DEMONSTRATIO MATHEMATICA
Vol. XXV No 1-2 1992

**Marian Dawidowski, Ireneusz Kubiaczyk**

# ON BOUNDED SOLUTIONS OF HYPERBOLIC DIFFERENTIAL INCLUSION IN BANACH SPACES

In this paper we extend the Artstein-Prikry selection theorem [2]. Using this result and a theorem on existence of a bounded solution of the Darboux problem for the hyperbolic differential equation $z''_{xy} = f(x,y,z)$ $(x\geq 0,\ y\geq 0)$, where f is a Caratheodory function with values in a separable Banach space satisfying some regularity conditions expressed in term of the measure of noncompactness $\alpha$ and a Stokes type assumption [8], we prove the existence theorem for the hyperbolic differential inclusion $z''_{xy}\in F(x,y,z)$ $(x\geq 0,\ y\geq 0)$, where the values of F are nonempty subsets of E.

## 1. Introduction

Let $R_+=[0,\infty)$ and $Q=R_+\times R_+$ be endowed with the Lebesgue (product) measure. Let $\langle E,\|\cdot\|\rangle$ be a separable Banach space and $2^E$ - the class of all nonempty subsets of E. The measure of noncompactness $\alpha(A)$ of nonempty bounded subset A of E is defined as the infimum of all $\varepsilon>0$ such that there exists a finite covering of A by sets of diameter less than $\varepsilon$. For the properties of $\alpha$ we refer the reader to [3]. Denote by $C(Q,E)$ the space of all continuous functions from Q to E endowed with the topology of almost uniform convergence. For $V\subset C(Q,E)$ we denote by $V(x,y)$ the set of all $z(x,y)$ with $z\in V$. Further, we will use the standard notation. The closure of a set A and its closed convex hull will be denoted, respectively, $\overline{A}$ and $\overline{co}A$. For $A,B\subset E$ we put $\|A\|=\sup\{\|a\| : a\in A\}$, $A\dot{-}B=A\setminus B\cup B\setminus A$.

The lemma below is an adaptation of the corresponding result of Ambrosetti [1]. It is special result of Heinz Lemma [5].

**Lemma.** If P is a compact subset of Q and V is a bounded equicontinuous subset of C(P,E), then

$$\alpha\left(\bigcup\{V(x,y) : (x,y)\in P\}\right) = \sup\{(\alpha(V(x,y) : (x,y)\in P\}.$$

Denote by $S_\infty$ the set of all nonnegative real sequences. For $\xi=(\xi_n)$, $\eta=(\eta_n)\in S_\infty$ we write $\xi<\eta$ if $\xi\le\eta$ (i.e. $\xi_n\le\eta_n$ for $n\in N$) and $\xi\ne\eta$. Let $\mathfrak{X}$ be a closed convex subset of C(Q,E) and $\Phi$ be a function which assigns to each nonempty subset Z of $\mathfrak{X}$ a sequence $\Phi(Z)\in S_\infty$ such that

(1) $\Phi(Z\cup\{z\}) = \Phi(Z)$ for $z\in\mathfrak{X}$,

(2) $\Phi(\overline{co}Z) = \Phi(Z)$,

(3) if $\Phi(Z)=\emptyset$ (the zero sequence), then $\overline{Z}$ is compact,

(4) if $Z_1\subset Z_2$, then $\Phi(Z_1)\le\Phi(Z_2)$.

Here we use the Sadovskii fixed point theorem [7] in the following form (cf. [6]): If $T:\mathfrak{X}\longrightarrow\mathfrak{X}$ is a continuous mapping satisfying $\Phi(T(Z))<\Phi(Z)$ for arbitrary nonempty subset Z of $\mathfrak{X}$ with $\Phi(Z)>\theta$, then T has a fixed point in $\mathfrak{X}$.

