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ON UNIFORMLY CONVEX AND UNIFORMLY 2-CONVEX 2-NORMED SPACES

In [6] the author investigated the generalizations of strictly convex and strictly 2-convex 2-normed spaces (defined in [3] and [4], respectively) without using the quotient spaces, or the bounded bilinear functionals techniques. In this paper we shall define a uniformly convex 2-normed space which is 2-dimensional analogue of the concept of a uniformly convex normed space defined in [1]. Consequently, as in the case of strictly 2-convex 2-normed space, it seems natural to define a uniformly 2-convex 2-normed space. The purpose of this paper is to present characterizations of these two spaces and to study relationships among other 2-normed spaces. Our main tools used here are basic properties of a 2-norm in a 2-normed space. As it is essential let us first repeat those definitions from [5].

Recall that, when X is a linear space of dimension greater than one and $\|.,.\|$ a real function on $X\times X$, then X is called a 2-normed space with a 2-norm $\|.,.\|$, if the following conditions are satisfied:

1° $\|x,y\|=0$ if and only if x and y are linearly dependent,

2° $\|x,y\|=\|y,x\|$,

3° $\|ax,y\|=|a|\,\|x,y\|$ for every real a,

4° $\|x+y,z\|\le\|x,z\|+\|y,z\|$.

We shall frequently use one of the basic properties that $\|ax+by,y\| = |a|\,\|x,y\|$ for any real numbers a and b ([5] p.5).

Definition 1. A 2-normed space X is said to be uniformly convex, if to each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,z\|=\|y,z\|=1$, $z\notin V(x,y)$ and $\|x-y,z\|\ge\varepsilon$, then $\frac{1}{2}\|x+y,z\|\le$

$\leq 1-\delta(\varepsilon)$, where $V(x,y)$ denotes the subspace of X generated by x and y.

Definition 1 implies that ε must be in the interval (0,2].

As an illustration, consider a 2-pre-Hilbert space, i.e., 2-inner product space, which is characterized by the parallelogram law (Theorem 4 and 5 [2]), i.e.,

$$\|x+y,z\|^2+\|x-y,z\|^2 = 2(\|x,z\|^2+\|y,z\|^2).$$

If $\|x,z\|=\|y,z\|=1$, $z\notin V(x,y)$ and $\|x-y,z\|\geq\varepsilon$, then

$$\frac{1}{4}\|x+y,z\|^2 = 1-\frac{1}{4}\|x-y,z\|^2 \leq 1-\frac{1}{4}\varepsilon^2.$$

Thus, by letting $\delta(\varepsilon)=1-\frac{1}{2}\sqrt{4-\varepsilon^2}$, we see that every 2-pre-Hilbert space is uniformly convex.

Theorem 1. For a 2-normed space X the following four statements are equivalent:

(1) X is uniformly convex;

(2) to each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,z\|=\|y,z\|$, $z\notin V(x,y)$ and $\|x-y,z\|\geq\varepsilon\|x,z\|$, then

$$\frac{1}{2}\|x+y,z\|\leq(1-\delta(\varepsilon))\|x,z\|;$$

(3) to each $\varepsilon>0$ there corresponds a $\delta'(\varepsilon)>0$ such that, if $\|x-ay,z\|\geq\varepsilon\|x,z\|$, where $a=\frac{\|x,z\|}{\|y,z\|}$, and $z\notin V(x,y)$, then

$$\frac{1}{2}\|x+ay,z\|\leq(1-\delta'(\varepsilon))\|x,z\|;$$

(4) to each $\varepsilon>0$ there corresponds a $\delta'(\varepsilon)>0$ such that, if $\|x-ay,z\|\geq\varepsilon\|x,z\|$, with a from (3) and $z\notin V(x,y)$, then

$$\|x+y,z\|\leq\|x,z\|+\|y,z\|-\delta'(\varepsilon)\|x,z\| \quad \text{or}$$

$$\|x+y,z\|\leq 3\|x,z\|-\|y,z\|-\delta'(\varepsilon)\|x,z\|.$$

Proof. (1)⇒(2): Let $\|x,z\|=\|y,z\|=c$, $c\neq 0$ and $z\notin V(x,y)$, then $\|x,\frac{z}{c}\|=\|y,\frac{z}{c}\|=1$ and $\|x-y,\frac{z}{c}\|\geq\varepsilon$ imply that $\frac{1}{2}\|x+y,\frac{z}{c}\|\leq$ $\leq 1-\delta(\varepsilon)$, by (1), i.e., $\frac{1}{2}\|x+y,z\|\leq(1-\delta(\varepsilon))\|x,z\|$.

