho(x,y)=\exp(-(x^2-2\rho xy+y^2)/(2(1-\rho^2)))/(2\pi\sqrt{1-\rho^2})$,

$$H_n(x)=n!\sum_{m=0}^{[n/2]}(-1)^m x^{n-2m}/(m!2^m(n-2m)!),$$

$h_n(x)=\exp(-x^2/4)/(2\sqrt{\pi})H_n(x)/(\sqrt{2\pi n!})$, $x\in\mathbb{R}$, $n=0,1,\ldots$.

Functions H_n and h_n will be called Hermite polynomials and Hermite (Hermite-Weber in fact) functions respectively.

If $\varphi\in\mathcal{S}$ and $\underline{g}\in\mathcal{S}'$ then $<\underline{g},\varphi>$ will denote the inner product. We will use also the following,more intuitive notation:

$$<\underline{g},\varphi>=\iint \underline{g}(x,y)\varphi(x,y)\,dxdy=\iint\varphi(x,y)\,dg(x,y).$$

Let (X,Y) be two standardized random variables defined on acommon probability space $(\Omega,\mathcal{F},P)$, such that $|EXY|<1$. Let $\underline{f}$ denote distribution of (X,Y) (in probabilistic sense). Notice that $\underline{f}$ is also a tempered distribution (in the sense of the theory of distributions) for we have:

$$\forall\ \varphi\in\mathcal{S}\ \ |<\underline{f},\varphi>|=|E\varphi(X,Y)|<\infty$$

because φ is a bounded function.

Expansion Theorem:

Suppose that $\exists\ C>0,k>0\ \forall\ n,m\ge 0$ either

(A) $\quad |EH_n(X)H_m((Y-\rho X)/\sqrt{1-\rho^2})|\le Cn^km^k\sqrt{n!m!}$

or

(A*) $\quad |EH_n(Y)H_m((X-\rho Y)/\sqrt{1-\rho^2})|\le Cn^km^k\sqrt{n!m!}$, then $\underline{f}(x,y)/\sqrt{p_\rho(x,y)}$ is a tempered distribution and moreover distribution $\underline{f}(x,y)$ can be expanded in the following unconditionally convergent Hermite series:

$$(1) \qquad (x,y)=p_\rho(x,y)\sum_{n=0}\sum_{m=0} c_{n,m}H_n(x)H_m((y-\rho x)/\sqrt{1-\rho^2})$$

if (A) is assumed

$$(1') \qquad \underline{f}(x,y)=p_\rho(x,y)\sum_{n=0}\sum_{m=0} \hat{c}_{n,m}H_n(y)H_m((x-\rho y)/\sqrt{1-\rho^2})$$

if (A*) is assumed where:

$$c_{n,m}=EH_n(X)H_m((Y-\rho X)/\sqrt{1-\rho^2}),\hat{c}_{n,m}=EH_n(Y)H_m((X-\rho Y)/\sqrt{1-\rho^2}).$$

Proof. First notice that under (A) we have:

$$2\pi\left|\iint\underline{f}(x,y)h_n(x)h_m((y-\rho x)/\sqrt{1-\rho^2})/(\sqrt{p_\rho(x,y)})dxdy\right| =$$

$$= |EH_n(X)H_m((Y-\rho X)/\sqrt{1-\rho^2})|/\sqrt{n!m!}|\le Cn^km^k.$$

Thus $\underline{g}=\underline{f}/\sqrt{p_\rho}$ is a tempered distribution (Thm.8.1.1 of [1]). Hence $\underline{g}$ can be expanded in the Hermite series

$$\underline{g}(x,y) = \sum_{n=0}\sum_{m=0} d_{n,m}h_n(x)h_m((y-\rho x)/\sqrt{1-\rho^2}) =$$

$$= \sqrt{p_\rho(x,y)}\sum_{n=0}\sum_{m=0} c_{n,m}H_n(x)H_m((y-\rho x)/\sqrt{1-\rho^2}),$$

where $d_{n,m}=c_{n,m}/\sqrt{(n!m!)}$.

