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GLUING OF DIFFERENTIAL SPACES

In this paper we study some geometric properties of the differential space which is obtained by gluing together differential spaces in the sense of Sikorski [13], [14]. In Section 1 we review some of the standard facts on Sikorski's differential spaces. In Section 2 we describe some basic notions and facts concerning the singularity which is obtained by gluing differential spaces together. In Section 3 some special cases are presented. The aim of the paper is to prepare mathematical methods of gluing differential spaces together for further applications in the theory of singularities of space-times [2], [1].

1. Preliminaries

Let M be any set and C be any non-empty set of real functions on M. By τ_C we denote the weakest topology on M in which all functions from C are continuous. For any subset $A \subset M$, let C_A be the set of all real functions β on A such that, for any $p \in A$, there exist an open neighborhood $U \in \tau_C$ of p and a function $\alpha \in C$ such that $\beta | A \cap U = \alpha | A \cap U$. By scC ([16]) we shall denote the family of all real functions on M of the form $\omega \circ (\alpha_1, \dots, \alpha_n)$, where $\omega \in \mathcal{E}_n$, $\alpha_1 \dots, \alpha_n \in C$, $n \in \mathbb{N}$ and $\mathcal{E}_n = C^\infty(\mathbb{R}^n)$.

A set C of real functions on M is called a differential structure on M if $C = C_M = \mathrm{sc}C$. The pair (M,C) is said to be a differential space [13]. A differential space (M,C) is said to be generated by C_0 if $C = (\mathrm{sc}C_0)_M$. If (M,C) is a differential space and A is a subset of M, then (A,C_A) is also a differential space, which is called the differential subspace of (M,C). By a tangent vector to (M,C), at a point $p \in M$, we shall mean any linear mapping $v: C \longrightarrow \mathbb{R}$ which satisfies the condition

$$v(\alpha\cdot\beta) = v(\alpha)\cdot\beta(p) + \alpha(p)\cdot v(\beta) \text{ for } \alpha,\beta \in C.$$

By T_pM we shall denote the tangent space to (M,C) at $p \in M$. A mapping $f: M \longrightarrow N$ of a differential space (M,C) into a differential space (N,D) is said to be smooth if $f^*(\alpha) := \alpha\circ f \in C$, for every $\alpha \in D$. A mapping $f: M \longrightarrow N$ is said to be a diffeomorphism of (M,C) onto (N,D) if f is a smooth bijection and f^{-1} is smooth.

If $f: M \longrightarrow N$ is smooth and $v \in T_pM$, then the formula

$$(f_{*p}v)\ (\alpha) = v(\alpha\circ f) \quad \text{for } \alpha \in D,$$

defines a vector $f_{*p}v$ tangent to (N,D) at f(p).

Let $\mathfrak{X}(M)$ be the C-module of all smooth vector fields tangent to (M,C). Now, let (M,C) be a differential space and $K \subset M$ a non-empty subset. By ι_K we denote the imbedding of the differential subspace (K,C_K) into (M,C).

Definition 1.1. A vector field $X\in\mathfrak{X}(M)$ is said to be tangent to the set K if for every $p\in K$ there exists a vector $v \in T_pK$ such that $X(p) = (\iota_K)_{*p}v$.

Let $\mathfrak{X}^K(M)$ be the set of all smooth vector fields tangent to K. Of course, $\mathfrak{X}^K(M)$ is a C-submodule of the C-module $\mathfrak{X}(M)$. If $K = \{p\}$ is a one - element set, then $\mathfrak{X}^K(M)$ is the C-submodule of all vector fields of $\mathfrak{X}(M)$, which vanish at p.

Definition 1.2. A subset $K \subset M$ is said to be full in (M,C) if $\dim T_pK = \dim T_pM$ for each $p \in K$.

It is easy to see that if K is full in (M,C), then $\mathfrak{X}^K(M) = \mathfrak{X}(M)$.

Proposition 1.3. For any vector field $X \in \mathfrak{X}^K(M)$ tangent to a subset K of M, there exists a unique vector field $Y \in \mathfrak{X}(K)$

such that

(1.1) $X(p) = (\iota_K)_{*p}\, Y(p)$ for any $p \in K$.

Proof. Let $Y: K \longrightarrow TK$ be the vector field defined by

(1.2) $Y(p) := (\iota_K)^{-1}_{*p}\,(X(p))$ for $p \in K$.

The smoothness of Y follows from

(1.3) $Y(f|K) = (Xf)|K$ for any $f \in C$.

Since $(\iota_K)_{*p}$ is a monomorphism for each $p \in K$, Y satisfying (1.1) is unique.

In the sequell the vector field Y defined by (1.1) will be called the restriction of $X \in \mathfrak{X}^K(M)$ to the subspace (K, C_K) and will be denoted by $X|K$.

Lemma 1.4. Let (M,C) be a differential space and $K \subset M$ be a non-empty subset. A vector field $X \in \mathfrak{X}(M)$ is tangent to K if and only if

(1.4) $\forall_{f,g \in C}\, [f|K = g|K \Longrightarrow Xf|K = Xg|K]$.

Proof. ($\Longrightarrow$) Let $X \in \mathfrak{X}^K(M)$. From Proposition 1.3 it follows that there exists a unique vector field $Y \in \mathfrak{X}(K)$ satisfying (1.1). Let $f,g \in C$ be arbitrary functions such that $f|K = g|K$. From (1.1) it follows that

$$X(p)(f) = Y(p)(f|K) = Y(p)(g|K) = X(p)(g),$$

for any $p \in K$. Hence $Xf|K = Xg|K$.

($\Longleftarrow$) Assume that $X \in \mathfrak{X}(M)$ is a vector field satisfying (1.4). It is enough to show that for an arbitrary point $p \in K$ there exists a vector $v \in T_pK$ such that $X(p) = (\iota_K)_{*p}v$. Indeed, let $v\colon C_K \longrightarrow \mathbb{R}$ be a mapping defined by

(1.5) $$v(\alpha) = \sum_{i=1}^{n} \omega'_{|i}(f_1(p),\dots,f_n(p))\cdot X(f_i)(p) \text{ for } \alpha \in C,$$

where $f_1,\dots,f_n \in C$, $\omega \in \mathcal{E}_n$ are functions such that $\alpha|U \cap K = \omega \circ (f_1,\dots,f_n)|U \cap K$ for some open neighborhood $U \in \tau_C$ of p.

From (1.4) it follows that (1.5) is correct. Of course, $v(f|K) = X(f)(p)$ for any $f \in C$ or equivalently $(\iota_K)_{*p}v = X(p)$.

Now, let ρ be an equivalence relation on (M,C) ([10], [16]). A function $f \in C$ is said to be consistent with ρ if $x\rho y$ implies $f(x) = f(y)$ for any $x,y \in M$. We denote by C_ρ the set of all $f \in C$ consistent with ρ. One can easily show that C is a differential structure on M. Let M/ρ denote the set of all equivalence classes of ρ and $\pi_\rho: M \longrightarrow M/\rho$ be the canonical mapping. We denote by $C/\rho := (\pi_\rho^*)^{-1}(C)$ the differential structure on M/ρ coinduced on M/ρ by the mapping π_ρ ([16], [10]). It is easy to show that $\pi_\rho^*|(C/\rho) : C/\rho \longrightarrow C_\rho$ is an isomorphism of algebras. A subset $A \subset M$ is called ρ-saturated if $\pi_\rho^{-1}(\pi_\rho(A)) = A$. Let us observe that the mapping $M/\rho \supset A \overset{I}{\longmapsto} \pi_\rho^{-1}(A) \subset M$ is a bijection between the family of ρ-saturated sets in M and the family of all subsets of M/ρ. Let us put $\mathfrak{M}_\rho := \left\{U \in \tau_C : U = \pi_\rho^{-1}(\pi_\rho(U))\right\}$. It is easy to see that $\mathfrak{M}_\rho = I(\tau_C/\rho)$, where τ_C/ρ is the quotient topology in the set M/ρ and $\tau_{C_\rho} = I(\tau_{C/\rho})$. We have $\tau_C/\rho = \tau_{C/\rho}$ if and only if $\mathfrak{M}_\rho = \tau_{C_\rho}$ [10]. Moreover, $\mathfrak{M}_\rho = \tau_{C_\rho}$ if and only if for any $U \in \mathfrak{M}_\rho$ and for any $p \in U$ there exists a function $\varphi \in C_\rho$ such that $\varphi(p) = 1$ and $\varphi|M-U = 0$ [9].

