DEMONSTRATIO MATHEMATICA
Vol. XXV No 1-2 1992

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# ON EXPLOSION OF ONE DIMENSIONAL PROCESS UNDER ADDITIVE NOISE

Let $f:R^1 \to R^1$ be a continuous function and $W=(W_t)_{t\geq 0}$ a one dimensional Wiener process on a probability space $(\Omega,\mathcal{F},\mathcal{P})$. In this note we study a relationship between existence of a global or exploding solution X of the deterministic equation

(1) $$dX_t = f(X_t)dt,\ X_0=x$$

and of a global or exploding solution Y of the stochastic equation

(2) $$dY_t = f(Y_t)dt + dW_t,\ Y_0 = x.$$

We show that explosion of the deterministic solution X of (1) always implies explosion of the stochastic solution Y of (2). We construct also an example of a function f such that there exists a global solution X of (1) but the solution Y of (2) explodes with probability one. We give general conditions under which the converse implication takes place.

The purpose of this note is to prove the following theorems

**Theorem 1.** (i) Assume that f is locally Lipschitz positive function such that the solution X of (1) explodes. Then the solution Y of (2) explodes as well.

(ii) For arbitrary $\alpha>1$ there exists a smooth positive function f satisfying $f(x)\leq x^{\alpha}$ for large x and such that the solution Y of (2) explodes but the solution X of (1) exists globally.

**Theorem 2.** Let f be positive and differentiable function such that

(3) $$\int_0^{\infty}\exp\Big(-2\int_0^{v}f(s)ds\Big)\,dv < +\infty$$

and $\exists M>0$ such that

(4) $$-\frac{df}{dx} \le Mf^2(x),$$

then existence of a global solution of (1) implies existence of a global solution of (2).

**Remark.** Note that (3) is satisfied if, for instance, there exists $C>1/2$ such that

(5) $$f(x) \ge \frac{C}{x},$$

for x sufficiently large.

**Remark.** Conditions (3) and (4) are satisfied, in particular, by non-decreasing positive function f.

Before proving the theorems we need some preliminary results. To simplify notation we introduce the following operators defined on the space of continuous functions:

$$\Psi(f) = \int_0^{\infty}\frac{1}{f(s)}\,ds,$$

$$\Phi(f) = \int_0^{\infty}\exp\Big(-2\int_0^{v}f(s)ds\Big)\int_0^{v}\exp\Big(2\int_0^{u}f(s)ds\Big)\,du\,dv.$$

The following theorem is well known, see [1]-[3].

**Theorem 3.** (i) Assume that f is positive. Equation (1) has a global solution iff $\Psi(f)=+\infty$.

(ii) Equation (2) has a global solution iff $\Phi(f)=+\infty$.

**Remark.** It is easy to see that, if for all $x$, $f(x)\le g(x)$, then

$$\Phi(f) \ge \Phi(g) \quad \text{and} \quad \Psi(f) \ge \Psi(g).$$

**Proof of Theorem 1.** (i) To prove the theorem it is enough to show that $\Phi(f)\le\Psi(f)$ because of Theorem 3.

We express $\Phi$ as follows

$$\Phi(f) = \int_0^{\infty}\int_0^{v}\exp\Big(-2\int_u^{v}f(s)ds\Big)\,du\,dv = \int_0^{\infty}z(v)\,dv,$$

where

$$z(v) = \int_0^{v}\exp\Big(-2\int_u^{v}f(s)ds\Big)\,du, \quad v\ge 0.$$

The differentiable function $z$ is a unique, non-negative solution of the following problem

$$z'(v) = 1-2f(v)z(v),\ z(0)=0. \tag{6}$$

From (6) we have that $1-z'(v)>0$ for $v>0$ and

$$\frac{1}{f(v)} = \frac{2z(v)}{1-z'(v)} \quad \text{for} \quad v>0.$$

On the other hand

$$2z(v) = \left[\frac{2z(v)}{1-z'(v)}\right]^{1/2}[2z(v)(1-z'(v))]^{1/2} = \tag{7}$$

