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Karol Pethe

THE ESTIMATION OF THE FOURTH COEFFICIENT IN THE CLASS OF p-VALENT FUNCTIONS

Let S_p be the class of functions

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + a_{p+3}z^{p+3} + \dots$$

holomorphic and p-valent in the unit circle $|z|<1$.

Theorem 1. If $f(z)$ is a function belonging to the class S_p then

$$|a_{p+3}| \le \frac{2}{3}\, p(p+1)(2p+1).$$

The equality sign is realised only for the function

$$f(z) = \frac{z^p}{(1+\eta\cdot z)^{2p}}, \quad |\eta|=1.$$

Proof. We take advantage of parametric presentation of the coefficients of the functions $f\in S_p$ [2], two identities of Z. Nehari [3] and the inequality of G.M. Goluzin [1].

K. Koseki in the work [2] examined a class of p-valent functions L defined as follows; function $f(z)$ belongs to the class L, if $f(z)=g(z,0)$, where $g(z,t)$ is the solution of the equation.

$$\frac{\partial g(z,t)}{\partial t} = \frac{\partial g(z,t)}{\partial z}\, z\, \frac{1+k(t)z}{1-k(t)z}, \quad 0\le t\le t_0<\infty$$

satisfying the condition $g(z,t_0)=z^p$, moreover $k(t)=e^{iQ(t)}$ and $Q(t)$ is real function and continuous and he showed, that the class of these functions is dense in the class S_p.

From this equation we obtain easily for the coefficient a_{p+3} the following formulas

(1) $$a_{p+3}= \int_0^{\infty} -2pe^{-3t}k^3(t)dt +$$

$$+ 4p(p+1)\left(\int_0^\infty e^{-t}k(t)dt\right)\left(\int_0^\infty e^{-2t}k^2(t)dt\right) -$$

$$- \frac{4}{3}p(p+1)(p+2)\left(\int_0^\infty e^{-t}k(t)dt\right)^3 + 4p\int_0^\infty e^{-2t}k^2(t)\left(\int_0^t e^{-s}k(s)ds\right)dt.$$

Z. Nehari [3] using the Loevner's method proved the inequality $|a_4| \le 4$ starting from the following identities:

$$(2) \qquad -\int_0^\infty u(u+v+A_1(\sigma-1))^2 dt = \frac{a_4}{2} - (\sigma+2)(2A_1A_2 - A_1^3) + \left(\frac{5}{3} - \sigma^2\right)A_1^3$$

$$(3) \qquad \int_0^\infty e^{-t}|u+v+A_1(\sigma-1)|^2 dt = \frac{1}{3} + |A_1|^2 \cdot |\sigma|^2,$$

where

$$(4) \qquad A_n = \int_0^\infty e^{-nt}k^n(t)dt, \quad n=1,2,\dots$$

$$(5) \qquad u = e^{-t}k(t), \quad v = \int_t^\infty e^{-s}k(s)ds$$

and σ is an arbitrary complex parameter.

From (1), using the notation (4) we obtain for a_{p+3} the formula

$$(6) \qquad a_{p+3} = -2pA_3 + 4p(p+1)A_1A_2 - \frac{4}{3}p(p+1)(p+2)A_1^3 +$$
$$+ 4p\int_0^\infty e^{-2t}k^2(t)\left(\int_t^\infty e^{-s}k(s)ds\right)dt.$$

Using the notations (4) and (5) we shall also base ourselves on two identities, which for p=1 will be identical with the identities (2) and (3). It can be easily verified that such identities have the form

$$(7) \qquad -\int_0^\infty u(u+v+A_1(\sigma-1))^2 dt = \frac{a_{p+3}}{2p} - (\sigma+p+1)(2A_1A_2 - A_1^3) +$$
$$+ \left(\frac{p(2p+3)}{3} - \sigma^2\right)A_1^3,$$

$$(8) \qquad \int_0^\infty e^{-t}|u+v+A_1(\sigma-1)|^2 dt = \frac{1}{3} + |A_1|^2|\sigma|^2,$$

where σ is also an arbitrary complex parameter.

From the identities (7) and (8) we obtain the inequality

$$(9) \qquad \left|a_{p+3} - 2p(\sigma+p+1)(2A_1A_2 - A_1^3) + 2p\left(\frac{p(2p+3)}{3} - \sigma^2\right)A_1^3\right| \le$$
$$\le \frac{2p}{3} + 2p|A_1|^2|\sigma|^2.$$

Without dimimishing the generalisation of the problem we can assume that $a_{p+3} \geq 0$. Then from (9) we obtain the inequality

$$a_{p+3} \leq \frac{2p}{3} + 2p|A_1|^2|\sigma|^2 + \mathrm{Re}\Big[2p(\sigma+p+1)(2A_1A_2-A_1^3) - - 2p\Big(\frac{p(2p+3)}{3} - \sigma^2\Big)A_1^3\Big]$$

from which follows

$$a_{p+3} \leq \frac{2p}{3} + 2p|A_1|^2|\sigma|^2 + 2p|\sigma+p+1||2A_1A_2-A_1^3| - - 2p\mathrm{Re}\Big[\Big(\frac{p(2p+3)}{3} - \sigma^2\Big)A_1^3\Big]. \tag{10}$$

Let us note that when $A_1=0$, then

$$|a_{p+3}| \leq \frac{2p}{3} \tag{11}$$

moreover we obtain the equality sign only for the function

$$f(z) = \frac{z^p}{(1+\eta z^3)^{\frac{2p}{3}}}, \quad |\eta|=1.$$

In the case when $A_1 \neq 0$ let us first conjecture the expression $|2A_1A_2 - A_1^3|$.