2. Some properties of gluing differential spaces together by diffeomorphism

Let (M_1,C_1) and (M_2,C_2) be differential spaces. Let $(N,D) = (M_1 \amalg M_2, C_1 \amalg C_2)$ be the disjoint union [15]. By definition, $f \in D$ iff $f|M_1 \in C_1$ and $f|M_2 \in C_2$. For $f_1 \in C_1$ and $f_2 \in C_2$ we denote by $f_1 \amalg f_2$ the real function on N such that $(f_1 \amalg f_2)|M_i = f_i$ for $i=1,2$. Let $\Delta_1 \subset M_1$ and $\Delta_2 \subset M_2$ be arbitrary non-empty subsets such that there exists a diffeomorphism $h: (\Delta_1, C_{1\Delta_1}) \longrightarrow (\Delta_2, C_{2\Delta_2})$. Let ρ_h be the equivalence relation on (N,D) identifying a point $p\in\Delta_1$ with $h(p) \in \Delta_2$. We denote by $[q]$ the equivalence class containing $q\in N$. Of course, $[p] = \{p,h(p)\}$ for $p\in\Delta_1$ and an equivalence class $[q]$ for $q\notin\Delta_1 \cup \Delta_2$ is a one-element set. The quotient space $(N/\rho_h, D/\rho_h)$ is called the gluing of the differential spaces (M_1,C_1) and (M_2,C_2) and it will be denoted by $(M_1\cup_h M_2, C_1\cup_h C_2)$. It can be seen that $D_{\rho_h} = \{f\in D: f|\Delta_1 = f\circ h\}$. For any $f_1\in C_1$ and $f_2\in C_2$ such that $f_1|\Delta_1 = f_2\circ h$, we denote by $f_1\cup_h f_2$ the function from $C_1\cup_h C_2$ corresponding to the function $f_1\amalg f_2\in D_{\rho_h}$ by the isomorphism $\pi^*_{\rho_h}|(D/\rho_h): D/\rho_h \longrightarrow D_{\rho_h}$.

Definition 2.1. Let K be a non-empty subset of a differential space (M,C). A differential subspace (K,C_K) is said to have the property of global extension in (M,C) if for any $f\in C_K$ there is a function $\bar{f}\in C$ such that $\bar{f}|K = f$.

Lemma 2.1. If (M,C) is a differential space with smooth partition of unity and $K\subset M$ is a non-empty closed subset of M, then (K,C_K) has the property of global extension in (M,C).

Proof is standard.

Lemma 2.2. Let (M_1,C_1) and (M_2,C_2) be disjoint differential spaces, and $h: (\Delta_1, C_{1\Delta_1}) \longrightarrow (\Delta_2, C_{2\Delta_2})$ be a diffeomorphism between respective differential subspaces. If $(\Delta_1, C_{1\Delta_1})$ and $(\Delta_2, C_{2\Delta_2})$ have the property of global extension in (M_1, C_1) and (M_2, C_2), respectively, then for any $f \in C_1$ there exists a function $\tilde{f} \in D_{\rho_h}$ such that $\tilde{f}|M_1 = f$ and for any $g \in C_2$ there is $\tilde{g} \in D_{\rho_h}$ such that $\tilde{g}|M_2 = g$.

Proof. Let $f \in C_1$. Since $h\colon (\Delta_1, C_{1\Delta_1}) \longrightarrow (\Delta_2, C_{2\Delta_2})$ is a diffeomorphism, $f \circ h^{-1} \in C_{2\Delta_2}$. From the property of global extension of $(\Delta_2, C_{2\Delta_2})$ in (M_2, C_2) it follows that there exists $\bar{f} \in C_2$ such that $\bar{f}|\Delta_2 = f \circ h^{-1}$. It is easy to see that the function $\tilde{f} := f \sqcup \bar{f} \in D_{\rho_h}$ and it satisfies $\tilde{f}|M_1 = f$.

Analogously, one can prove the existence of $\tilde{g} \in D_{\rho_n}$ for a function $g \in C_2$.

Now we prove

Proposition 2.3. Let (M_1, C_1) and (M_2, C_2) be differential spaces and $h\colon (\Delta_1, C_{1\Delta_1}) \longrightarrow (\Delta_2, C_{2\Delta_2})$ a diffeomorphism of respective differential subspaces.

Then the mappings defined by

$$\hat{\iota}_1 := \pi_{\rho_h}|M_1 \quad \text{and} \quad \hat{\iota}_2 := \pi_{\rho_h}|M_2, \tag{2.1}$$

are injective. Moreover, if $(\Delta_i, C_{i\Delta_i})$ has the property of global extension in (M_i, C_i), for $i=1,2$, then $\hat{\iota}_1$ and $\hat{\iota}_2$ are diffeomorphisms onto their images.

Proof. One can easily verify that $\hat{\iota}_1$ and $\hat{\iota}_2$ are monomorphisms. Let us put

$$\hat{M}_j = \hat{\iota}_j(M_j)\ , \quad \hat{C}_j = (C_1 \cup_h C_2)_{\hat{M}_j}, \text{ for } j=1,2. \tag{2.2}$$

Assume that $(\Delta_j, C_{j\Delta_j})$ has the property of global extension in (M_j, C_j), for $j=1,2$. We shall show that $\hat{\iota}_1\colon (M_1, C_1) \longrightarrow (\hat{M}_1, \hat{C}_1)$ and $\hat{\iota}_2\colon (M_2, C_2) \longrightarrow (\hat{M}_2, \hat{C}_2)$ are diffeomorphisms.

Let $\psi_j\colon \hat{M}_j \longrightarrow M_j$ be the inverse of $\hat{\iota}_j$, for $j=1,2$. Clearly,

$$f \circ \psi_j = \hat{f}|\hat{M}_j \text{ for } f \in C_j,\ j=1,2,$$

where $\hat{f} \in C_1 \cup_h C_2$ is a function such that $\hat{f} \circ \pi_{\rho_h}|M_j = f$. Therefore, ψ_j is smooth for $j=1,2$. Thus $\hat{\iota}_1$ and $\hat{\iota}_2$ are diffeomorphisms.

One can prove

Lemma 2.4. $f \in C_1 \cup_h C_2$ if and only if $f|\hat{M}_j \in \hat{C}_j$, for $j=1,2$.
Now, we prove

Proposition 2.5. Let (M_1,C_1) and (M_2,C_2) be differential spaces and $h: (\Delta_1, C_{1\Delta_1}) \longrightarrow (\Delta_2, C_{2\Delta_2})$ be a diffeomorphism between respective differential subspaces.
Then for any $p_1 \in \Delta_2$ and $p_2 = h(p_1) \in \Delta_2$,

(i) $$(\hat{\iota}_1)_{*p_1} T_{p_1} M_1 \cap (\hat{\iota}_2)_{*p_2} T_{p_2} M_2 = (\hat{\iota}_1|\Delta_1)_{*p_1} T_{p_1}\Delta_1 = (\hat{\iota}_2|\Delta_2)_{*p_2} T_{p_2}\Delta_2,$$

(ii) $$\dim\left((\hat{\iota}_1)_{*p_1} T_{p_1} M_1 + (\hat{\iota}_2)_{*p_2} T_{p_2} M_2\right) = \dim T_{p_1} M_1 + \dim T_{p_2} M_2 - \dim T_{p_1}\Delta_1.$$

Proof. (i) Since h is a diffeomorphism and $(\hat{\iota}_2|\Delta_2)\circ h = \hat{\iota}_1|\Delta_1$, we have $(\hat{\iota}_1|\Delta_1)_{*p_1} T_{p_1}\Delta_1 = (\hat{\iota}_2|\Delta_2)_{*p_2} T_{p_2}\Delta_2$.
The following inclusion is evident:

$$(\hat{\iota}_1|\Delta_1)_{*p_1} T_{p_1}\Delta_1 \subset (\hat{\iota}_1)_{*p_1} T_{p_1} M_1 \cap (\hat{\iota}_2)_{*p_2} T_{p_2} M_2.$$

It suffices to show that

$$(\hat{\iota}_1)_{*p_1} T_{p_1} M_1 \cap (\hat{\iota}_2)_{*p_2} T_{p_2} M_2 \subset (\hat{\iota}_{1|\Delta_1})_{*p_1} T_{p_1}\Delta_1.$$

Let $v \in (\hat{\iota}_1)_{*p_1} T_{p_1} M_1 \cap (\hat{\iota}_2)_{*p_2} T_{p_2} M_2$. Then there exists a unique pair of vectors $(u_1,u_2) \in T_{p_1} M_1 \times T_{p_2} M_2$ such that $v = (\hat{\iota}_1)_{*p_1} u_1 = (\hat{\iota}_2)_{*p_2} u_2$.