Multiplying both sides of the above equality by $\sqrt{p_\rho}$ we get (1). We argue similarly when assuming (A*). ■

Remark 1. Assumptions (A) and (A*) are rather complicated. We will give thus an example of joint distribution that satisfy (A) and (A*).

Namely since $|H_n(x)|\le$const $\sqrt{n}!\exp(x^2/4)$ (Sasone [8] p.324) if (X,Y) are such that: $E\exp(X^2/4)$, $E\exp(Y^2/4)<\infty$ then (A) and (A*) are satisfied.

Proof of Lemma 1.

(i) Since $H_m(z)p(z)\sqrt{(2\pi)}=(-1)^m d^m(\exp(-z^2/2))/dz^m$ we get immediately

$$J_{k,n,m}(y)=\int H_n(\alpha z+\beta y)H_k(\gamma z+\delta y)d^m((-1)^m\exp(-z^2/2)).$$

After integrating this expression by parts (m times) we obtain:

$$J_{k,n,m}(y) = \int \exp(-z^2/2) \times$$

$$\times \left(\sum_{j=0}^{m} \binom{m}{j} \frac{n!}{(n-j)!}\alpha^j H_{n-j}(\alpha z+\beta y)\frac{k!}{(k-m+j)!}\gamma^{m-j} H_{k-m+j}(\gamma z+\delta y)\right) =$$

$$= m! \sum_{j=0}^{m} \binom{n}{j}\binom{k}{m-j}\alpha^j\gamma^{m-j}\int \exp(-z^2/2)H_{n-j}(\alpha z+\beta y)H_{k-m+j}(\gamma z+\delta y)dz.$$

Now we use formula: $H_k(\alpha z+\beta y)=\sum_{j=0}^{k}\binom{k}{j}\alpha^j\beta^{k-j}H_j(z)H_{k-j}(y)$

twice and get:

$$\int \exp(-z^2/2)H_{n-j}(\alpha z+\beta y)H_{k-m+j}(\gamma z+\delta y)dz=$$

$$= \sum_{t=0}^{n-j}\sum_{s=0}^{k-m+j}\binom{m-j}{t}\binom{k-m+j}{s}\alpha^t\delta^{k+j-m-s}\gamma^s\beta^{n-j-t}H_{n-j-t}(y)H_{k-m+j-s}(y)\times$$

$$\times\int \exp(-z^2/2)H_t(z)H_s(z)dz$$

which after some computation ends the proof of (i),

(ii)is a well known property of Hermite polynomials,

(iii) First let us make some remarks concerning functions $D_{n,m}(r,\varphi)$.

(a) If m is even then $D_{n,m}(r,.)$ is even. If m is odd then $D_{n,m}(r,.)$ is odd.

(b) Since H_n is a polynomial of degree n, $D_{n,m}(r,.)$ is a trigonometric polynomial of order at most n+m containing either only sinuses (m odd) or only cosines (m even).

(c) Since $H_n(x)$ contains only either even or only odd powers of x $D_{n,m}(r,\varphi)$ must be of the form:

$$D_{n,m}(r,\varphi)=\begin{cases}\sum_{t=0}^{[n/2]+[m/2]} P_t^{n,m}(r)\ \sin(n+m-2t)\varphi & \text{if m is odd}\\ \sum_{t=0}^{[n/2]+[m/2]} P_t^{n,m}(r)\ \cos(n+m-2t)\varphi & \text{if m is even.}\end{cases}$$

