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PSEUDO-SEMIDIRECT PRODUCT OF THE ABSTRACT OBJECTS

In the present paper we define a pseudo-semidirect product of the abstract objects and state its properties. Especially, we prove, sufficient and necessary conditions for effectivity (Theorems 3, 4) and transitivity (Theorem 5) for the product.

1. Introduction

This paper is an attempt to give a definition of some kind of product of abstract objects.

Previous attempts were made by E. Kapustka ([2]) and B. Szociński ([7], [8]). This definition is connected with semidirect product of groups. Properties of the defined product are the motivation for the definition.

2. Preliminaries

2.1. An abstract object (object) is a triple (X,G,F), where X is a nonempty set, G is a group and $F:X\times G\longrightarrow X$ is a mapping satisfying the conditions:

(1) $$F(F(x,g_1),g_2) = F(x,g_2g_1), \text{ for } x\in X,\ g_1,g_2\in G$$

and

(2) $$F(x,e) = x, \text{ for } x\in X,$$

where e denotes the neutral element of the group G (cf. [4], p. 12).

An equivariant mapping from the object (X,G,F) into the object (Y,H,f) is a pair (α,φ), where $\alpha:X\longrightarrow Y$ is a mapping from X into Y and $\varphi:G\longrightarrow H$ is an homomorphism which satisfies the condition:

(3) $$\alpha(F(x,g)) = f(\alpha(x),\varphi(g)), \text{ for } x\in X,\ g\in G$$

(cf. [4], p. 18).

An object (X,G,F) is equivariant with an object (Y,H,f) if there exists a morphism $(\alpha,\varphi):(X,G,F)\longrightarrow(Y,H,f)$ such that α is a bijective mapping and φ is an isomorphism of groups.

2.2. A triple $\left(X,H,F|_{X\times H}\right)$ is a subobject of the object (X,G,F) if H is a subgroup of the group G.

A triple $\left(Y,G,F|_{Y\times G}\right)$ is a partial object of the object (X,G,F) if $Y\subset X$, $Y\neq\emptyset$ and $F(x,g)\in Y$, for all $x\in Y$ and $g\in G$ (cf. [4], p. 35, 37).

2.3. Let H and G denote groups. A mapping

$$\tau:H\times G\longrightarrow H \tag{4}$$

is called a homomorphism from the group G into the group of automorphism of H if it satisfies the conditions:

$$\tau(\tau(h,g_1),g_2)=\tau(h,g_2\cdot g_1),\ \text{for}\ h\in H,\ g_1,g_2\in G, \tag{5}$$

$$\tau(h,e_G)=h,\ \text{for}\ f\in H, \tag{6}$$

$$\tau(h_1*h_2),g) = \tau(h_1,g)*\tau(h_2,g),\ \text{for}\ h_1,h_2\in H,\ g\in G \tag{7}$$

(cf. [5], p. 10, [1], p.66).

If in the set $H\times G$ we define an action by the formula

$$(h,g)_\tau\lambda(h_1,g_1) := (h\cdot\tau_g(h_1),gg_1),$$

we obtain a group $H_\tau\lambda G$ called the semidirect product of the groups H and G (cf. [5], p. 12).

Conditions (4)-(6) mean that

$$(H,G,\tau) \tag{8}$$

is an object. The condition (7) means that for every fixed $g\in G$ the mapping $\tau_g:H\longrightarrow H$, $\tau_g(h)=\tau(h,g)$ is an automorphism of the group H (cf. [5], p. 11).

Let F denotes a mapping in the object (X,G,F). We will denote by F_g a mapping $F_g:X\longrightarrow X$, $F_g(x)=F(x,g)$, where $g\in G$ is fixed.

2.4. The direct product of the objects (X_1,G_1,F_1) and (X_2,G_2,F_2) is defined as a triple

$$(X_1\times X_2,G_1\times G_2,F_1\times F_2),$$

where

$$(F_1\times F_2)((x_1,x_2),(g_1,g_2)) := (F_1(x_1,g_1),F_2(x_2,g_2))$$

for $(x_1,x_2)\in X_1\times X_2$ and $(g_1,g_2)\in G_1\times G_2$ (cf. [5], p. 16). The G-product of the objects (X_1,G,F_1) and (X_2,G,F_2) is defined as a triple

$$(X_1\times X_2,G,F),$$

where $F((x_1,x_2),g) := (F_1(x_1,g),F_2(x_2,g))$ for $(x_1,x_2)\in X_1\times X_2$ and $g\in G$ (cf. [5], p. 18).