It can be seen that the following condition is satisfied:

(•) $\forall f_1 \in C_1, f_2 \in C_2,\ [(f_1|\Delta_1 = f_2\circ h) \Rightarrow u_1(f_1) = u_2(f_2)]$.

Now let $w: C_{1\Delta_1} \longrightarrow \mathbb{R}$ be a function given by

(2.3) $$w(g) = u_1(\tilde{g}|M_1) \quad \text{for } g \in C_{1\Delta_1},$$

where $\tilde{g} \in C_1$ is a function such that $\tilde{g}|\Delta_1 \cap U = g|\Delta_1 \cap U$ for some open neighborhood $U \in \tau_{C_1}$ of p_1. The correctness of definition (2.3) is a consequence of condition (•). It is easy to verify the equality

$$(2.4) \qquad (\hat{\iota}_1|\Delta_1)_{*p_1} w = (\hat{\iota}_1)_{*p_1} u_1 .$$

Thus $v = (\hat{\iota}_1)_{*p_1} u_1 = (\hat{\iota}_1|\Delta_1)_{*p_1} w \in (\hat{\iota}_1|\Delta_1)_{*p_1} T_{p_1}\Delta_1$. This finishes the proof of (i). It is evident that (ii) is a consequence of (i).

Proposition 2.6. If there exist homomorphisms $\bar{-}: C_{1\Delta_1} \longrightarrow C_1$ and $\bar{\bar{-}}: C_{2\Delta_2} \longrightarrow C_2$ satisfying the condition

$$(2.5) \qquad \bar{\alpha}|\Delta_1 = \alpha \text{ and } \bar{\bar{\beta}}|\Delta_2 = \beta \text{ for any } \alpha \in C_{1\Delta_1},\ \beta \in C_{2\Delta_2},$$

then for an arbitrary $p \in \hat{\Delta} := \hat{\iota}_1(\Delta_1)$,

$$T_p(M_1 \cup_h M_2) = (\hat{\iota}_1)_{*p_1} T_{p_1} M_1 + (\hat{\iota}_2)_{*p_2} T_{p_2} M_2,$$

where $p_1 \in \Delta_1$, $p_2 \in \Delta_2$ are points such that $p = [p_1] = [p_2]$.

Proof. Let $w \in T_p(M_1 \cup_h M_2)$. Let $v_1: C_1 \longrightarrow \mathbb{R}$, $v_2: C_2 \longrightarrow \mathbb{R}$, $v_3: C_{1\Delta_1} \longrightarrow \mathbb{R}$ be the functions defined by

$$(2.6) \qquad v_1(f) = w(f \cup_h \overline{\overline{f \circ h^{-1}}}) \text{ for } f \in C_1,$$

$$(2.7) \qquad v_2(g) = w(\overline{g \circ h} \cup_h g) \quad \text{for } g \in C_2,$$

$$(2.8) \qquad v_3(\alpha) = w(\bar{\alpha} \cup_h \overline{\overline{\alpha \circ h^{-1}}}) \text{ for } \alpha \in C_{1\Delta_1}.$$

It is easy to see that $v_1 \in T_{p_1}M_1$, $v_2 \in T_{p_2}M_2$ and $v_3 \in T_{p_1}\Delta_1$. Let us notice that every function $f_1 \cup_h f_2 \in C_1 \cup_h C_2$ can be presented in the form

$$(2.9) \qquad f_1 \cup_h f_2 = f_1 \cup_h \overline{\overline{f_1 \circ h}} + \overline{f_2 \circ h} \cup_h f_2 - \overline{f_2 \circ h} \cup_h \overline{\overline{f_1 \circ h^{-1}}}.$$

Hence

$$w(f_1 \cup_h f_2) = w(f_1 \cup_h \overline{\overline{f_1 \circ h}}) + w(\overline{f_2 \circ h} \cup_h f_2) - w(\overline{f_2 \circ h} \cup_h \overline{\overline{f_1 \circ h^{-1}}})$$

or equivalently

(2.10) $w(f_1\cup_h f_2) = (\hat{\iota}_1)_{*p_1} v_1 (f_1\cup_h f_2) + (\hat{\iota}_2)_{*p_2} v_2 (f_1\cup_h f_2) -$
$- (\hat{\iota}_1|\Delta_1)_{*p_1} v_3 (f_1\cup_h f_2),$

for any $f_1\cup_h f_2 \in C_1\cup_h C_2$.

Therefore,

$$w = (\hat{\iota}_1)_{*p_1} v_1 + (\hat{\iota}_2)_{*p_2} v_2 - (\hat{\iota}_1|\Delta_1)_{*p_1} v_3.$$

Thus $T_p(M_1\cup_h M_2)$ is generated by $(\hat{\iota}_1)_{*p} T_p M_1 \cup (\hat{\iota}_2)_{*p_2} T_{p_2} M_2$. This finishes the proof.

Denote by $V_1 = \hat{\iota}_1^{-1}(\mathrm{Int}\Delta_1)$ and $V_2 = \hat{\iota}_2^{-1}(\mathrm{Int}\Delta_2)$. Of course, $V_1 \in \tau_{C_1}$ is an open subset of Δ_1 and $V_2 \in \tau_{C_2}$ is an open subset of Δ_2. From Proposition 2.3 it follows that if $(\Delta_i, C_{i\Delta_i})$ has the property of global extension in (M_i, C_i), for $i=1,2$, then $\hat{\iota}_1|V_1 : V_1 \longrightarrow \mathrm{Int}\hat{\Delta}$ and $\hat{\iota}_2|V_2 : V_2 \longrightarrow \mathrm{Int}\hat{\Delta}$ are diffeomorphisms of the respective differential subspaces.

Proposition 2.7. Let (M_1, C_1) and (M_2, C_2) be disjoint differential spaces and $h: (\Delta_1, C_{1\Delta_1}) \longrightarrow (\Delta_2, C_{2\Delta_2})$ be a diffeomorphism of respective differential subspaces.
Then $\pi_{\rho_h} | (M_1 - \mathrm{cl}\Delta_1) \cup (M_2 - \mathrm{cl}\Delta_2)$ is a diffeomorphism of the subspace $\big((M_1 - \mathrm{cl}\Delta_1) \cup (M_2 - \mathrm{cl}\Delta_2), (C_1 \amalg C_2)_{(M_1 - \mathrm{cl}\Delta_1)\cup(M_2 - \mathrm{cl}\Delta_2)}\big)$ onto its image.