(d) We have:

$$r\partial D_{n,m}(r,\varphi)/\partial r = (n+m)D_{n,m}(r,\varphi)+n(n-1)D_{n-2,m}(r,\varphi)+ + m(m-2)D_{n,m-2}(r,\varphi),$$

$$\partial D_{n,m}(r,\varphi)/\partial\varphi = -nD_{n-1,m+1}(r,\varphi)+mD_{n+1,m-1}(r,\varphi),$$

$$D_{n,m}(r,\pi/2-\varphi) = D_{n,m}(r,\varphi).$$

From the above equations we get the following ones:

$$\text{(A1)}\quad \begin{cases} r(P_0^{n,m})'=(n+m)P_0^{n,m}\\ r(P_i^{n,m})=(n+m)P_i^{n,m}+n(n-1)P_{i-1}^{n-2,m}+m(m-1)P_{i-1}^{n,m-2}\\ \text{for } 0 < i \le [n/2]+[m/2].\end{cases}$$

$$\text{(A2)}\quad \begin{cases} \text{If } m=2h \text{ then}\\ (n+2h-2i)P_i^{n,2h}=nP_i^{n-1,2h+1}-2hP_i^{n+1,2h-1}\\ \text{If } m=2h+1 \text{ then}\\ (n+2h+1-2i)P_i^{n,2h+1}=-nP_i^{n-1,2h}+(2h+1)P_i^{n+1,2h}.\end{cases}$$

$$\text{(A3)}\quad P_i^{m,n}(r)=(-1)^{[n/2]+[m/2]+i}P_i^{n,m}(r);\ r\ge 0.$$

(e) We get also the following equalities by straightforward calculations:

$$P_i^{n,0}(r)=(-1)^i n!r^{n-2i}L_i^{(n-2i)}(r^2/2)/(2^{n-i-1}(n-i)!) \text{ for } 2i<n$$

$$P_n^{2n,0}(r)=(2n)!L_n(r^2/2)/(2^n n!).$$

From the first of the equations (A1) we get $P_0^{n,m}(r)=c_{n,m}r^{n+m}$.

Now since $P_0^{n,0}(r)=r^n 2^{-n+1}$, $P_0^{0,n}(r)=(-1)^{[n/2]}r^n 2^{-n+1}$ and because of (A2) we deduce that: $P_0^{n,m}(r)=(-1)^{[m/2]}r^{n+m}2^{-n-m+1}$.

Having $P_0^{n,m}(r)$, $P_i^{n,0}(r)$ $i\le n+m$ and $P_i^{0,m}(r)$, $n,m\in\mathbb{N}$ after some computations we can check that $P_i^{n,m}$ given by (3) satisfy (A1). ■

Proof of Corollary 1.2.

(i) $$f(p,q)=E\exp(ipU+iqY) =$$

$$= \exp(-(p^2+q^2)/2)\sum_{k,j\ge 0} c_{k,j}(ip)^j(iq)^k(\sqrt{(1-\rho^2)})^k/(k!j!),$$

where $U=(X-\rho Y)/\sqrt{(1-\rho^2)}$, $i^2=-1$.

It is easy to notice that $\varphi_{X,Y}(t,s) = f(t\sqrt{(1-\rho^2)},s+\rho t) =$

$$= q(t,s)\sum_{m\ge 0}\sum_{n\ge 0}((is)^m/m!)((it)^n/n!)\sum_{k=0}^{n}\binom{k}{n}\rho^k c_{k+m,n-k}\ .$$

(ii) $$f(r,\varphi)=(2\pi)^{-1}r\exp(-r^2/2)\sum_{k,n\ge 0} c_{k,n}D_{k,n}(r,\varphi)(\sqrt{(1-\rho^2)})^k=$$

$$= (2\pi)^{-1}r\exp(-r^2/2)\times$$

$$\times\Big\{\sum_{j\ge 0}\cos(2j\varphi)\sum_{s=j}\sum_{k=0}^{s} c_{2k,2s-2k}P_{s-j}^{2k,2s-2k}(r)(1-\rho^2)^k/((2k)!(2s-2k)!)+$$