(2)⇒(3): $\|x,z\|=\|ay,z\|$ obviously. Replace y by ay in (2) to get (3).

(3)⇒(4): Since $\frac{1}{2}\|x+ay,z\|\leq(1-\delta'(\varepsilon))\|x,z\|$, by (3), and if

$\|y,z\|\geq\|x,z\|$, then

$$\|x+y,z\| = \|x+ay+y(1-a),z\| \leq \|x+ay,z\|+(1-a)\|y,z\| \leq$$
$$\leq 2(1-\delta'(\varepsilon))\|x,z\|+\|y,z\|-\|x,z\| = \|x,z\|+\|y,z\|-2\delta'(\varepsilon)\|x,z\|.$$

Let $\delta(\varepsilon)=2\delta'(\varepsilon)$, so that the first inequality follows. In the case of $\|y,z\|\leq\|x,z\|$ we have

$$\|x+y,z\|\leq\|x+ay,z\|+(a-1)\|y,z\|\leq 3\|x,z\|-\|y,z\|-2\delta'(\varepsilon)\|x,z\|.$$

(4)⇒(1): If $\|x,z\|=\|y,z\|=1$, then $\|x+y,z\|\leq 2-\delta'(\varepsilon)$, by (4). We may let $\delta(\varepsilon)=\frac{1}{2}\delta'(\varepsilon)$.

Corollary 1. Let X be an uniformly convex 2-normed space. If $x_i\in X$, $i=1,2,\ldots,n$, $y=\sum_{i=1}^{n}x_i$ and $\|x_k,z\|\leq\|y-\sum_{j=1}^{k}x_j,z\|$ for $k=1,\ldots,n-1$, then

$$\|y,z\| \leq \sum_{i=1}^{n}\|x_i,z\| - \sum_{k=1}^{n-1}\delta(\varepsilon_k)\|x_k,z\|$$

for $z\in X$ with $z\notin V(x_1,x_2,\ldots,x_n)$, where

$$\varepsilon_k = \left\| \frac{x_k}{\|x_k,z\|} - \frac{y-\sum_{j=1}^{k}x_j}{\left\|y-\sum_{j=1}^{k}x_j,z\right\|}, z \right\|, \quad k=1,2,\ldots,n-1,$$

and δ is the function from Definition 1 with $\delta(0)=0$.

Proof. Use the first inequality of (4) in Theorem 1 and induction.

Corollary 2. Let X be a uniformly convex 2-normed space. If $x_i\in X$, $i=1,2,\ldots,n$, and $y=\sum_{i=1}^{n}x_i$, then

$$\|y,z\|\leq\sum_{i=1}^{n}(1-2\delta(\varepsilon_i))\|x_i,z\| \quad \text{for } z\in X \text{ with } z\notin V(x_1,x_2,\ldots,x_n),$$

where $\varepsilon_i = \left\|\frac{x_i}{\|x_i,z\|} - \frac{y}{\|y,z\|},z\right\|$ and δ is the function from Definition 1 with $\delta(0)=0$.

Proof. Clearly,

$$\left\| \frac{x_i}{\|x_i,z\|}, z \right\| = \left\| \frac{y}{\|y,z\|}, z \right\| = 1, \quad i=1,2,\ldots,n.$$

By the definition there exists $\delta(\varepsilon_i)>0$ such that

$$\left\| \frac{x_i}{\|x_i,z\|} + \frac{y}{\|y,z\|}, z \right\| \le 2(1-\delta(\varepsilon_i)),$$

or $\| x_i\|y,z\| + y\|x_i,z\|, z \| \le 2(1-\delta(\varepsilon_i))\|x_i,z\|\|y,z\|$. Hence,

$$\left\| \|y,z\|y + y\sum_{i=1}^{n}\|x_i,z\|, z \right\| = \left\| \sum_{i=1}^{n}(x_i\|y,z\|+y\|x_i,z\|), z \right\| \le$$

$$\le \sum_{i=1}^{n}\| x_i\|y,z\| + y\|x_i,z\|, z \| \le \sum_{i=1}^{n}2(1-\delta(\varepsilon_i))\|x_i,z\|\|y,z\|,$$

or

$$\|y,z\|\left(\|y,z\| + \sum_{i=1}^{n}\|x_i,z\|\right) \le \sum_{i=1}^{n}2(1-\delta(\varepsilon_i))\|x_i,z\|\|y,z\|.$$