Proof. Let ψ be the inverse to the bijection $\pi_\rho | (M_1 - \mathrm{cl}\Delta_1) \cup (M_2 - \mathrm{cl}\Delta_2)$. It is enough to show that ψ is smooth. Let $f_1 \amalg f_2 \in C_1 \amalg C_2$ be an arbitrary element. It remains to show the smoothness of $(f_1 \amalg f_2) \circ \psi$. For any point $p \in (M_1 - \mathrm{cl}\Delta_1)) \cup (M_2 - \mathrm{cl}\Delta_2)$ there exist an open neighborhood U of p disjoint with $\Delta_1 \cup \Delta_2$ and a function $g \in D$ such that

$f_1 \amalg f_2 | U = g | U$ and $g | \Delta_1 \amalg \Delta_2 = 0$. Of course $U \in \mathfrak{M}_{\rho_h}$ and $g \in D_{\rho_h}$. It is easy to see that

$$(f_1 \amalg f_2)\circ\psi|\pi_\rho(U) = \hat{g}|\pi_\rho(U),$$

where $\hat{g} \in D/\rho_h$ is the function corresponding to g by isomorphism $\pi^*_{\rho_h}|(D/\rho_h)$ (i.e. $\hat{g} \circ \pi_{\rho_h} = g$).

Thus the composition $(f_1 \amalg f_2)\circ\psi$ is smooth for any $f_1 \amalg f_2 \in C_1 \amalg C_2$.

Now, we prove

Lemma 2.8. Let (M_1,C_1) and (M_2,C_2) be differential spaces and $h\colon \Delta_1 \longrightarrow \Delta_2$ be a diffeomorphism of differential subspaces. Then for any dense subset A of Δ_1, the linear ring $C_1 \cup_{h|A} C_2$ is isomorphic to the linear ring $C_1 \cup_h C_2$.

Proof. It is enough to prove that $D_{\rho_h} = D_{\rho_{h|A}}$. It is clear that $D_{\rho_h} \subset D_{\rho_{h|A}}$. Let $f_1 \amalg f_2 \in D_{\rho_{h|A}}$. It means that $f_1|A = (f_2 \circ h)|A$. Since A is a dense subset of Δ_1 and $f_1, f_2 \circ h$ are continuous, $f_1|\Delta = f_2 \circ h|\Delta$. Thus $f_1 \amalg f_2 \in D_{\rho_h}$. Hence $D_{\rho_{h|A}} \subset D_{\rho_h}$ and consequently $D_{\rho_{h|A}} = D_{\rho_h}$. Now, it is evident that the composition $(\pi^*_{\rho_{h|A}}|(D/\rho_{h|A}))^{-1} \circ \pi^*_{\rho_h}|((D/\rho_h))$ is an isomorphism of $C_1 \cup_h C_2$ and $C_1 \cup_{h|A} C_2$.

Remark. From Proposition 2.3 and Lemma 2.8 it follows that one can obtain "a new singular point" only on the boundary $\mathrm{Fr}\Delta$ of the set $\Delta := \hat{\iota}_1(\Delta_1)$. One can assume that Δ_1 and Δ_2 are closed subsets of M_1 and M_2, respectively.

In general, the topology $\tau_{C_1 \amalg C_2/\rho_h}$ is weaker than the quotient topology $\tau_{C_1 \amalg C_2}/\rho_h$.

Lemma 2.9. Let $h\colon (M_1,C_1) \longrightarrow (M_2,C_2)$ be a diffeomorphism. If the set Δ_1 is not closed in (M_1,C_1) and $\mathrm{Int}\Delta_1 \neq \emptyset$, then $\tau_{C_1 \amalg C_2/\rho_{h|\Delta_1}}$ is weaker than $\tau_{C_1 \amalg C_2}/\rho_{h|\Delta_1}$.

Proof. Let $p_0 \in \mathrm{cl}\Delta_1 - \Delta_1$ be an arbitrary point. There exist an open neighborhood $U_1 \in \tau_{C_1}$ of p_0 and an open set $U_2 \in \tau_{C_2}$ such that $h(p_0) \notin U_2$ and $U_2 \cap h(\Delta_1) = h(\Delta_1 \cap U_1)$.

Then the set $V := U_1 \cup h(U_1 \cap \mathrm{Int}\Delta_1) \in \mathfrak{M}_{\rho_h}$ is open in τ_D and $\rho_{h|\Delta_1}$-saturated but it does not belong to $\tau_{D_{\rho_{h|\Delta_1}}}$. In fact, there is no function $\varphi \in (C_1 \amalg C_2)_{\rho_{h|\Delta_1}}$ such that $\varphi(p_0) = 1$ and $\varphi(q) = 0$ for $q \notin V$.

Example. Let $h = id_{\mathbb{R}}: \mathbb{R} \longrightarrow \mathbb{R}$ and $\Delta_1 = (-\infty, 0)$. Then the set $(-1,1) \cup (-1,0) \in \mathfrak{M}_{h|\Delta_1}$, but it does not belong to $\tau_{D_{\rho_{h|\Delta_1}}}$.

Now, we describe the module of smooth vector fields tangent to $(M_1 \cup_h M_2, C_1 \cup_h C_2)$.

Definition 2.1. A pair $(X_1, X_2) \in \mathfrak{X}^{\Delta_1}(M_1) \times \mathfrak{X}^{\Delta_2}(M_2)$ is said to be h-consistent if $h_*(X_1|\Delta_1) = X_2|\Delta_2$.

Denote by $\mathfrak{X}_h(M_1, M_2)$ the subset of $\mathfrak{X}^{\Delta_1}(M_1) \times \mathfrak{X}^{\Delta_2}(M_2)$ of all h-consistent pairs. It is easy to see that $\mathfrak{X}_h(M_1, M_2)$ is a $C_2 \cup_h C_2$-module with the standard addition and multiplication defined by

$$(2.11) \qquad \varphi \cdot (X_1, X_2) = (\varphi \circ \hat{\iota}_1 \cdot X_1, \varphi \circ \hat{\iota}_2 \cdot X_2),$$

for $\varphi \in D/\rho_h$, $(X_1, X_2) \in \mathfrak{X}_h(M_1, M_2)$.

For any $(X_1, X_2) \in \mathfrak{X}_h(M_1, M_2)$, let $X_1 \cup_h X_2$ be the vector field tangent to $(M_1 \cup_h M_2, C_1 \cup_h C_2)$ defined by:

$$(2.12) \qquad (X_1 \cup_h X_2)([p]) = \begin{cases} (\hat{\iota}_1)_{*p} X_1(p) & \text{for } p \in M_1, \\ (\hat{\iota}_2)_{*p} X_2(p) & \text{for } p \in M_2. \end{cases}$$

It is easy to see that the vector field $X_1 \cup_h X_2$ is tangent to $\hat{M}_1, \hat{M}_2$ and $\Delta = \hat{M}_1 \cap \hat{M}_2$.

Lemma 2.10. Assume that $\hat{\iota}_1$ and $\hat{\iota}_2$ are imbeddings. Then the D/ρ_h-module $\mathfrak{X}^{\Delta}(M_1 \cup_h M_2)$ is isomorphic to the D/ρ_h-module $\mathfrak{X}_h(M_1, M_2)$.

Proof. Let $H : \mathfrak{X}_h(M_1,M_2) \longrightarrow \mathfrak{X}^{\Delta}(M_1 \cup_h M_2)$ be the mapping defined by

(2.13) $$H(X_1,X_2) = X_1 \cup_h X_2 \text{ for } (X_1,X_2) \in \mathfrak{X}_h(M_1,M_2).$$

For any $Y \in \mathfrak{X}^{\Delta}(M_1 \cup_h M_2)$, let $Y_1 \in \mathfrak{X}^{\Delta_1}(M_1)$ and $Y_2 \in \mathfrak{X}^{\Delta_2}(M_2)$ be the vector fields given by

(2.14) $$Y_1(p) = (\hat{\iota}_1)^{-1}_{*p}\, Y_1([p]) \quad \text{for } p \in M_1,$$

(2.15) $$Y_2(p) = (\hat{\iota}_2)^{-1}_{*p}\, Y_2([p]) \quad \text{for } p \in M_2.$$

It is clear that $H(Y_1,Y_2) = Y$. Thus H is an isomorphism.

Proposition 2.11. Assume that (M_1,C_1) and (M_2,C_2) are differential spaces of constant differential dimension. If $(\Delta_i, C_{i\Delta_i})$ has the property of global extension in (M_i,C_i), for $i=1,2$, then $\mathfrak{X}(M_1 \cup_h M_2) = \mathfrak{X}^{\Delta}(M_1 \cup_h M_2)$.