$$= \left[\frac{1}{f(v)}\right]^{1/2}[2z(v)(1-z'(v))]^{1/2}.$$

Integrating z and applying Hölder's Inequality for decomposition (7) we obtain for arbitrary $T>0$

$$\left(\int_0^T 2z(v)dv\right)^2 \le \int_0^T \frac{1}{f(v)}\,dv \int_0^T z(v)(1-z'(v))\,dv =$$

$$= \int_0^T \frac{1}{f(v)}\,dv \left[\int_0^T 2z(v)dv - z^2(T)\right] \le$$

$$\le \int_0^T \frac{1}{f(v)}\,dv \int_0^T 2z(v)dv.$$

Thus, as T converges to $\infty$, we get

$$2\int_0^\infty z(v)dv \le \int_0^\infty \frac{1}{f(v)}\,dv.$$

Hence $\Phi(f)\le\Psi(f)$ and so the proof is finished.

(ii) Now we show that the positive function $f$ such that the solution of (2) explodes and the solution of (1) does not explode can be found. Let $\alpha>1$ and

$$f(x) = \frac{8(x+1)^\alpha(1+\cos(x+1)^{\alpha+1})}{2+\sin(x+1)^{\alpha+1}+\cos(x+1)^{\alpha+1}},$$

for $x\ge0$, but for $x<0$ f is any positive continuous function. Since, for each starting point x, we can find such $x_0$ that $f(x_0)=0$ and $x<x_0$, then (1) has a global solution for every starting point. We will prove that $\Phi(f)<\infty$ and so solution of (2) explodes. To simplify calculations we take $\alpha=3$.

We will need the following function

$$\xi(x) = \exp(x+1)^4(2+\sin(x+1)^4 + \cos(x+1)^4).$$

By easy calculation we have

$$f(x) = \frac{d}{dx}(\ln\xi(x)).$$

Hence

$$\Phi(f) = \int_0^\infty \xi^{-2}(v)\int_0^v \xi^2(x)dx\, dv.$$

Since $\frac{1}{2} < 2+\sin(x+1)^4+\cos(x+1)^4 < 4$, then

$$\Phi(f) \le 64\int_0^\infty \exp(-2(v+1)^4)\int_0^v \exp(2(x+1)^4)dx\, dv =$$

$$= 64\int_0^\infty \exp(-2(v+1)^4)\int_0^v \sum_{k=0}^\infty \frac{(2(x+1)^4)}{k!}\, dx\, dv =$$

$$64\sum_{k=0}^\infty \frac{1}{k!(4k+1)}\int_0^\infty \exp(-2(v+1)^4)[2^k(v+1)^{4k+1}-1]dv.$$

To show that the above expression is finite it suffices to verify that

$$I_1 = \sum_{k=0}^\infty \frac{1}{k!(4k+1)}\int_0^\infty \exp(-2(v+1)^4)2^4(v+1)^{4k+1}dv$$

is finite. Putting $t=2(v+1)^4$ we have

$$I_1 = B\sum_{k=0}^\infty \frac{1}{k!(4k+1)}\int_0^\infty \exp(-t)\, t^{k-1/2}\, dt =$$

$$= B\sum_{k=0}^\infty \frac{1}{(4k+1)}\frac{\Gamma(k+1/2)}{\Gamma(k+1)},$$

where $B=2^{-5/2}$. Since $\Gamma^{(n)}(x)=(\ln x)^n\Gamma(x)$, for $n\in N$, then

$$\Gamma(k+1) \ge \Gamma(k+1)(1+\tfrac{1}{2}\ln(k+1/2) + \tfrac{1}{8}\ln^2(k+1/2)) \ge$$

$$\ge \tfrac{1}{8}\ln^2(k+1/2)\Gamma(k+1/2)$$

for $k\ge1$. Hence

$$I_1 \le B\Pi^{1/2} + B\sum_{k=1} \frac{1}{k+1/4}\frac{1}{\ln^2(k+1/4)}$$

but

$$\int_2^\infty \frac{1}{x\ln^2 x}\, dx < \infty .$$

For that reason $I_1<\infty$ and so $\Phi(f)<\infty$.