An object $(X,H_\tau\lambda\ G,f)$, where $H_\tau\lambda\ G$ is a semidirect product of groups, is called the semidirect product object.

We have

Theorem 1. (cf. [2]; [5], p. 19).

Let $(X,H_\tau\lambda\ G,f)$ be a semidirect product object. Then

$$f(x,(h,g)) = F_1(F(x,g),h), \tag{9}$$

where $F(x,g):=f(x,(e_H,g))$ and $F_1(x,h):=f(x,(h,e_G))$, $x\in X$, $g\in G$, $h\in H$.

The mappings F and F_1 satisfy the condition

$$F(F_1(x,h),g) = F_1(F(x,g),\tau_g(h)), \tag{10}$$

for $x\in X$, $h\in H$, $g\in G$ and the triples (X,G,F) and (X,H,F_1) are the objects.

Conversely, if (X,G,F) and (X,H,F_1) are objects such that F and F_1 satisfy (10) and f is a map defined by (9), then $(X,H_\tau\lambda G,f)$ is a semidirect product object. ∎

3. Definition of the pseudo semidirect-product of abstract objects

Let us assume that the homomorphism (4) and the objects:

$$(X,H,f_1), \tag{11}$$

$$(Y,G,f_2), \tag{12}$$

$$(X,G,\alpha), \tag{13}$$

where for each fixed $g\in G$ a pair (α_g,τ_g) is a morphism:

$$(\alpha_g,\tau_g):(X,H,f_1)\longrightarrow(X,H,f_1),$$

satisfy the condition

$$\alpha_g(f_1(x,h)) = f_1(\alpha_g(x),\tau_g(h)),\ \text{for}\ x\in X,\ h\in H. \tag{14}$$

Definition. An object

$$(X\times Y,\ H_\tau\lambda G,F), \tag{15}$$

where F is defined by the formula

(16) $$F((x,y),(h,g)):=(f_1(\alpha_g(x),h),f_2(y,g)$$

$$\text{for } (x,y)\in X\times Y,\ (h,g)\in H_\tau\lambda G,$$

is called a pseudo-semidirect product of the object (11) by the object (12) in relation to objects (13) and (8) and is denoted by

$$(X,H,f_1)_{(\alpha,\tau)}\lambda\ (Y,G,f_2) := (X\times Y,H_\tau\lambda G,F).$$

It is easy to verify that F given by (16) satisfies conditions (1) and (2).

Example. Let us consider the objects (8), (H,H,L_H), (G,G,L_G) where L_H and L_G are the left translations in H and G respectively (cf. [5], p. 27). In pseudo-semidirect product

(17) $$(H,H,L_H)_{(\tau,\tau)}\lambda\ (G,G,L_G) := (H\times G,H_\lambda G,L)$$

L is given by

$$L((h_1,g_1),(h,g)) := (L_H(\tau_g(h_1),h),L_G(g_1,g)) =$$

$$= (h\cdot\tau_g(h_1),g\cdot g_1) = (h,g)_\tau\lambda(h_1,g_1),$$

for $(h_1,g_1)\in H\times G$, $(h,g)\in H_\tau\lambda G$. So, L is the left translation in the group $H_\tau\lambda G$ and the object (17) is equivariant with the object $(H_\tau\lambda G,H_\tau\lambda G,L_{H_\tau\lambda G})$.

4. Properties of the pseudo-semidirect product of the abstract objects

Proposition 1. The object (11) is equivariant with the partial subobject

(18) $$\left(X\times\{y_0\},H_\tau\lambda\{e_G\},F\big|_{(X\times\{y_0\})\times(H_\tau\lambda\{e_G\})}\right)$$

of the object (15), where y_0 is an arbitrary fixed point of Y.

Property 1. A subobject

(19) $$\left(X\times Y,H_\tau\lambda\{e_G\},F\big|_{(X\times Y)\times(H_\tau\lambda\{e_G\})}\right)$$

of the object (15) is equivariant with the product

$$\left(X\times Y,H\times\{e_G\},f_1\times\left(f_2\big|_{Y\times\{e_G\}}\right)\right)$$

of the object (11) and scalar $\left(Y,\{e_G\},f_2\big|_{Y\times\{e_G\}}\right)$.