Proof. Let $X \in \mathfrak{X}(M_1 \cup_h M_2)$. Let $p_1 \in \Delta_1$ be an arbitrary point. We will show that $X([p_1]) \in (\hat{\iota}_1)_{*p_1} T_{p_1} M_1$. Let $U \in \tau_{C_1}$ be a neighborhood of p_1 such that there is a local vector basis $W_1,\dots,W_n \in \mathfrak{X}(U)$ of the C_1-module $\mathfrak{X}(M_1)$. Denote by $\partial(M_1 \cup_h M_2)$ the set of all singular points in the space $(M_1 \cup_h M_2, C_1 \cup_h C_2)$. From Lemma 2.8 it follows that the set $\partial(M_1 \cup_h M_2)$ is a boundary set. By Proposition 2.3, $\hat{\iota}_1$ and $\hat{\iota}_2$ are imbeddings. Let us put

$$\hat{W}_i = \hat{\iota}_{i*} W_i \quad \text{for } i=1,\dots,n.$$

Then

(2.16) $$X|\hat{U} - \partial(M_1 \cup_h M_2) = \sum_{i=1}^{n} \varphi^i \hat{W}_i,$$

where $\varphi^i : \hat{U} - \partial(M_1 \cup_h M_2) \longrightarrow \mathbb{R}$ is a real function for $i=1,\dots n$ and $\hat{U} = \hat{\iota}_1(U)$. Without loosing generality we may assume that there exist functions $\alpha_1,\dots,\alpha_n \in C_1$ such that $W_i(\alpha_j)=\delta_{ij}$, for $i,j=1,\dots,n$. For any $i \in \{1,\dots,n\}$, let $\hat{\alpha}_i$ be the function of $\hat{C}_1$ corresponding to α_i by the isomorphism $\hat{\iota}_1^*$. It is clear that $\hat{W}_i(\hat{\alpha}_i) = \delta_{ij}$, for i,j=1,...,n. Hence and from (2.16) it follows that

$$X(p)(\hat{\alpha}_j) = \varphi^j(p) \text{ for } p \in \hat{U} - \partial(M_1 \cup_h M_2), \quad j=1,\dots,n.$$

The set $\hat{U} \cap \partial(M_1 \cup_h M_2)$ is a boundary set in the subspace $\hat{U}$. Thus the function φ^i has the unique extension $\bar{\varphi}_i := (X|\hat{U})(\hat{\alpha}_i)$ on $\hat{U}$, for i=1,...,n. Therefore $X|\hat{U} = \sum_{i=1}^{n} \bar{\varphi}^i \hat{W}_i$. Hence $X([p_1]) \in (\hat{\iota}_1)_{*p_1} T_{p_1} M_1$. Analogously, one can prove that $X([p_2]) \in (\hat{\iota}_2)_{*p_2} T_{p_2} M_2$, where $p_2 = h(p_1)$. From Proposition 2.5 (i) and Proposition 2.3 it follows that $X([p_1]) \in T_{[p_1]}\Delta$.

3. Some special cases

Now let (M_1,C_1) and (M_2,C_2) be differential spaces and let $p_i \in M_i$, i=1,2, be arbitrary points. Let $\bullet$: $\{p_1\} \longrightarrow \{p_2\}$ be the natural diffeomorphism of one-element differential subspaces.

Let ρ_* be the equivalence relation on $(M_1 \amalg M_2, C_1 \amalg C_2)$ identifying the points p_1 and p_2. Denote by $(M_1 \cup_* M_2, C_1 \cup_* C_2)$ the quotient differential space modulo ρ_*.

For any $f_1 \in C_1$, let $\tilde{f}_1$: $M_1 \amalg M_2 \longrightarrow \mathbb{R}$ be the function defined by

$$\text{(3.1)} \qquad \tilde{f}_1(p) = \begin{cases} f_1(p) & \text{for } p \in M_1, \\ f_1(p_1) & \text{for } p \in M_2. \end{cases}$$

For any $f_2 \in C_2$, let $\tilde{f}_2$: $M_1 \amalg M_2 \longrightarrow \mathbb{R}$ be the function defined by

$$\text{(3.2)} \qquad \tilde{f}_2(p) = \begin{cases} f_2(p_2) & \text{for } p \in M_1, \\ f_2(p) & \text{for } p \in M_2. \end{cases}$$

Of course, $\tilde{f}_1$ and $\tilde{f}_2$ are consistent with ρ_*. Let $\hat{f}_1$ and $\hat{f}_2$ be the functions of $C_1 \cup_* C_2$ corresponding to $\tilde{f}_1$ and

$\tilde{f}_2$, respectively, by the isomorphism $\pi^*_{\rho_*}|(C_1 \cup_* C_2)$. Of course, $\hat{f}_i$ satisfies the condition

$$(3.3) \qquad \tilde{f}_i = \hat{f}_i \circ \pi_{\rho_*}, \quad i=1,2.$$

Now, we prove

Proposition 3.1 Let (M_1,C_1) and (M_2,C_2) be differential spaces, $p_i \in M_i$ an arbitrary point, for $i=1,2$. Then

(i) $\quad T_{p_*}(M_1 \cup_* M_2) = (\hat{\iota}_1)_{*p_1} T_{p_1} M_1 \oplus (\hat{\iota}_2)_{*p_2} T_{p_2} M_2$,

where $p_* = [p_1] = [p_2]$,

(ii) $\quad$ if p_1 and p_2 are non-isolated in $(M_1 \amalg M_2, C_1 \amalg C_2)$, then $X(p_*) = 0$, for any vector field $X \in \mathcal{X}(M_1 \cup_* M_2)$,

(iii) $\quad \tau_{C_1 \cup_* C_2} = \tau_{C_1 \amalg C_2}/\rho_*$,

(iv) $\quad$ if (M_i,C_i) is generated by C_i^0, $i=1,2$, then $(M_1 \cup_* M_2, C_1 \cup_* C_2)$ is generated by the set

$$\{\hat{f}_1 \colon f_1 \in C_1^0\} \cup \{\hat{f}_2 \colon f_2 \in C_2^0\}.$$

Proof. (i) Let $-\colon C_{1\Delta_1} \longrightarrow C_1$ and $=\colon C_{2\Delta_2} \longrightarrow C_2$, where $\Delta_1 = \{p_1\}$ and $\Delta_2 = \{p_2\}$, be the homomorphisms which are the identities (the image of a real number k is the constant function identically equal to k). It is obvious that these homomorphisms satisfy condition (2.5). From Proposition 2.6 it follows that

$$T_{p_*}(M_1 \cup_* M_2) = (\hat{\iota}_1)_{*p_1} T_{p_1} M_1 + (\hat{\iota}_2)_{*p_2} T_{p_2} M_2 .$$

By Proposition 2.5(i) we have

$$(\hat{\iota}_1)_{*p_1} T_{p_1} M_1 \cap (\hat{\iota}_2)_{*p_2} T_{p_2} M_2 = (\hat{\iota}_1 | \Delta_1)_{*p_1} T_{p_1} \Delta_1 = \{0\}.$$

Thus $T_{p_*}(M_1 \cup_* M_2)$ is a direct sum of $(\hat{\iota})_{*p_1} T_{p_1} M_1$ and $(\hat{\iota})_{*p_2} T_{p_2} M_2$.

