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ON SOME APPROXIMATE METHOD FOR FREDHOLM'S INTEGRAL EQUATION OF THE SECOND KIND WITH BOUNDED KERNEL

1. Introduction

Application of the boundary element method to the elliptic boundary problem (c.f. [1]) requires an approximate solution of the Fredholm's integral equation of the second kind which, in the 1-dimensional case (m=1), has the form

$$\phi(x) - \int_a^b N(x,y)\phi(y)dy = f(x) \quad , \quad x \in \langle a;b\rangle \ .$$

The above mentioned approximate solution of this equation bases on solving the system of equations

$$\phi(x_i) - \sum_{j=1}^{n} \phi(x_j)\cdot\int_{a_{j-1}}^{a_j} N(x_i,y)dy = f(x_i)$$

with variables $\phi(x_1),\phi(x_2),\dots,\phi(x_n)$, where $a = a_0<a_1<a_2<\dots<a_{n-1}<a_n = b$, $x_1\in\langle a_0;a_1\rangle$, $x_i\in(a_{i-1};a_i\rangle$ for $i=2,3,\dots,n$.

In this paper the m-dimensional case, $m \ge 1$, is considered. We examine existence and uniqueness of the approximate solution and we estimate its error with respect to given approximate values of the functions N and f.

2. Notation and assumptions

Let m be a fixed positive integer and let R^m be the m-dimensional Euclidean space.

We assume that

(H_1) A set A is a bounded subset of R^m. Its boundary ∂A satisfies the condition: for every $\varepsilon > 0$ there exists a finite number of m-dimensional closed intervals $P_1, P_2, \dots, P_k$ whose Lebesgue's measures are equal to $|P_1|, |P_2|, \dots, |P_k|$, respectively and such that

$$\partial A \subset \bigcup_{i=1}^{n} P_i \quad \text{and} \quad |P_1| + |P_2| + \dots + |P_k| < \varepsilon .$$

We assume that the Lebesgue's measure of A is positive.

Let R(A) be the set of all functions $f:A \longrightarrow R$ Riemann integrable over A and let $\|f\| = \sup_{x\in A}|f(x)|$. We define in R(A) the operations of adding elements and multiplying an element by a real number in the usual way. Moreover let $\|f - g\|$ be the distance between f and g, $f,g \in R(A)$.

The set R(A) with the above defined operations, the norm and the distance is a Banach space.

Let $\|K\|$ be the norm of a linear operator $K:P \longrightarrow R(A)$, where P is a linear subspace of the space R(A).

Denote by Δ_n a partition of a set A into subsets $A_1, A_2, \dots, A_n$ such that the following assumption is satisfied:

(H_2) $A = \bigcup_{i=1}^{n} A_i$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and every set $A_1, A_2, \dots, A_n$ satisfies (H_1).

For a partition Δ_n of A, denote by $C_n(A)$ the set of all functions $h:A \longrightarrow R$ such that

$$h(x) = \sum_{i=1}^{n} c_i \cdot \chi_i(x),$$

where $(c_1, c_2, \dots, c_n) \in R^n$ and $\chi_i:A \longrightarrow R$, $i=1,2,\dots,n$, is the characteristic function for A_i, i.e.

$$\chi_i(x) = \begin{cases} 1, \text{if} x \in A_i \\ 0, \text{if} x \in A - A_i \end{cases} .$$

Let $x_i \in A_i$ be a given point of the set A_i, $i=1,2,\ldots,n$ and assume that the following assumptions are satisfied.

(H_3) $f \in R(A)$ and $N: A\times A \longrightarrow R$ is a Riemann integrable function, i.e. there exist the integrals: $\int_{A\times A} N(x,y)\,dxdy$, $\int_A N(x,y)\,dx$ for every $y \in A$ and $\int_A N(x,y)\,dy$ for every $x \in A$.

(H_4) For every function $f \in R(A)$ the Fredholm's equation of the second kind

$$(1) \qquad \phi(x) - \int_A N(x,y)\phi(y)\,dy = f(x)\ , \quad x \in A$$

($(I - N)\phi = f$ for short) has in $R(A)$ a unique solution ϕ.

(H_5) The number

$$(2) \qquad \delta(N,n) = \max_{i\in\{1,\ldots,n\}} \left[\sup_{x\in A_i} \int_{A_i} |N(x,y) - N(x_i,y)|\,dy \right]$$

is sufficiently small.