Thus,

$$\|y,z\| \le \sum_{i=1}^{n}(1-2\delta(\varepsilon_i))\|x_i,z\|.$$

Definition 2. A 2-normed space is said to be uniformly 2-convex, if to each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,y\|=\|y,z\|=\|z,x\|=1$ and $\|x+y,z\|\ge\varepsilon$, then

$$\frac{1}{3}\|x+z,y+z\| \le 1-\delta(\varepsilon).$$

Definition 2 implies that ε must be in the interval $(0,2]$.

Theorem 2. For a 2-normed space X the following six statements are equivalent:

(1) X is uniformly 2-convex.

(2) To each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,y\|=\|y,z\|=\|z,x\|\ne0$ and $\|x+y,z\|\ge\varepsilon\|x,z\|$, then

$$\frac{1}{3}\|x+z,y+z\| \le (1-\delta(\varepsilon))\|x,z\|.$$

(3) To each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,y\|=\|y,z\|\ne0$ and $\|x+ay,z\|\ge\varepsilon\|x,z\|$, where $a=\frac{\|x,z\|}{\|y,z\|}$, then

$$\frac{1}{3}\|x+z,ay+z\| \le (1-\delta(\varepsilon))\|x,z\|.$$

(4) To each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|y,z\|\|x,z\|\neq 0$, $\|x,y\|=\|y,z\|$ and $\|x+ay,z\|\ge\varepsilon\|x,z\|$, where $a=\frac{\|x,z\|}{\|y,z\|}$, then

$$\|x+z,y+z\|\le\|x,z\|+2\|y,z\|-\delta(\varepsilon)\|x,z\|,$$

or $$\|x+z,y+z\|\le 5\|x,z\|-2\|y,z\|-\delta(\varepsilon)\|x,z\|.$$

(5) To each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,y\|\|y,z\|\|z,x\|\neq 0$ and $\|bx+cy,z\|\ge\varepsilon\|bx,z\|$, where $b=\frac{\|y,z\|}{\|x,y\|}$ and $c=\frac{\|x,z\|}{\|x,y\|}$, then $\frac{1}{3}\|bx+z,cy+z\| \le (1-\delta(\varepsilon))\|bx,z\|$.

(6) To each $\varepsilon>0$ there corresponds a $\delta(\varepsilon)>0$ such that, if $\|x,y\|\|y,z\|\|z,x\|\neq 0$ and $\|bx+cy,z\|\ge\varepsilon\|bx,z\|$, where $b=\frac{\|y,z\|}{\|x,y\|}$ and $c=\frac{\|x,z\|}{\|x,y\|}$, then

$$\|x+z,y+z\| \le \|x,y\|+\|y,z\|+\|z,x\|-\delta(\varepsilon)\|bx,z\|,$$

or $$\|x+z,y+z\| \le \|x,y\|(8bc+1)-3(\|y,z\|+\|x,z\|)-\delta(\varepsilon)\|bx,z\|,$$

or $$\|x+z,y+z\| \le \|x,y\|(2bc-1)+3\|x,z\|-\|y,z\|-\delta(\varepsilon)\|bx,z\|,$$

or $$\|x+z,y+z\| \le \|x,y\|(2bc-1)+3\|y,z\|-\|x,z\|-\delta(\varepsilon)\|bx,z\|.$$

Proof. (1)⇒(2): Let $\|x,z\|=d^2$ for some $d\neq 0$, then $\|\frac{x}{d},\frac{y}{d}\|= =\|\frac{y}{d},\frac{z}{d}\|=\|\frac{z}{d},\frac{x}{d}\|=1$ and $\|\frac{x}{d} + \frac{y}{d},\frac{z}{d}\|\ge\varepsilon$, by (2). Hence, $\frac{1}{3}\|\frac{x}{d} + \frac{z}{d},\frac{y}{d} + \frac{z}{d}\|\le 1-\delta(\varepsilon)$, by (1), i.e., $\frac{1}{3}\|x+z,y+z\|\le(1-\delta(\varepsilon))\|x,z\|$.