The fact that the function f can reach zero is not important. We show that we can find the positive function g such that $\Phi(g)<+\infty$ and $\Psi(g)=+\infty$.

Let $\{x_k\}_{k\in N}$ be a sequence of zeros of f (i.e. $f(x_k)=0$) and $\{I_k\}_{k\in N}$ be a sequence of intervals such that $x_k\in I_k$ for $k\in N$ and $I_k\cap I_l=0$ if $k\neq l$. One can verify that $f'(x_k)=0$, $k\in N$, therefore

$$\int_{I_k} \frac{1}{f(v)}\, dv = +\infty \quad \text{for each } k\in N.$$

Let $\gamma$ be positive and continuous function such that

$$\int_{I_k} \frac{1}{f(v)+\gamma(v)}\, dv = \alpha_k\ , \ k\in N,$$

with $\sum_{k=1}^{\infty}\alpha_k = +\infty$.

If we take $g=f+\gamma$, then g is positive and continuous function such that

$$\Psi(g) = \int_0^{\infty} \frac{1}{g(v)}\, dv \geq \sum_{k=1}^{\infty}\alpha_k = +\infty.$$

On the other hand $g\geq f$, so we obtain, $\Phi(g)\leq\Phi(f)$. Hence $\Phi(g)<+\infty$.

**Proof of Theorem 2.** We show that conditions (3) and (4) imply that there are positive constants $C_1$, $C_2$ such that

$$\Phi(f) + C_1 \geq C_2\Psi(f).$$

At first we notice that

$$\frac{d}{dv}\Big(\frac{1}{2f(v)} \exp\Big(2\int_0^v f(s)ds\Big)\Big) = \tag{8}$$

$$= \left(1 - \frac{f'(v)}{2f^2(v)}\right) \exp\Big(2\int_0^v f(s)ds\Big).$$

Condition (4) implies

$$\frac{d}{dv}\Big(\frac{1}{2f(v)} \exp\Big(2\int_0^v f(s)ds\Big)\Big) \leq \tag{9}$$

$$\leq(1+M) \exp\Big(2\int_0^v f(s)ds\Big).$$

Integrating both sides of (5) we obtain

(10) $$\frac{1}{2(1+M)f(v)}\exp\left(2\int_0^v f(s)ds\right) - \frac{1}{2(1+M)f(\emptyset)} \le \int_0^v \exp\left(2\int_0^u f(s)ds\right)du,$$

for each positive v. Hence,

(11) $$\Phi(f) \ge \int_0^\infty \exp\left(-2\int_0^v f(s)ds\right)\left[\frac{1}{2(1+M)f(v)}\exp\left(2\int_0^v f(s)ds\right) + - \frac{1}{2(1+M)f(\emptyset)}\right]dv = \frac{1}{2(1+M)}\Psi(f) + - \frac{1}{2(1+M)f(\emptyset)}\int_0^\infty \exp\left(-2\int_0^v f(s)ds\right)dv = C_2\Psi(f) - C_1$$

as a consequence of (3).

Applying Theorem 3 we complete the proof.

**Remark.** We can obtain similar result for f negative.

## Final comments

When the paper was ready the author has learned (indirectly) that the habilitation by M.Scheutzow [4] contains Theorem 1. Our proof of the first part of the theorem seems to be much shorter than in the habilitation and also the example presented in the proof of the second part of the theorem has additional properties.

Acknowledgement: I would like to thank Prof. J. Zabczyk for stimulating and helpful conversations.

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Received January 9, 1990, revised version November 9, 1990.