3. Existence of an approximate solution and error estimation

Theorem. If the assumptions (H_1)-(H_5) hold then for every function $f \in R(A)$ the system of equations

$$(3) \qquad c_i - \sum_{j=1}^{n} c_j \cdot \int_{A_j} N(x_i,y)\,dy = f(x_i)\ ,$$

$x_i \in A_i$, $i=1,2,\ldots,n$, has a unique solution $(c_1,c_2,\ldots,c_n) \in R^n$. If $\phi \in R(A)$ is a solution of the equation (1) then there exist constants d_1 and d_2 depending on functions N and f such that

$$\max_{i\in\{1,\ldots,n\}} |\phi(x_i) - c_i| \le \sup_{x\in A} |\phi(x) - \sum_{i=1}^{n} c_i \chi_i(x)|$$

$$\le d_1 \delta(N,n) + d_2 \cdot \max_{i\in\{1,\ldots,n\}} \sup_{x\in A_i} |f(x) - f(x_i)|$$

where $x_i \in A_i$, $i=1,2,\ldots,n$.

Proof. The system of equations (3) is equivalent to the equation

$$\phi_n(x) - \int_A N_n(x,y)\phi_n(y)dy = f_n(x),\ x\in A, \tag{5}$$

where

$$N_n(x,y) = \sum_{i=1}^{n} N(x_i,y)\chi_i(x), \tag{6}$$

$$f_n(x) = \sum_{i=1}^{n} f(x_i)\chi_i(x), \tag{7}$$

$$\phi_n(x) = \sum_{i=1}^{n} c_i\chi_i(x). \tag{8}$$

Notice that if functions N_n and f_n are defined by equalities (6) and (7), respectively and the function ϕ_n satisfies the equation (5) then ϕ_n has the form of (8). Since $C_n(A) \subset R(A)$, to prove existence and uniqueness of a solution of the equation (5) it suffices to show that for every function $g\in R(A)$ the Fredholm's equation

$$\psi(x) - \int_A N_n(x,y)\psi(y)dy = g(x),\ x\in A, \tag{9}$$

has a unique solution $\psi \in R(A)$.

Since the functions N, N_n, f and g are Riemann integrable, the Fredholm's Theorem applies to equations (1) and (9) (c.f. [2], pp. 47-67). It follows from these theorems and from the assumed uniqueness of a solution of (1) that the sum D(1) of the first Fredholm's series for the kernel N(x,y) is not equal to zero. Hence the only solution of (1) is

$$\phi(x) = f(x) + \int_A \mathfrak{R}(x,y,1)f(y)dy\ ,\ x \in A\ , \tag{10}$$

where the resolvent kernel

$$\mathfrak{R}(x,y,1) = \frac{D(x,y,1)}{D(1)}$$

is Riemann integrable and D(x,y,1) is the sum of the second

Fredholm's series for the kernel $N(x,y)$. Therefore for the linear operator $(I-N):R(A)\longrightarrow R(A)$ (where I is the identity operator) there exists a continuous inverse operator

$$(11)\qquad (I-N)^{-1} = I + \mathfrak{R}.$$

If the number $\delta(N,n)$ given by (2) satisfies the condition:

$$(12)\qquad \delta(N,n) \le \frac{1}{\|(I-N)^{-1}\|}$$

then, since $\|N-N_n\| \le \delta(N,n)$, uniqueness of the solution of the equation (9) in $R(A)$ follows by Theorem 4 in ([3], p.212). Indeed, put $X = Y = R(A)$, $U_0 = I - N$ and $U = N - N_n$. Then, for $V = U_0 + U = I - N_n$ there exists a continuous inverse operator $V^{-1} = (I - N_n)^{-1}$. Moreover

$$(13)\qquad \|(I-N)^{-1}\| = \|V^{-1}\| \le \frac{\|(I-N)^{-1}\|}{1-\|U_0^{-1}\|\cdot\|U\|} = \frac{\|(I-N)^{-1}\|}{1-\|(I-N)^{-1}\|\cdot\|N-N_n\|}\,.$$