(ii) For an arbitrary vector field $X \in \mathcal{X}(M_1 \cup_* M_2)$ let $X_1 \in \mathcal{X}(M_1)$ and $X_2 \in \mathcal{X}(M_2)$ be vector fields defined by

(3.4) $$X_1(\alpha) = X(\hat{\alpha})\circ\hat{\iota}_1 \quad \text{for } \alpha \in C_1,$$

(3.5) $$X_2(\beta) = X(\hat{\beta})\circ\hat{\iota}_2 \quad \text{for } \beta \in C_2,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are functions defined by (3.3). It is easy to see that

(3.6) $$X(\hat{\iota}_j(p)) = (\hat{\iota}_j)_{*p}X_j(p) \quad \text{for } p \in M-\{p_j\},\ j=1,2,$$

(3.7) $$X(p_*) = (\hat{\iota}_1)_{*p_1}X_1(p_1) + (\hat{\iota}_2)_{*p_2}X_2(p_2).$$

We will show that $X_j(p_j) = 0$, for $j=1,2$. From (3.6) it follows that

$$X(\hat{\alpha})\circ\pi_{\rho_*}|M_2-\{p_2\} = 0 \quad \text{for } \alpha \in C_1$$

and

$$X(\hat{\beta})\circ\pi_{\rho_*}|M_1-\{p_1\} = 0 \quad \text{for } \beta \in C_2.$$

Since p_1 and p_2 are non-isolated, $X(\hat{\alpha})\circ\pi_{\rho_*}|M_2 = 0$ and $X(\hat{\beta})\circ\pi_{\rho_*}|M_1 = 0$. Of course, $X(\hat{\alpha})\circ\pi_{\rho_*}$ and $X(\hat{\beta})\circ\pi_{\rho_*}$ are ρ_*-consistent functions. Hence $X(\hat{\alpha})\circ\pi_{\rho_*}(p_1) = 0$ and $X(\hat{\beta})\circ\pi_{\rho_*}(p_2) = 0$ for any $\alpha \in C_1, \beta \in C_2$. Thus by definition (3.4) and (3.5) we have

$$X_1(\alpha)(p_1) = X(\hat{\alpha})\circ\hat{\iota}_1(p_1) = 0 \quad \text{for any } \alpha \in C_1,$$

$$X_2(\beta)(p_2) = X(\hat{\beta})\circ\hat{\iota}_2(p_2) = 0 \quad \text{for any } \beta \in C_2.$$

Therefore, $X_1(p_1) = 0$ and $X_2(p_2) = 0$. Hence (3.7) gives $X(p_*) = 0$.

(iii) Let $U \in \mathfrak{M}_{\rho_*}$. It suffices to show that for any point $p \in U$ there exists a function $\varphi \in D_{\rho_*}$ such that

(3.8) $$\varphi(p) = 1 \text{ and } \varphi(q) = 0 \quad \text{for } q \notin U.$$

Assume that $p \in \{p_1,p_2\}$. For $i=1,2$, there exists a function $f_i \in C_i$ such that $f_i(p_i) = 1$ and $f_i|M_i-(U\cap M_i) = 0$. It is clear that the function $\varphi = f_1 \sqcup f_2$ is consistent with ρ_* and satisfies (3.8).

Now for instance, let $p \notin \{p_1,p_2\}$ and let $p \in U\cap M_1$.

There exists a function $g \in C_1$ such that $g(p)=1$, $g(p_1)=0$ and $g|M_1-(U\cap M_1)=0$. Let φ: $M_1 \amalg M_2 \longrightarrow \mathbb{R}$ be the function such that $\varphi|M_1 = g$ and $\varphi|M_2 = 0$. It is evident that φ is consistent with ρ_* and satisfies (3.8).

(iv) Let $f \in D/\rho_*$ be an arbitrary function. It suffices to show that f smoothly depends on a finite number of functions from the set $\left\{\hat{f}_1: f_1 \in C_1^o\right\} \cup \left\{\hat{f}_2: f_2 \in C_2^o\right\}$, in a neighborhood of p_*. There exist an open neighborhood $U_1 \in \tau_{C_1}$ of p_1, an open neigborhood $U_2 \in \tau_{C_2}$ of p_2, and functions $\alpha_1,\dots,\alpha_n \in C_1^o$, $\beta_1,\dots,\beta_n \in C_2^o$, $\theta_1 \in \mathcal{E}_n$, $\theta_2 \in \mathcal{E}_n$ such that

$$f \circ \pi_{\rho_*}|U_1 = \theta_1 \circ (\alpha_1,\dots,\alpha_n)|U_1,$$

$$f \circ \pi_{\rho_*}|U_2 = \theta_2 \circ (\beta_1,\dots,\beta_n)|U_2.$$

Clearly, the set $V := \pi_{\rho_*}^{-1}(U_1 \cup U_2)$ is an open neighorhood of p_*. It is easily seen that

$$f|U = \left(\theta_1 \circ (\hat{\alpha}_1,\dots,\hat{\alpha}_n) + \theta_2(\hat{\beta}_1,\dots,\hat{\beta}_n) - k\right)|U,$$

where $k = \theta_1 \circ \left(\alpha_1(p_1),\dots,\alpha_n(p_1)\right)$.

This finishes the proof.

Now, we prove

Proposition 3.2. Let (M_1,C_1) and (M_2,C_2) be differential spaces, $p_i \in M_i$ an arbitrary point, i=1,2. For a differential space $(Z,\mathcal{Z})$, let h: $\{p_1\}\times Z \longrightarrow \{p_2\}\times Z$ be the diffeomorphism defined by

(3.9) $$h(p_1,z) = (p_2,z) \quad \text{for } z \in Z.$$

Then

(i) $\tau_{C_1\times\mathcal{Z}\cup_h C_2\times\mathcal{Z}} = \tau_{C_1\times\mathcal{Z}\amalg C_2\times\mathcal{Z}/\rho_h}$,

(ii) the mapping $\Phi : M_1\times Z\cup_h M_2\times Z \longrightarrow (M_1\cup_* M_2)\times Z$ given by:

(3.10) $$\Phi\left([(p,z)]\right) = ([p],z) \quad \text{for } [(p,z)] \in M_1\times Z\cup_h M_2\times Z,$$

is a diffeomorphism of the differential space $(M_1\times Z\cup_h M_2\times Z, C_1\times\mathcal{Z}\cup_h C_2\times\mathcal{Z})$ onto the differential space

$((M_1 \cup_\bullet M_2)\times Z, (C_1 \cup_\bullet C_2)\times Z)$,

(iii) $T_{[(p_1,z)]}(M_1\times Z\cup_h M_2\times Z) = T_{p_1}M_1 \oplus T_{p_2}M_2 \oplus T_z Z$, for any $z \in Z$.

Proof. (i) Consider a set of the form $U\times W\cup V\times W$, where $U \in \tau_{C_1}$ is a neighborhood of p_1, $V \in \tau_{C_2}$ is a neighborhood of p_2 and $W \in \tau_Z$. Clearly, the set $U\times W\cup V\times W$ is ρ_h-saturated and open in $\tau_{C_1\times Z \sqcup C_2\times Z}$. We shall show that

$$\pi_{\rho_h}(U\times W\cup V\times W) \in \tau_{C_1\times Z\sqcup C_2\times Z}/\rho_h.$$

It suffices to show that for any point $(p_0,z_0) \in U\times W\cup V\times W$ there exists a function $\varphi \in C_1\times Z\sqcup C_2\times Z$ consistent with ρ_h such that

$$\varphi(p_0,z_0) = 1 \quad\text{and}\quad \varphi(p,z) = 0 \quad \text{for } (p,z) \notin U\times W\cup V\times W. \tag{3.11}$$

Assume that $p_0 \neq p_1$, $p_0 \neq p_2$ and $p_0 \in U$. There exists a function $f_1 \in C_1$ such that $f_1(p_0)=1$, $f_1(p_1)=0$ and $f_1|M_1-U = 0$. It is clear that the function $\varphi = \overline{f}_1\sqcup 0$ satisfies (3.11), where $\overline{f}_1\colon M_1\times Z \longrightarrow \mathbb{R}$ is the constant prolongation of f_1 with respect to Z onto $M_1\times Z$.

Now, let $p_0 = p_1$ or $p_0 = p_2$. There exist functions $\varphi_1 \in C_1$ and $\varphi_2 \in C_2$ such that $\varphi_1(p_1) = 1$, $\varphi_1|M_1-U=0$, $\varphi_2(p_2) = 1$ and $\varphi_2|M_2-V = 0$. It is evident that the function $\varphi = \overline{\varphi}_1\sqcup\overline{\varphi}_2$ satisfies (3.11), for the point (p_1,z_0) and (p_2,z_0), where $\overline{\varphi}_i$ is the constant extension of φ_i onto $M_i\times Z$, for $i=1,2$.