Thus let Σ_p be the class of function obtained from the class S_p by the known transformation

$$F(\zeta) = f^{-1}\Big(\frac{1}{\zeta}\Big) = \zeta^p\Big(1 + \frac{b_{-p+1}}{\zeta} + \frac{b_{-p+2}}{\zeta} + \dots\Big), \quad |\zeta|>1$$

$$\begin{cases} b_{-p+1} = -a_{p+1} = -2pA_1 \\ b_{-p+2} = -a_{p+2} + a_{p+1}^2 = 2pA_2 + 2p(p-1)A_1^2. \end{cases} \tag{12}$$

G.M. Goluzin [1] proved that if $F \in \Sigma_p$ then for $\lambda>0$ and $\rho>1$ we obtain the inequality

$$\frac{\partial}{\partial\rho}\int_0^{2\pi}|F(\rho e^{i\psi})|^\lambda d\psi \geq 0. \tag{13}$$

From this inequality for $\lambda=\frac{1}{p}$ and $\rho\longrightarrow 1$ we obtain

$$|b_{-p+1}|^2 + 3\Big|b_{-p+2} - \frac{2p-1}{4p}b_{-p+1}^2\Big|^2 \leq 4p^2, \tag{14}$$

whence after taking into account (12) and multiplying both sides by $|A_1|$ we obtain the inequality

$$|2A_1A_2-A_1^3| \leq \frac{2}{\sqrt{3}}|A_1|\sqrt{1-|A_1|^2} \tag{15}$$

moreover the equality sign is realised for the function

$$f(z) = \frac{z^p}{(1+\eta z)^{2p}}, \quad |\eta|=1.$$

The inequality (10) taking into account (15) takes the form

$$(16) \quad a_{p+3} \le \frac{2p}{3} + 2p|A_1|^2|\sigma|^2 + \frac{4p}{\sqrt{3}}|\sigma+p+1||A_1|\sqrt{1-|A_1|^2} - $$
$$-2p\cdot \mathrm{Re}\left[\left(\frac{p(2p+3)}{3} - \sigma^2\right)A_1^3\right].$$

Let $\sigma = -(p+1)e^{-\frac{3}{2}\varphi i}\cos\frac{3}{2}\varphi$, $A_1=-|A_1|e^{i\varphi}$ where $0<|A_1|\le 1$, then $|\sigma+p+1| = (p+1)|\sin\frac{3}{2}\varphi|$, $\mathrm{Re}\left[\left(\frac{p(2p+3)}{3} - \sigma^2\right)A_1^3\right] = \frac{p^2+3p+3}{3}|A_1|^3 + \frac{p^2-3}{3}\sin^2\frac{3}{2}\varphi$, and taking into account (16) we obtain the inequality

$$(17) \quad a_{p+3} \le \frac{2p}{3} + 2p(p+1)^2|A_1|^2 - \frac{2p}{3}(p^2+3p+3)|A_1|^3 -$$
$$- \frac{2p}{3}|A_1|^2\left(3(p+1)^2 - (p^2-3)|A_1|\right)\sin^2\frac{3}{2}\varphi +$$
$$+ \frac{4\sqrt{3}p(p+1)}{3}|A_1|\sqrt{1-|A_1|^2}\left|\sin\frac{3}{2}\varphi\right|.$$

Now if we determine maximum of the right side (17) over $|\sin\frac{3}{2}\varphi|$ then we obtain

$$(18) \quad a_{p+3} \le \frac{2p}{3} + 2p(p+1)^2|A_1|^2 - \frac{2p}{3}(p^2+3p+3)|A_1|^3 +$$
$$+ \frac{2p(p+1)^2(1-|A_1|^2)}{3(p+1)^2+(p^2-3)|A_1|}.$$

Now it will be sufficient to prove that

$$(19) \quad \frac{2}{3}p(p+1)(2p+1) \ge \frac{2p}{3} + 2p(p+1)^2|A_1|^2 - \frac{2p}{3}(p^2+3p+3)|A_1|^3 +$$
$$+ \frac{2p(p+1)^2(1-|A_1|^2)}{3(p+1)^2+(p^2-3)|A_1|}$$

for $0<A_1\le 1$ and $p\ge 1$.

The inequality (19) is equivalent the inequality

$$[(13p^4+33p^3-39p-15)|A_1|^2 + (14p^4+45p^3+36p^2-3p-6)|A_1| +$$
$$+(6p^4+21p^3+21p^2+3p-3)](1-|A_1|)^2 +$$

$$+ 6(2p^4+5p^3-5p-1)|A_1|^3(1-|A_1|) \geq 0.$$

It is evidently true and the equality sign is realised only for $|A_1|=1$. It denotes that

$$a_{p+3} \leq \tfrac{2}{3}p(p+1)(2p+1)$$

and the equality sign is realised only for the function

$$f(z) = \frac{z^p}{(1+\eta z)^{2p}}, \quad |\eta|=1.$$

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INSTITUTE OF MATHEMATICS, SILESIAN TECHNICAL UNIVERSITY,
44-100 GLIWICE, POLAND

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