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THE STRUCTURE OF THE SECOND SYMMETRIC TENSOR OF THE BICONFORMAL MAPPING

Let M^n denote an n-dimensional Riemannian space. For any point $x \in M^n$ by a decomposition of the tangent space at the point x we mean the decomposition of T_xM^n into a sum of two orthogonal subspaces Λ^p and Λ^q. The decomposition of p--dimensional tangent subspaces of a smooth n-dimensional manifold M represent a classical example of simple geometrical structures. In order to be defined they do not need any additional assumptions on a manifold M. The geometry of decompositions in the case of a smooth manifold is simply a geometry of the system of Pfaf differential equations. Among many problems here the most important one is the question concerning integrability of the decomposition depending on the tensor of non-holonomy.

The purpose of this paper is to investigate the structure of biconformal mapping of Riemannian spaces and to distinguish different types of Riemannian spaces connected with this notion. We obtain the following results:

1) The structure of the second symmetric tensor of the biconformal mapping and all its components are described and geometric characterizations are given.
2) The connection between the second symmetric tensor of the biconformal mapping Φ and its inverse Φ^{-1} is found.
3) Several types of mappings are characterized depending on the structure of the second symmetric tensor.

This article is a part of my doctoral dissertation ([2])

written at the Department of Mathematics and Mechanics of Moscow State University under a supervision of prof. A.M. Vasiliev. I would like to express my deep gratitude for posing the problem and for his invaluable assistance during the preparation of the thesis.

We begin by recalling some notions and facts from differential geometry in the form needed in this paper.

Let M^n, $\overline{M}^n$ be two Riemannian spaces of dimension n and $\Phi: M^n \longrightarrow \overline{M}^n$ a diffeomorphism.

The structural equations of these spaces are of the form ([4] pp. 63 and 89):

$$d\omega^\lambda = \omega^\lambda_\mu \wedge \omega^\mu$$

on M^n and

$$d\overline{\omega}^\lambda = \overline{\omega}^\lambda_\mu \wedge \omega^\mu$$

on $\overline{M}^n$, with

$$\omega^\lambda_\mu + \omega^\mu_\lambda = 0,\ \overline{\omega}^\lambda_\mu + \overline{\omega}^\mu_\lambda = 0,\ \lambda,\mu=1,2,\dots,n.$$

Here ω^λ, $\overline{\omega}^\lambda$ denote forms defined on sub-bundles $O(M^n)$ and $O(\overline{M}^n)$ of orthonormal repers of manifolds M^n and $\overline{M}^n$ respectively and ω^λ_μ, $\overline{\omega}^\lambda_\mu$ denote forms defined on sub-bundles $O^2(M^n)$ and $O^2(\overline{M}^n)$ of orthonormal repers of second order ([4] chap. IV, §2).

The map Φ can be defined by equations

$$\overline{\omega}^\lambda = h^\lambda_\mu \omega^\mu$$

where, in the case of conformal mapping h,

$$h^\lambda_\mu = a\cdot\delta^\lambda_\mu$$

with natural obvious identifications given by diffeomorphism Φ, where h^λ_μ denotes a coefficient matrix.

Now suppose we have given for any $x \in M^n$ the decomposition of the tangent space $T_x M^n$ into the sum of two orthogonal subspaces Λ^p and Λ^q and similarly $T_y\overline{M}^n = \overline{\Lambda}^p \oplus \overline{\Lambda}^q$ for $y \in \overline{M}^n$, where p+q=n. An orthogonal frame (e_α, e_i) such that $e_\alpha \in \Lambda^p$, $e_i \in \Lambda^q$ is called a frame adapted to the space M^n.

The following definition was suggested by A.M. Vasiliev

([4]).

Definition 1. Diffeomorphism $\Phi:M^n \longrightarrow \bar{M}^n$ is called a *bi-conformal mapping* if at each point of the space M^n there exists a decomposition of its tangent space into the pair of orthogonal subspaces Λ^p, Λ^q; $\Lambda^p \perp \Lambda^q$, $p+q=n$ such that

1° $\bar{\Lambda}^p = \Phi_*(\Lambda^p)$, $\bar{\Lambda}^q = \Phi_*(\Lambda^q)$ where $\phi_*:TM^n \longrightarrow T\bar{M}^n$ and

2° restrictions of Φ_* to Λ^p and Λ^q are conformal ones i.e.

$$\Phi_*|_{\Lambda^p} = a\cdot id, \quad \Phi_*|_{\Lambda^q} = b\cdot id,$$

where a and b are positive smooth functions on M^n.

The decomposition on the manifold M^n in the bundle of adapted frames is given by equations:

$$\text{(1)} \qquad \begin{aligned} &\omega^\alpha=0 \quad \text{on } \Lambda^p, \ \alpha,\beta,\gamma=1,2,\ldots,q, \\ &\omega^i=0 \quad \text{on } \Lambda^q, \ i,k,l=q+1,\ldots,n, \end{aligned}$$

where ω^α, ω^i denote basic forms on M^n, i.e. (ω^α,ω^i) is a co-frame of the frame adapted to M^n. Similarly we have decompositions $\bar{\omega}^\alpha=0$ on $\bar{\Lambda}^p$ and $\bar{\omega}^i=0$ on $\bar{\Lambda}^q$ for the manifold $\bar{M}^n$.