## 2. Existence Theorem for hyperbolic differential equation

Let $f:Q\times E\longrightarrow E$. By (+) we shall denote the problem of finding a solution of the hyperbolic differential equation

$$z''_{xy} = f(x,y,z),$$

$$z(x,0)=0,\ z(0,y)=0 \quad \text{for} \quad x>0,\ y>0.$$

By a solution of the problem (+) we mean a function $z:Q\longrightarrow E$ such that $z(0,y)=z(x,0)=0$ and

$$z''_{xy} = f(x,y,z(x,y)) \quad \text{for almost all} \quad (x,y)\in Q.$$

**Theorem 1.** Assume that $f:Q\times E\longrightarrow E$ satisfies the following conditions:

($1^o$) $f(.,z):Q\longrightarrow E$ is measurable for each $z\in E$,

($2^o$) $f(x,y,.):E\longrightarrow E$ is continuous for $(x,y)\in Q$,

($3^o$) $\|f(x,y,z)\|\le G(x,y,\|z\|)$ for $(x,y,z)\in Q\times E$,

where $G:R_+^3\longrightarrow R_+$ is nondecreasing in the last variable and such that $(x,y)\mapsto G(x,y,u)$ is locally integrable for any fixed $u \in R_+$, and $(x,y) \mapsto G(x,y,u(x,y))$ is measurable for every continuous bounded function $u: Q \to R_+$,

($4^o$) the scalar inequality

$$g(x,y) \geq \int_0^x \int_0^y G(t,s,g(t,s))\,dtds$$

has a bounded solution $g_0$ existing on Q,

($5^o$) $\alpha(f(P \times Z) \leq \sup\{L(x,y,\alpha(Z) : (x,y) \in P\}$

for any compact subset $P \subset Q$ and each nonempty bounded subset Z of E, where $L: R_+^3 \to R_+$ is a function such that

(a) the mapping $(x,y) \mapsto L(x,y,u)$ is continuous for each $u \in R$,

(b) $L(x,y,0)=0$ on Q,

(c) $\int_0^\infty \int_0^\infty L(t,s,u)\,dtds < u$ for all $u>0$.

Then there exists a solution z of the problem (+) such that

$$\|z(x,y)\| \leq g_0(x,y) \quad \text{for} \quad (x,y) \in Q.$$

Proof. Denote by $\mathfrak{X}$ the set of all $z \in C(Q,E)$ with $\|z(x,y)\| \leq$ $\leq g_0(x,y)$ on Q and

$$\|z(x_1,y_1) - z(x_2,y_2)\| \leq \int\int_A G(t,s,g_0(t,s))\,dtds$$

for $(x_1,y_1),(x_2,y_2) \in Q$, where $A=[0,x_1]\times[0,y_1] \dot{-} [0,x_2]\times[0,y_2]$.

The set $\mathfrak{X}$ is closed convex and almost equicontinuous subset of C(Q,E).

We define a continuous mapping $T: \mathfrak{X} \to \mathfrak{X}$ as follows:

$$(Tz)(x,y) = \int_0^x \int_0^y f(t,s,z(t,s))\,dtds \quad \text{for} \quad (x,y) \in Q, \ z \in \mathfrak{X}.$$

Let n be a positive integer and $P_n=[0,n]\times[0,n]$. Let Z be a nonempty subset of $\mathfrak{X}$ and $W=\bigcup\{Z(x,y) : (x,y) \in P_n\}$.

Fix (x,y) in $P_n$. For any given $\varepsilon>0$ there exists $\delta>0$ such that $u',u'' \in [0,x]$, $v',v'' \in [0,y]$ with $|u'-u''|<\delta$ and $|v'-v''|<\delta$ implies $|L(u',v',\alpha(W)) - L(u'',v'',\alpha(W))|<\varepsilon$. We divide the intervals [0,x] and [0,y] into m parts

$$x_0 = 0 < x_1 < \ldots < x_m = x, \quad y_0 = 0 < y_1 < \ldots < y_m = y$$

in such a way that $|x_i - x_{i-1}| < \delta$ and $|y_i - y_{i-1}| < \delta$ for $i=1,2,\dots,m$. Put $P_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $(i,j=1,2,\dots,m)$. Let $(\sigma_i, \tau_j)$ be a point in $P_{ij}$ such that

$$L(x,y,\alpha(W)) \le L(\sigma_i, \tau_j, \alpha(W)) \quad \text{for} \quad (x,y) \in P_{ij}.$$