(2)⇒(3): $\|x,ay\|=\|ay,z\|=\|x,z\|\neq 0$ obviously. Replace y by ay in (2) to get (3).

(3)⇒(4): Assume that $\|y,z\|\ge\|x,z\|$, then by (3)

$$\|x+z,y+z\| = \|x+z,ay+z+y(1-a)\| \le \|x+z,ay+z\|+\|x+z,y\|(1-a) \le$$

$$\le 3(1-\delta'(\varepsilon))\|x,z\| + 2(1-a)\|y,z\| = \|x,z\|+2\|y,z\|-3\delta'(\varepsilon)\|x,z\|.$$

Let $\delta(\varepsilon)=3\delta'(\varepsilon)$, so that the first inequality follows. If $\|y,z\|\le\|x,z\|$, then

$$\|x+z,y+z\| \le 3(1-\delta'(\varepsilon))\|x,z\|+2(a-1)\|y,z\| =$$
$$= 5\|x,z\|-2\|y,z\|-3\delta'(\varepsilon)\|x,z\|.$$

(4)⇒(1): If $\|x,y\|=\|y,z\|=\|z,x\|=1$, then $\|x+z,y+z\|\le 3-\delta'(\varepsilon)$, by (4). We may let $\delta(\varepsilon)=\frac{1}{3}\delta'(\varepsilon)$.

(3)⇒(5): Since $\|bx,y\|=\|y,z\|\neq 0$ and $\|bx+cy,z\| \geq \varepsilon\|bx,z\|$, where $c=\frac{\|x,z\|}{\|x,y\|}=\frac{\|bx,z\|}{\|y,z\|}$. Hence, $\frac{1}{3}\|bx+z,cy+z\| \leq (1-\delta(\varepsilon))\|bx,z\|$, by (3).

(5)⇒(6): We must consider the following four cases.

Case 1. If $\|x,y\|\geq\|x,z\|$ and $\|x,y\|\geq\|y,z\|$, then

$$\|x+z,y+z\| = \|bx+z-x(b-1),cy+z+y(1-c)\| \leq$$
$$\leq \|bx+z,cy+z\|+\|bx+z,y\|(1-c)+\|x,cy+z\|(1-b)+(1-b)(1-c)\|x,y\|\leq$$
$$\leq 3(1-\delta'(\varepsilon))\|bx,z\|+2(1-c)\|y,z\|+2(1-b)\|x,z\|+\|x,y\|-\|y,z\|-$$
$$-\|x,z\|+\|bx,z\| = \|x,y\|+\|y,z\|+\|z,x\|-3\delta'(\varepsilon)\|bx,z\|.$$

Let $\delta(\varepsilon)=3\delta'(\varepsilon)$, so that the first inequality follows.

Case 2. If $\|x,y\|\leq\|x,z\|$ and $\|x,y\|\leq\|y,z\|$, then
$\|x+z,y+z\| \leq 3(1-\delta'(\varepsilon))\|bx,z\|+2(c-1)\|y,z\|+2(b-1)\|x,z\|+\|x,y\|-$
$-\|y,z\|-\|x,z\|+\|bx,z\|=\|x,y\|(8bc+1)-3(\|y,z\|+\|x,z\|)-3\delta'(\varepsilon)\|bx,z\|$.

Case 3. If $\|x,z\|\geq\|x,y\|\geq\|y,z\|$, then
$\|x+z,y+z\| \leq 3(1-\delta'(\varepsilon))\|bx,z\|+2(c-1)\|y,z\|+2(1-b)\|x,z\|-\|x,y\|+$
$+\|y,z\|+\|x,z\|-\|bx,z\| = \|x,y\|(2bc-1)+3\|x,z\|-\|y,z\|-3\delta'(\varepsilon)\|bx,z\|$.

Case 4. If $\|y,z\|\geq\|x,y\|\geq\|x,z\|$, then we should obtain the result directly by interchanging x and y in Case 3. Of course, b and c are interchanged, too.

(6)⇒(1): If $\|x,y\|=\|y,z\|=\|z,x\|=1$, then b=c=1 in (6). Thus, $\|x+z,y+z\| \leq 3-\delta'(\varepsilon)$, by (6). Let $\delta(\varepsilon)=\frac{1}{3}\delta'(\varepsilon)$. The proof is now completed.