$$+\sum_{j\ge 0}\cos((2j+1)\varphi)\sum_{s=j}\sum_{k=0}^{s} c_{2k+1,2s-2k}P_{s-j}^{2k+1,2s-2k}(r)(\sqrt{(1-\rho^2)})^{2k+1}/((2k+1)!(2s-2k)!)+$$

$$+\sum_{j\ge 0}\sin((2j+1)\varphi)\sum_{s=j}\sum_{k=0}^{s} c_{2k,2s-2k+1}P_{s-j}^{2k,2s-2k+1}(r)(1-\rho^2)^k/((2k)!(2s-2k+1)!)+$$

$$+\sum_{j\ge 0}\sin((2j+2)\varphi)\sum_{s=j}\sum_{k=0}^{s} c_{2+1,2s-2k+1}P_{s-j}^{2k+1,2s-2k+1}(r)(\sqrt{(1-\rho^2)})^{2k+1}/((2k+1)!(2s-2k+1)!)\Big\}.$$

Now we use Lemma 1 and change variables from s to s-i to get our assertion. ■

Proof of Lemma 3. Let $U=(X-\rho Y)/\sqrt{(1-\rho^2)}$, $V=(Y-\rho X)/\sqrt{(1-\rho^2)}$. First notice that $(X,Y)\in Y\text{-GPRN}$ (X-GPRN) means that

$$\forall\ k\geq 1\ E(H_k(U)\,|\,Y)=Q_k(Y)\ \text{a.s.}\ \ (E(H_k(V)\,|\,X)=Q'_k(X)\,\text{a.s.})$$

Similarly $(X,Y)\in Y\text{-PRN}$ ($(X,Y)\in X\text{-PRN}$) means that:

$$\forall\ k\geq 1\ E(H_k(U)\,|\,Y)=Q_{k-1}(Y)\ \text{a.s.}\ \ (E(H_k(V)\,|\,X)=Q'_{k-1}(X)\,\text{a.s.})$$

Secondly since $Y\sim N(0,1)$ $(X\sim N(0,1))$ we have $c_{k,0}=0$ & $b_{0,k}=0$ for $k\geq 1$ ($\hat{c}_{k,0}=0$, & $b_{k,0}=0$ for $k\geq 1$) and also that $\forall\ k\geq 1;\ y\in\mathbb{R}$ $\int H_k(u)f(u,y)du = p(y)Q_k(y)$ in case of $(X,Y)\in Y\text{-GPRN}$ and $\forall k\geq 1; y\in\mathbb{R}$ $\int H_k(u)f(u,y)du = p(y)Q_{k-1}(y)$ in case of $(X,Y)\in Y\text{-PRN}$.

Similarly for X-GPRN and X-PRN. Now we use expansion (1) and get:

$(X,Y)\in Y\text{-GPRN} \Leftrightarrow c_{k,n}=0$ for $k>n\geq 0$ and

$(X,Y)\in Y\text{-PRN} \Leftrightarrow c_{k,n}=0$ for $k\geq n\geq 0, k+n\geq 1$ since $c_{0,0}=0$ has to be excluded. Statements concerning $(X,Y)\in X\text{-GPRN}$ and $(X,Y)\in X\text{-PRN}$ are proved in the similar way. Now we use first assertion of Corollary 1.1 and get the desired assertions. ∎

Proof of Lemma 4. Our assumptions assure that $b_{n,m}=0$ for $|n-m|\geq 1$. Thus the first assertion of our lemma we get immediately. Corollary 1.1 gives $b_{n,m}$ in terms of $c_{k,n}$. It is easy to check that:

$$c_{k,n}=\sum_{j=0}^{n}\binom{n}{j}(-\rho)^j b_{n-j,k+j} \qquad k,n\geq 0$$

We get ii) from this formula. Since $c_{0,n}=0$ for $n\geq 1$ are the coefficients which appear in the expansion of $f(u)$ we get iii) from ii). To get iv) we insert ii) into (1) and after some algebra we get the desired formula. ∎

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