Now, by the integral mean value theorem, our comparison condition $3^{\circ}$ and Lemma, we obtain

$$\alpha(T(Z)(x,y)) \le \alpha\Big( \sum_{i,j=1}^{m} \operatorname{mes}(P_{ij}) \overline{\operatorname{co}}(f(P_{ij} \times W)) \Big) \le$$

$$\le \sum_{i,j=1}^{m} \operatorname{mes}(P_{ij}) L(\sigma_i, \tau_j, \alpha(W)) \le \sum_{i,j=1}^{m} \Big( \int\!\!\int_{P_{ij}} L(u,v,\alpha(W)) \Big) dudv +$$

$$+ \sum_{i,j=1}^{m} \int\!\!\int_{P_{ij}} |L(u,v,\alpha(W)) - L(\sigma_i, \tau_j, \alpha(W))| dudv <$$

$$< \int_0^x\!\!\int_0^y L(u,v,\alpha(W)) dudv + \varepsilon \cdot xy = \varepsilon \cdot xy +$$

$$+ \int_0^x\!\!\int_0^y L(u,v,\sup\{\alpha(Z(x,y)) : (x,y) \in P_n\}) dudv.$$

As $\varepsilon > 0$ is arbitrary, this implies

(*) $$\sup\{\alpha(T(Z)(x,y)) : (x,y) \in P_n\} \le$$

$$\le \sup_{(x,y) \in Q} \int_0^x\!\!\int_0^y L(u,v,\sup\{\alpha(Z(x,y)) : (x,y) \in P_n) dudv$$

Define

$$\Phi(Z) = \Big( \sup_{(x,y) \in P_1} \alpha(Z(x,y)), \quad \sup_{(x,y) \in P_2} \alpha(Z(x,y)), \dots \Big)$$

for nonempty subset $Z$ of $\mathfrak{X}$. Evidently $\Phi(Z) \in S_{\infty}$. By the corresponding properties of $\alpha$ the function $\Phi$ satisfies conditions (1)-(4) listed above. From our assumption on $L$ and inequality (*) it follows that $\Phi(F(Z)) < \Phi(Z)$ whenever $\Phi(Z) > \theta$. Thus all assumptions of Sadovski's fixed point theorem are satisfied, $F$ has a fixed point in $\mathfrak{X}$ and the proof is complete.

Remark. This theorem extends the result of [4].

## 3. Existence Theorem for hyperbolic differential inclusion

Recall that a set-valued map F from a metric space Z into nonempty subsets of a metric space Y is lower semicontinuous if $z_k \longrightarrow z_0$ in Z and $y_0 \in F(z_0)$, then there are $y_k \in F(z_k)$ so that $y_k \longrightarrow y_0$ in Y.

We say that $F:Z \longrightarrow 2^Y$ is an M-mapping if the restriction of F to $Z_1$ has a continuous selection whenever the restriction of F to $Z_1$ is lower semicontinuous.

Z. Artstein and K. Prikry proved the following theorem [2]

**Theorem 2.** Let P be a separable complete metric space and let $\mu$ be a finite measure on P. Let E be a separable complete metric space, equipped with its Borel structure. It is assume that $F:P\times E \longrightarrow 2^E$ is an M-mapping and

(i) $F(t,.)$ is lower semicontinuous for each fixed $t \in P$,

(ii) $F(.,.)$ is measurable on $P\times E$.