This completes the proof of existence and uniqueness of solutions of equations (9) and (5) and the system of equations (3). Thus the sum $D_n(1)$ of the first Fredholm's series for the kernel $N_n(x,y)$ of the equation (9) is not equal to 0. Hence the only solutions ψ and ϕ_n of the equations (9) and (5), respectively, are

$$(14)\qquad \psi(x) = g(x) + \int_A \mathfrak{R}_n(x,y,1)g(y)dy,\ x\in A,$$

$$(15)\qquad \phi_n(x) = f_n(x) + \int_A \mathfrak{R}_n(x,y,1)f_n(y)dy,\ x\in A,$$

where $\mathfrak{R}_n(x,y,1) = \dfrac{D_n(x,y,1)}{D_n(1)}$ is a Riemann integrable function and $D_n(x,y,1)$ is the sum of the second Fredholm's series for the kernel $N_n(x,y)$. By (14) the linear operator $(I-N_n):R(A)\longrightarrow RA)$ has a continuous inverse operator

$$(16)\qquad (I-N_n)^{-1} = I + \mathfrak{R}_n.$$

According to equality (143) in [2], (p.78) we get

(17) $$\mathfrak{K}(x,y,1) = N(x,y) + \int_A \mathfrak{K}(x,s,1)N(s,y)ds$$

(18) $$\mathfrak{K}_n(x,y,1) = N_n(x,y) + \int_A \mathfrak{K}_n(x,s,1)N_n(s,y)ds$$

for $x,y \in A$. Notice that the equalities (10), (15)-(18) imply

(19) $$\phi(x) - \phi_n(x) = f(x) - f_n(x) + \int_A \mathfrak{K}_n(x,y,1)[f(y)-f_n(y)]dy + \int_A [\mathfrak{K}(x,y,1)-\mathfrak{K}_n(x,y,1)]f(y)dy$$

and

(20) $$\mathfrak{K}(x,y,1) - \mathfrak{K}_n(x,y,1) - \int_A N(s,y)[\mathfrak{K}(x,s,1)-\mathfrak{K}_n(x,s,1)]ds = r_n(x,y) ,$$

where

(21) $$r_n(x,y) = N(x,y) - N_n(x,y) + \int_A \mathfrak{K}_n(x,s,1)[N(s,y)-N_n(s,y)]ds$$

For an arbitrary fixed $x \in A$ the function

$$\psi_x(s) = [\mathfrak{K}(x,s,1) - \mathfrak{K}_n(x,s,1)] \in R(A)$$

and

$$r_n(x,y) \in R(A) .$$

Treating the equality (20) as a Fredholm's equation with respect to the function ψ_x and the associate to the equation (1) we get the equality

(22) $$\mathfrak{K}(x,y,1) - \mathfrak{K}_n(x,y,1) = r_n(x,y) + \int_A \mathfrak{K}(x,s,1)r_n(x,s)]ds$$

equivalent to (20).

It follows from (19), (22), (21), (11), (16) and (13) that the solutions $\phi \in R(A)$ and $\phi_n \in C_n(A)$ of the equations (1) and (5), respectively, satisfy the condition

$$\|\phi-\phi_n\| \le \|I+\mathfrak{K}_n\|\cdot\|f-f_n\| + \|(\mathfrak{K}-\mathfrak{K}_n)f\|$$
$$= \|I+\mathfrak{K}_n\|\cdot\|f-f_n\| + \|(I+\mathfrak{K}_n)\{(N-N_n)[(I+\mathfrak{K})f]\}\|$$

$$\le \|I+\mathfrak{K}_n\|\cdot\|f-f_n\| + \|I+\mathfrak{K}_n\|\cdot\|N-N_n\|\cdot\|I+\mathfrak{K}\|\cdot\|f\|$$

$$\le \frac{\|(I-N)^{-1}\|}{1-\|(I-N)^{-1}\|\cdot\|N-N_n\|}\cdot(\|f-f_n\|+\|N-N_n\|\cdot\|(I-N)^{-1}\|\cdot\|f\|).$$

This yields the inequality (4). If, for example,

$$\delta(N,n) < \frac{1}{2\cdot\|(I-N)^{-1}\|},$$

then, since $\|N-N_n\| \le \delta(N,n,)$, we can take $d_1 = 2\cdot\|f\|\cdot\|(I-N)^{-1}\|^2$ and $d_2 = 2\cdot\|(I-N)^{-1}\|$.

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