Notice that the object (18) is a partial object of the object (19).

Property 2. A subject

$$(X\times Y, \{e_H\}_\tau\lambda G, F|_{(X\times Y)\times(\{e_H\}_\tau\lambda G)}) \tag{20}$$

of the object (15) is equivariant with G-product

$$(X\times Y, G, F_G) \tag{21}$$

of the objects (13) and (12) where

$$F_G((x,y),g) := (\alpha(x,g), f_2(y,g)) \tag{22}$$

for $(x,y)\in X\times Y$ and $g\in G$.

Moreover, notice that if the object (11) is a scalar, then F given by (16) is the same as F_G given by (22), but the objects (21) and (15) are not equivariant (if not $H=\{e\}$).

Property 3. Assume that the object (13) contains a one--element orbit $\{x_0\}$. Then a partial subobject

$$(\{x_0\}\times Y, \{e_H\}_\tau\lambda G, F|_{(\{x_0\}\times Y)\times(\{e_H\}_\tau\lambda G)}) \tag{23}$$

of the object (15) is equivariant with the object (12).

Property 4. If $H_\tau\lambda G$ is a direct product $H\times G$ or the object (11) is a Klein space and the object (13) is a scalar then the object (15) is a direct product of the objects (11) and (12).

As a consequence of the Theorem 1 we have

Proposition 2. If

$$(X\times Y, H_\tau\lambda G, F_3) \tag{24}$$

is a $H_\tau\lambda G$ - product of the semidirect product objects $(X, H_\tau\lambda G, F_1)$ and $(Y, H_\tau\lambda G, F_2)$, where $F_3 := F_1\times F_2$, then $F_3((x,y),(h,g)) = (f_1(\alpha(x,g),h), \beta(f(y,g),h))$, for $(x,y)\in X\times Y$ and $(h,g)\in H_\tau\lambda G$, where f_1, α, β, f are mappings in the objects (X,H,f_1), (X,G,α), (Y,H,β), (Y, G, f) satisfying the condition (10).

If the object (Y,H,β) is a scalar then mapping F_3 is of the form of mapping in the pseudo-semidirect product of the object (X,H,f_1) by the object (Y,G,f) in relation to the objects (X,G,α) and (H,G,τ).

In the notation of the above Proposition 2 we have

Theorem 2. The following statements are equivalent

a) For every $y_0 \in Y$ a triple

(25) $$(X \times \{y_0\}, H_\tau \lambda \{e_G\}, \bar{F})$$

where $\bar{F} := F_3 \big|_{(X \times \{y_0\}) \times (H_\tau \lambda \{e_G\})}$ is a partial subobject of (24).

b) The object (Y, H, β) is a scalar.

c) The object (24) is a pseudo-semidirect product of the object (X, H, f_1) by the object (Y, G, f) in relation to the objects (X, G, α) and (H, G, τ).

Proof.

b)$\Longrightarrow$c) was proved in the Proposition 2 and c)$\Longrightarrow$a) follows from the Proposition 1.

a)$\Longrightarrow$b). For every fixed $y_0 \in Y$ and every $x \in X$, $h \in H$ we have

$$\bar{F}((x,y_0),(h,e_G)) = F_3((x,y_0),(h,e_G)) =$$
$$= (f_1(\alpha_{e_G}(x),h), \beta(f(y_0,e_G),h) = (f_1(x,h), \beta(y_0,h)).$$

Since (25) is a partial subobject of (24), $(f_1(x,h), \beta(y_0,h)) \in X \times \{y_0\}$, so $\beta(y_0,h) = y_0$. Therefore the object (Y, H, β) is a scalar. ∎

5. Effectivity of the pseudo-semidirect product of abstract object

M. Kucharzewski stated that if a semidirect product object is effective then its elements are effective (cf. [5], p. 22), but he posed a problem if the converse implication was true.

We will prove that it is not.

Let an object (X, G, f) be effective. Let the object (8) be of the form

(26) $$(G, G, \tau)$$

where $\tau_g(h) := ghg^{-1}$, for $g, h \in G$.

In a semidirect product object

(27) $$(X, G_\tau \lambda G, F)$$

we define the action F by the formula

$$F(x,(h,g)) := f(f(x,g),h), \quad \text{for} \quad x \in X,\ h,g \in G_\tau \lambda G.$$

It is easy to verify that F satisfies the condition (10). Since for every $g \in G$ and $x \in X$ we obtain $F(x,(g,g^{-1})) = f(x, g^{-1} \cdot g) = f(x,e) = x$, the object (25) is not effective (if not $G = \{e\}$).