Then F has a Caratheodory selection i.e. there exists a function $f:P\times E \longrightarrow E$ such that

(.) $f(.,z)$ is measurable for each fixed $z \in E$,

(..) $f(t,.)$ is continuous for each $t \in P$,

(...) $f(t,x) \in F(t,x)$ for $(t,x) \in P\times E$.

**Remark.** Above theorem is true if P is separable complete metric space such that $P=\bigcup_{n=1}^{\infty} P_n$, where $P_n$ is closed, measurable and $\mu(P_n)<\infty$ for $n \in N$.

**Proof.** By Theorem 2 for any $n \in N$ there exists a Caratheodory selection $f_n$ of F on $P_n$. We define

$$f(t,z) = f_n(t,z) \quad \text{if} \quad t \in P_n \quad \text{and} \quad t \notin P_m \quad \text{for} \quad m<n.$$

Clearly f is a Caratheodory selection of F on P.

Now, let $F:Q\times E \longrightarrow 2^E$. By (++) we shall denote the problem of finding a solution of the hyperbolic differential inclusion

$$z''_{xy} \in F(x,y,z)$$

$$z(x,0) = z(0,y) = 0 \quad \text{for} \quad (x,y) \in Q.$$

**Theorem 3.** It is assumed that

(1′) $F(x,y,.)$ is lower semicontinuous on $E$ for each fixed $(x,y)\in Q$,

(2′) $F$ is measurable on $Q\times E$,

(3′) $F$ is an M-mapping,

(4′) $\|F(x,y,z)\| \le G(x,y,\|z\|)$ for each $z\in E$, $(x,y)\in Q$,

(5′) $\alpha(F(P\times Z)) \le \sup\{L(x,y,\alpha(Z)) : (x,y)\in P\}$ for any compact subset $P\subset Q$ and each nonempty bounded subset $Z\subset E$, where the functions $G,L:R_+^3 \longrightarrow R_+$ satisfy the conditions $3^o$, $4^o$, $5^o$ of Theorem 4.

Then there exists a solution $z$ of (++) such that

$$\|z(x,y)\| \le g_0(x,y) \quad \text{for} \quad (x,y)\in Q.$$

**Proof.** By Theorem 2 and Remark there exists a Caratheodory selection $f$ of $F$ on $Q$. It is easy to see that $f$ satisfies assumptions $1^o$-$5^o$ of Theorem 1. The solution of (+) is a solution of (++).

REFERENCES

[1] **A. Ambrosetti:** Un teorema di esistenza per le equazioni differenziali negli spazi di Banach, Rend. Sem. Mat. Univ. Padova 39(1967), 349-360.

[2] **Z. Artstein, K. Prikry:** Caratheodory selections and the Scorza Dragoni property, J. Math. Anal. Appl. 127(1987), 540-547.

[3] **J. Banaś, K. Goebel:** Measures of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Math., Marcel Dekker Inc., Vol. 60, New York and Basel 1980.

[4] **M. Dawidowski, I. Kubiaczyk, B. Rzepecki:** An existence theorem for the hyperbolic equation $z''_{xy}=f(x,y,z)$ in Banach spaces, Demonstratio Math. 20(1987), 489-493.

[5] **H.P. Heinz:** Theorems of Ascoli type involving measures of noncompactness, Nonlinear Analysis, 5(1983) 277-286.

[6] **B. Rzepecki:** Remarks on Schauders fixed point principle and its applications, Bull. Acad. Polon. Sci., Ser. Sci.

Math. 27(1979), 473-480.

[7] **B.N. Sadovskii:** On fixed point principle (in Russian), Function, Analiz. Prilozen, (1967), 74-76.

[8] **A. Stokes:** The application of a fixed point theorem to a variety of nonlinear stability problems, Proc. Nat. Acad. Sci. USA 45(1959), 231-235.

INSTITUTE OF MATHEMATICS, ADAM MICKIEWICZ UNIVERSITY,
ul. Matejki 48/49, 60-769 POZNAŃ, POLAND

Received June 6, 1989.