Recall that a 2-normed space is said to be strictly convex, if $\frac{1}{2}\|x+y,z\|=\|x,z\|=\|y,z\|=1$ and $z\notin V(x,y)$, imply x=y [3].

Theorem 3. Every uniformly convex 2-normed space is strictly convex.

Proof. Let $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|=1$ and $z\notin V(x,y)$. If $\varepsilon=\|x-y,z\|>0$, there exists a $\delta(\varepsilon)>0$ such that $2=\|x+y,z\|\leq 2-2\delta(\varepsilon)$, as the space is uniformly convex. This is impossible unless $\delta(\varepsilon)=0$. It follows that $\varepsilon=0$. Consequently x=y, since $z\notin V(x,y)$.

We include here an alternative proof of the theorem. From

Corollary 1 we see that, if $x,y\in X$, then

$$\|x+y,z\| \le \|x,z\|+\|y,z\|-\delta(\varepsilon)\|x,z\|$$

for $z\in X$ with $z\notin V(x,y)$, where $\varepsilon = \left\|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|},z\right\|$. If $\|x+y,z\|=\|x,z\|+\|y,z\|$, then $\delta(\varepsilon)=0$, and so $\varepsilon=0$. Hence, $\frac{x}{\|x,z\|}= =\frac{y}{\|y,z\|}$. By Theorem 1 in [3] (or Theorem 1, (4) in [6]), X is strictly convex.

Recall that a 2-normed space is said to be strictly 2-convex, if conditions $\frac{1}{3}\|x+z,y+z\|=\|x,y\|=\|y,z\|=\|z,x\|=1$ imply that $z=x+y$ [4]. It is known that a strictly convex 2-normed space is strictly 2-convex, but not conversely (cf. [4] Example 2).

Theorem 4. Every uniformly 2-convex 2-normed space is strictly 2-convex.

Proof. Let $\frac{1}{3}\|x+z,y+z\|=\|x,y\|=\|y,z\|=\|z,x\|=1$, also let $\|x+y,z\|=\varepsilon$. Then, there exists a $\delta(\varepsilon)>0$ such that $3=\|x+z,y+z\|\le \le 3-3\delta(\varepsilon)$. This is impossible unless $\delta(\varepsilon)=0$. It follows that $\varepsilon=0$, i.e., $z=a(x+y)$ for some real a and so $1=\|x,z\|= =\|x,a(x+y)\|=|a|\,\|x,y\|=|a|$, that $a=-1$ leads to a contradiction, as $1=\frac{1}{3}\|x+z,y+z\|=\frac{1}{3}\|-y,-x\|=\frac{1}{3}$, and that $a=1$ implies $z=x+y$.

We mention that a different proof of the theorem is possible by using (6) in Theorem 2 and Theorem 6 in [2] (or Theorem 2, (6) in [6]). The arguments are similar to that of the alternative proof of Theorem 3.

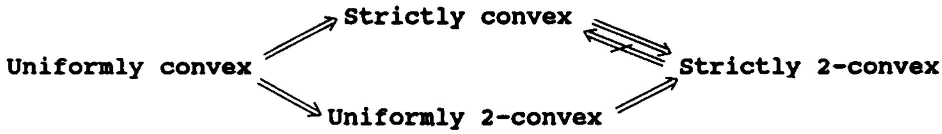
Theorem 5. Every uniformly convex 2-normed space is uniformly 2-convex.

Proof. To each $\varepsilon>0$ there exists a $\delta'(\varepsilon)>0$ such that, if $\|x,y\|=\|y,z\|=\|z,x\|=1$, $z\notin V(x,y)$ and $\|x+y,z\|\ge\varepsilon$, then $\|x-y,z\|\le \le 2-2\delta'(\varepsilon)$, since the space is uniformly convex. But

$$\|x+z,y+z\| = \|y+z+(x-y),y+z\| = \|x-y,y+z\| \le$$
$$\le \|x-y,y\|+\|x-y,z\| = \|x,y\|+\|x-y,z\| \le 1+2-2\delta'(\varepsilon) = 3-2\delta'(\varepsilon).$$

The proof is finished, if we let $\delta(\varepsilon) = \frac{2}{3}\delta'(\varepsilon)$.

In conclusion we remark that the following implications hold for a 2-normed space:



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