In the case of pseudo-semidirect product of objects we prove

Theorem 3. If the object (15) is effective then the object (11) is effective.

Proof. For every $(x,y)\in X\times Y$, from $F((x,y),(h,g))=(x,y)$ it follows that $(h,g)=(e_H,e_G)$, because the object (15) is effective. Therefore, for $g=e_G$ we obtain that from

$$F((x,y),(h,e_G))=(f_1(\alpha(x,e_G),h),f_2(y,e_G))=(f_1(x,h),y)=(x,y)$$

it follows that $(h,e_G)=(e_H,e_G)$. So, for each $x\in X$ from $f_1(x,h)=x$ we have $h=e_H$. That proves effectivity of the object (11). ∎

Notice that if in the above proof we set $h=e_H$ we obtain that it follows from $(\alpha(x,g),f_2(y,g))=(x,y)$ that $(e_H,g)=$ $=(e_H,e_G)$, but we cannot prove that the object (12) or (13) is effective.

Theorem 4. If the objects (11) and (12) are effective then the object (15) is effective.

Proof. Assume that (11) and (12) are effective objects and $(h,g)\in H_\tau\lambda G$ is such that for each $(x,y)\in X\times Y$:

$$F((x,y),(h,g)) = (f_1(\alpha(x,g),h),f_2(y,g)) = (x,y).$$

So we obtain that from $f_2(y,g)=y$, for each $y\in Y$, it follows that $g=e_G$ and from $f_1(\alpha(x,e_G),h)=f_1(x,h)=x$, for each $x\in X$, it follows that $h=e_H$. ∎

6. Transitivity of the pseudo-semidirect product of abstract objects

M. Kucharzewski stated (cf. [5], p. 22) that a semidirect product object is transitive if at least one of its elements is transitive. As in the case of effectivity he posed a problem if the converse implication was true.

We will prove that it is not so.

Let an object

$$(X,G,f) \tag{28}$$

be transitive.

Let us consider two objects (not transitive)

(29) $$(X\times X, G, f_i),\ i=1,2$$

where $f_1((x,y),g):=(f(x,g),y)$ and $f_2((x,y),g):=(x,f(y,g))$, for $(x,y)\in X\times X$ and $g\in G$.

The homomorphism $\tau: G\times G \longrightarrow G$ we define by $\tau(h,g):=h$, for $h,g\in G$.

In the semidirect product object

(30) $$(X\times X, G_\tau\lambda G, F)$$

the mapping F is given by

$$F((x,y),(h,g)):=f_1((f_2(x,y),g),h)=(f(x,h),f(y,g)).$$

It is easy to verify that the object (30) is transitive.

In case of pseudo-semidirect product we have

Theorem 5. If the objects (11) and (12) are transitive then the object (15) is transitive.

If the object (15) is transitive then the object (12) is transitive.

Proof. Assume that (11) and (12) are the transitive objects. Transitivity of (12) implies that for every $(x,y),(x_1,y_1)\in X\times Y$ there exists $g\in G$ such that $f_2(y,g)=y_1$.

Let $\alpha(x,g):=x_2$. From transitivity of (11) we obtain that there exists $h\in H$ such that $f_1(x_2,h)=x_1$. That shows transitivity of the object (15).

If the object (16) is transitive then for $y,y_1\in Y$ and $x\in X$ there exists $(h,g)\in H_\tau\lambda G$ such that

$$F((x,y),(h,g)) = (f_1(\alpha(x,g),h),f_2(y,g)) = (x,y_1).$$

So, we have $g\in G$ such that $f_2(y,g)=y_1$. ∎

Notice that the object (15) could be transitive while (11) is not transitive.

Indeed, let (11) be an arbitrary scalar (X,H,f_1) such that there are $x_1,x_2\in X$, $x_1\neq x_2$, (12) be a scalar $(\{y\},G,f_2)$ and (13) be a transitive object (X,G,α). For every object (8) a mapping F of the object $(X\times\{y\},H_\tau\lambda G,F)$ is given by

$$F((x,y),(g,h)) = (\alpha(x,g),y)$$

and such object is transitive.

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