(ii) We will show that Φ and the inverse of Φ are smooth. To show the smoothness of Φ we shall verify the smoothness of Φ_1 and Φ_2, where Φ_1 and Φ_2 are the coordinates of $\Phi = (\Phi_1,\Phi_2)$. For any $f \in C_1$, let $f \in C_1\times Z\cup_h C_2\times Z$ be the function defined by

$$f([(p,z)]) = \begin{cases} f(p) & \text{for}(p,z)\in M_1\times Z,\\ f(p_1) & \text{for}(p,z)\in M_2\times Z.\end{cases} \tag{3.12}$$

For any $g \in C_2$, let $g \in C_1\times Z\cup_h C_2\times Z$ be the function defined by

$$(3.13)\qquad \hat{g}([(p,z)]) = \begin{cases} g(p_2) & \text{for}(p,z)\in M_1\times Z, \\ f(p) & \text{for}(p,z)\in M_2\times Z. \end{cases}$$

It is easy to see that, for any $f \in C_1$ and $g \in C_2$,

$$\hat{f}\circ\Phi_1 = f \text{ and } \hat{g}\circ\Phi_1 = g.$$

Since $C_1\cup_* C_2$ is generated by the set $\{\hat{f}\colon f \in C_1\} \cup \{\hat{g}\colon g \in C_2\}$ (see Proposition 3.1(iv)), Φ_1 is smooth.

For any $\psi \in \mathcal{Z}$, the composition $\check{\psi} \circ \Phi_2 = \psi$, where $\check{\psi} \in C_1 x\mathcal{Z}\cup_h C_2 x\mathcal{Z}$ is the function given by

$$(3.14)\qquad \check{\psi}([(p,z)]) = \psi(z) \quad \text{for } (p,z) \in M_1\times Z\amalg M_2\times Z.$$

Therefore, Φ_2 is smooth.

Denote by Ψ the inverse of Φ. It remains to show that, for any $F \in C_1 x\mathcal{Z}\cup_h C_2 x\mathcal{Z}$, the composition $F\circ\psi$ is smooth. It suffices to show that $F \circ \psi$ is smooth in a neighborhood of a point $[(p_1,z_0)]$, where $z_0\in Z$. It is clear that $F\circ\pi_{\rho_h} \in (C_1 x\mathcal{Z}\amalg C_2 x\mathcal{Z})_{\rho_h}$.

There exist a neighborhood $U \in \tau_{C_1}$ of p_1, a neighborhood $V \in \tau_{C_2}$ of p_2, a neighborhood $W \in \tau_{\mathcal{Z}}$ of z_0, functions $\psi_1,\dots\psi_k \in \mathcal{Z}$, $\alpha_1,\dots,\alpha_k \in C_1$, $\beta_1,\dots,\beta_k \in C_2$, $\omega,\theta \in \mathcal{E}_{2k}$, for some $k \in \mathbb{N}$, such that

$$F \circ \pi_{\rho_h}|U\times W = \omega \circ (\alpha_1,\dots,\alpha_k,\ \psi_1,\dots,\psi_k)|U\times W,$$
$$F \circ \pi_{\rho_h}|V\times W = \theta \circ (\beta_1,\dots,\beta_k,\ \psi_1,\dots,\psi_k)|V\times W.$$

It can be seen that

$$F \circ \pi_{\rho_h}|U\times W\cup V\times W = (\omega \circ (\tilde{\alpha}_1,\dots,\tilde{\alpha}_k,\psi_1,\dots\psi_k) +$$
$$+ \theta \circ (\tilde{\beta}_1,\dots,\tilde{\beta}_k,\psi_1,\dots\psi_k) -$$
$$- \omega(\alpha_1(p_1),\dots,\alpha_k(p_1),\psi_1,\dots\psi_k))|U\times W\cup V\times W,$$

where $\tilde{\alpha}_1,\dots,\tilde{\alpha}_k,\tilde{\beta}_1,\dots,\tilde{\beta}_k$ are the functions defined by (3.1) and (3.2).

Hence we have the following equality

$$F \circ \Psi|\pi_{\rho_h}(U\times W \cup V\times W) = (\omega \circ (\alpha,\dots,\alpha_k,\psi_1,\dots,\psi_k) +$$
$$+ \theta \ (\beta_1,\dots,\beta_k,\psi_1,\dots\psi_k) -$$
$$- \omega(\alpha_1(p_1),\dots,\alpha_k(p_1),\psi_1,\dots,\psi_k))|\pi_{\rho_h}(U\times W \cup V\times W).$$

Clearly, by (i) the set $\pi_{\rho_h}(U\times W \cup V\times W)$ is an open neighborhood of the point $[(p_1,z_0)]$. Thus $F \circ \psi$ is smooth in a neighborhood of $[(p_1,z_0)]$. Therefore $F \circ \psi$ is smooth.

(iii) Since by (ii) Φ is a diffeomorphism,

$\Phi_{*[(p_1,z)]} : T_{[(p_1,z)]}(M_1\times Z\cup_h M_2\times Z) \longrightarrow T_{([p_1],z)}((M_1\cup_* M_2)\times Z)$ is an isomorphism, for every $z \in Z$. Evidently, the tangent space $T_{([p_1],z)}(M_1\cup_* M_2\times Z)$ is isomorphic to the direct sum $T_{[p_1]}(M_1\cup_* M_2) \oplus T_z Z$ [15]. From Proposition 3.1 (i) it follows that $T_{[p_1]}(M_1\cup_* M_2)$ is isomorphic to $T_{p_1}M_1\oplus T_{p_2}M_2$. Therefore, $T_{[(p_1,z)]}(M_1\times Z\cup_h M_2\times Z)$ is isomorphic to the direct sum $T_{p_1}M_1\oplus T_{p_2}M_2\oplus T_z Z$.

Now, we prove

Proposition 3.3. Let $h\colon (M_1,C_1) \longrightarrow (M_2,C_2)$ be a diffeomorphism of differential spaces. Let $\Delta_1 \subset M_1$ be an arbitrary subset and $\Delta_2 = h(\Delta_1) \subset M_2$.
Then

(i) if Δ_1 is closed in τ_{C_1}, then $\tau_{C_1\cup_{h|\Delta_1}C_2} = \tau_{C_1 \amalg C_2}/\rho_{h|\Delta_1}$,

(ii) the mappings $\hat{\iota}_1\colon M_1 \longrightarrow M_1\cup_{h|\Delta_1}M_2$ and $\hat{\iota}_2\colon M_2 \longrightarrow M_1\cup_{h|\Delta_1}M_2$ defined by (2.1) are embeddings,

(iii) for any vector field $X \in \mathfrak{X}^{\Delta_1}(M_1)$, if $c\colon [-\varepsilon,\varepsilon] \longrightarrow M_1$, $\varepsilon > 0$, is an integral curve of X such that $c(0) \in \mathrm{Fr}\Delta_1$, then $\hat{\iota}_1 \circ c$ and $\hat{\iota}_1 \circ h \circ c$ are different integral curves of the vector field $\hat{X}:=X\cup_{h|\Delta_1}(h_*\circ X\circ h^{-1})$ corresponding to the pair

$(X,h_*\circ X\circ h^{-1}) \in \mathfrak{X}_{h|\Delta_1}(M_1,M_2)$ by (2.13).

Proof. We shall prove that $\mathfrak{M}_{\rho_{h|\Delta_1}} = \tau_{D_{\rho_{h|\Delta_1}}}$. Let $V \in \mathfrak{M}_{\rho_{h|\Delta_1}}$ and $p\in V$ be an arbitrary point. It is enough to find a function $\varphi \in D_{\rho_{h|\Delta_1}}$ such that

(•) $\varphi(p) = 1$ and $\varphi|(M_1 \amalg M_2)-V = 0$.

Without loosing generality let us assume that $p \in M_1$. There are sets $U_1 \in \tau_{C_1}$ and $U_2 \in \tau_{C_2}$ such that $p \in U_1$, $h(U\cap\Delta_1) = U_2\cap h(\Delta_1)$ and $h(U_1) \subset U_2$. Let $f \in C_1$ be a function such that $f(p_1) = 1$ and $f|M_1-U_1 = 0$. Then the function $\varphi := f\amalg(f\circ h^{-1}) \in D_{\rho_{h|\Delta_1}}$ and it satisfies (•). From Proposition 2 in [9] it follows that $\tau_{C_1\cup_{h|\Delta_1}C_2} = \tau_{C_1\amalg C_2}/\rho_{h|\Delta_1}$.

(ii) For any $f \in C_1$, let $\hat{f}$ be the function from $C_1\cup_{h|\Delta_1}C_2$ correspoding to the function $f\amalg(f\circ h^{-1}) \in D_{\rho_{h|\Delta_1}}$. For any $g \in C_2$, let $\hat{g} \in C_1\cup_{h|\Delta_1}C_2$ be the function corresponding to the function $(g\circ h)\amalg g \in D_{\rho_{h|\Delta_1}}$.One can easy verify the equalities

$$f \circ \psi_1 = \hat{f} \text{ and } g \circ \psi_2 = \hat{g} \text{ for any } f \in C_1,\ g \in C_2,$$

where ψ_j is the inverse of $\hat{\iota}_j$, for $j = 1,2$.

Now, anologously to the proof of Proposition 2.3 one can show that $\hat{\iota}_1$ and $\hat{\iota}_2$ are embeddings.

(iii) The proof of (iii) is straightforward.

4. Some remarks about quasiregular singularities

In the rewiev article [2] a classification of singularities of space-times is described. If a space-time is modelled by a differential space, space-time singularities can be regarded as points of the differential space. The theory of differential spaces opens some possibilities to classify singularities of a space-time. Such a classification is presented in [12]. The differential space methods turn out to be a very efficient tool to investigate the classical singularities of a space-time.

In [2] quasiregular singularities are presented: both elementary quasiregular singularities and more complicated ones. The basic idea in producing complicated quasi-regular singularities is to "glue" together elementary quasiregular singularities. Therefore, the gluing differential spaces together seems to be very important as far as applications of differential spaces to analysis of singularities of space-times are concerned. Now, we recall

Definition 4.1. [12] A pair $\big((\bar{M},\bar{C}),(M.g)\big)$ is said to be the differential space-time if $(\bar{M},\bar{C})$ is a differential space and (M,g) is a Lorentz submanifold, which is a dense differential subspace of $(\bar{M}.\bar{C})$.

The set $\partial M = \bar{M}-M$ is called the boundary of the differential space-time.

Example 4.1. Let (M_1,g_1) and (M_2,g_2) be space–times such that $\dim M_1 = \dim M_2$. Let $h\colon \Delta_1 \longrightarrow \Delta_2$ be a diffeomorphism of a closed boundary differential subspace $(\Delta_1,C^\infty(M_1)_{\Delta_1})$ of M_1 onto a closed boundary differential subspace $(\Delta_2,C^\infty(M_2)_{\Delta_2})$ of M_2. The pair $\big((M_1\cup_h M_2,C_1\cup_h C_2),(\hat{M}_1-\Delta,\tilde{g}_1)\amalg(\hat{M}_2-\Delta,\tilde{g}_2)\big)$ is a differential space-time with the boundary Δ, where $\tilde{g}_j := \psi_j^* g_j|\hat{M}_j-\Delta$, ψ_j is the inverse of $\hat{\iota}_j$ defined by (2.1), $\hat{M}_j = \hat{\iota}_j(M_j)$, for $j = 1,2$ and $\Delta := \hat{\iota}_1(\Delta_1)$.

Definition 4.2. A boundary point $p \in \partial M$ of $\big((\bar{M},\bar{C}),(M,g)\big)$ is said to be the strongly quasiregular singularity if for any rictifiable smooth curve $\gamma\colon [0,a] \longrightarrow \bar{M}$ satisfying $\gamma(a) = p$ and

$\gamma([0,a)) \subset M$ there exists a set U open in M and $\varepsilon>0$ such that $\gamma((\varepsilon,a)) \subset M$ and $(U \cup \{p\}, \overline{C}_{U\cup\{p\}})$ is a differential subspace of constant differential dimension $n = \dim M$.

A differential space-time $((\overline{M},\overline{C}),(M,g))$ is said to be a differential space-time with a strongly quasi-regular boundary if every point of the boundary ∂M is strongly quasiregular.

Now, we prove

Lemma 4.1. Let $((\overline{M}_1,\overline{C}_1),(M_1,g_1))$ and $((\overline{M}_2,\overline{C}_2),(M_2,g_2))$ be differential spacetimes with strongly quasiregular boundaries. Let $\Delta_1 \subset \overline{M}_1$ and $\Delta_2 \subset \overline{M}_2$ be closed subsets and
$h: (\Delta_1,\overline{C}_{1\Delta_1}) \longrightarrow (\Delta_2,\overline{C}_{2\Delta_2})$ be a diffeomorphism. Assume that $(\overline{M}_1,\overline{C}_1)$ and $(\overline{M}_2,\overline{C}_2)$ are differential spaces with smooth partition of unity and $\dim M_1 = \dim M_2 = n$. Then the pair
$((\overline{M}_1\cup_h\overline{M}_2,\overline{C}_1\cup_h\overline{C}_2),(\hat{M}_1-\mathrm{Fr}\Delta,\tilde{g}_1)\amalg(\hat{M}_2-\mathrm{Fr}\Delta,\tilde{g}_2))$,
where
$\tilde{g}_j:=\psi_j^* g_j|\hat{M}_j-\mathrm{Fr}\Delta$, $\hat{M}_j = \hat{\iota}_j(M_j)$, for $j = 1,2$, is a differential space-time with a strongly quasiregular boundary.

Proof. Of course, the set $\hat{\iota}_1(\partial M_1) \cup \hat{\iota}_2(\partial M_2) \cup \hat{\iota}_1(\mathrm{Fr}\Delta_1)$ is the boundary of $((\overline{M}_1\cup_h\overline{M}_2,\overline{C}_1\cup_h\overline{C}_2),(\hat{M}_1-\mathrm{Fr}\Delta,\tilde{g}_1)\amalg(\hat{M}_2-\mathrm{Fr}\Delta,\tilde{g}_2))$. Let $\gamma:[0,a] \longrightarrow \overline{M}_1\cup_h\overline{M}_2$ be a smooth rectitfiable curve ending at a boundary point p. From Lemma 2.1 and Proposition 2.3 it follows that the mappings $\hat{\iota}_1$ and $\hat{\iota}_2$ are embeddings.

Clearly, $\gamma([0,a)) \subset \hat{M}_1 - \mathrm{Fr}\Delta$ or $\gamma([0,a)) \subset \hat{M}_2-\mathrm{Fr}\Delta$. If $\gamma([0,a)) \subset \hat{M}_1-\mathrm{Fr}\Delta$, then $\hat{\iota}_1^{-1}\circ\gamma$ is a smooth curve in M_1 ending at $p_1 = \hat{\iota}_1^{-1}(p)$. There exist a set U open in M_1 and $\varepsilon>0$ such that $(\hat{\iota}_1^{-1}\circ\gamma)((\varepsilon,a)) \subset U$ and $(U\cup\{p\},C_{1U\cup\{p\}})$ is of constant differential dimension n. The set $\hat{U} = \hat{\iota}_1(U)$ is open in $\hat{\iota}_1(M_1)-\hat{\iota}_1(\mathrm{Fr}\Delta_1)$ and contains the set $\gamma((\varepsilon,a))$. Since $\hat{\iota}_1$ is a diffeomorphism of $\bar{M}_1$ onto $\hat{\iota}_1(\bar{M}_1)$, the differential space $(\hat{U}, (\bar{C}_1 \cup_h \bar{C}_2)_{\hat{U}\cup\{p\}})$ has the constant differential dimension $n = \dim M_1 = \dim (M_1 - \mathrm{Fr}\Delta_1)$. The prove in the case when $\gamma([0,a)) \subset \hat{M}_2 - \mathrm{Fr}\Delta$ is analogous.

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