The structural equations in this orthonormal frame have the form

$$\text{(2)} \qquad M^n : \begin{cases} d\omega^\alpha = \omega^\alpha_\beta\wedge\omega^\beta + \omega^\alpha_k\wedge\omega^k \\ d\omega^i = \omega^i_k\wedge\omega^k + \omega^i_\alpha\wedge\omega^\alpha, \end{cases}$$

$$\text{(3)} \qquad \bar{M}^n : \begin{cases} d\bar{\omega}^\alpha = \bar{\omega}^\alpha_\beta\wedge\bar{\omega}^\beta + \bar{\omega}^\alpha_k\wedge\bar{\omega}^k \\ d\bar{\omega}^i = \bar{\omega}^i_k\wedge\bar{\omega}^k + \bar{\omega}^i_\alpha\wedge\bar{\omega}^\alpha, \end{cases}$$

where

$$\omega^\alpha_\beta + \omega^\beta_\alpha = 0, \ \omega^\alpha_k + \omega^k_\alpha = 0, \ \omega^i_k + \omega^k_i = 0,$$

$$\bar{\omega}^\alpha_\beta + \bar{\omega}^\beta_\alpha = 0, \ \bar{\omega}^\alpha_k + \bar{\omega}^k_\alpha = 0, \ \bar{\omega}^i_k + \bar{\omega}^k_i = 0.$$

The structural equations of the first tensor h_1 with the local components h^α_β, h^α_k, h^k_α, h^k_l of any map $\Phi:M^n \longrightarrow \bar{M}^n$ are of the form ([2], pp. 43-50)

$$
(4)\quad \begin{cases}
dh^{\alpha}_{\beta} + h^{\alpha}_{\gamma}\omega^{\gamma}_{\beta} + h^{\alpha}_{k}\omega^{k}_{\beta} - h^{\gamma}_{\beta}\bar{\omega}^{\alpha}_{\gamma} - h^{k}_{\beta}\bar{\omega}^{\alpha}_{k} = h^{\alpha}_{\beta\gamma}\omega^{\gamma} + h^{\alpha}_{\beta k}\omega^{k} \\
dh^{\alpha}_{k} + h^{\alpha}_{\beta}\omega^{\beta}_{k} + h^{\alpha}_{l}\omega^{l}_{k} - h^{\beta}_{k}\bar{\omega}^{\alpha}_{\beta} - h^{l}_{k}\bar{\omega}^{\alpha}_{l} = h^{\alpha}_{k\gamma}\omega^{\gamma} + h^{\alpha}_{kl}\omega^{l} \\
dh^{k}_{\beta} + h^{k}_{\gamma}\omega^{\gamma}_{\beta} + h^{k}_{l}\omega^{l}_{\beta} - h^{\gamma}_{\beta}\bar{\omega}^{k}_{\gamma} - h^{l}_{\beta}\bar{\omega}^{k}_{l} = h^{k}_{\beta\gamma}\omega^{\gamma} + h^{k}_{\beta l}\omega^{l} \\
dh^{k}_{l} + h^{k}_{\beta}\omega^{\beta}_{l} + h^{k}_{i}\omega^{i}_{l} - h^{\beta}_{l}\bar{\omega}^{k}_{\beta} - h^{i}_{l}\bar{\omega}^{k}_{i} = h^{k}_{l\gamma}\omega^{\gamma} + h^{k}_{li}\omega^{i} .
\end{cases}
$$

Definition 2. Expression of the form $h_2(h^{\alpha}_{\beta\gamma}, h^{\alpha}_{\beta k}, h^{\alpha}_{k\gamma}, h^{k}_{\beta\gamma}, h^{k}_{\beta l}, h^{k}_{li})$ appearing in the above formulae is called a *second symmetric tensor of the mapping* Φ with the local components $h^{\alpha}_{\beta\gamma}$, $h^{\alpha}_{\beta k}$, $h^{\alpha}_{k\gamma}$, $h^{k}_{\beta\gamma}$, $h^{k}_{\beta l}$, h^{k}_{li}.

Now we want to determine components of this second symmetric tensor. Taking into account the biconformality condition

$$h^{\alpha}_{\beta} = a\cdot\delta^{\alpha}_{\beta} \quad , \quad h^{i}_{k} = b\cdot\delta^{i}_{k}$$

we get

$$(5)\qquad \delta^{\alpha}_{\beta}\cdot da + a\left(\omega^{\alpha}_{\beta} - \bar{\omega}^{\alpha}_{\beta}\right) = h^{\alpha}_{\beta\gamma}\omega^{\gamma} + h^{\alpha}_{\beta k}\omega^{k},$$

$$(6)\qquad a\omega^{\alpha}_{k} - b\bar{\omega}^{\alpha}_{k} = h^{\alpha}_{k\gamma}\omega^{\gamma} + h_{kl}\omega^{l},$$

$$(7)\qquad b\omega^{k}_{\alpha} - a\bar{\omega}^{k}_{\alpha} = h^{k}_{\alpha\gamma}\omega^{\gamma} + h^{k}_{\alpha l}\omega^{l},$$

$$(8)\qquad \delta^{k}_{l}\cdot db + b\left(\omega^{k}_{l} - \bar{\omega}^{k}_{l}\right) = h^{k}_{l\gamma}\omega^{\gamma} + h^{k}_{li}\omega^{i}.$$

Symmetrization of (5) with respect to α and β gives rise to:

$$\delta^{\alpha}_{\beta}\cdot da = \left(h^{\alpha}_{\beta\gamma} + h^{\beta}_{\alpha\gamma}\right)\omega^{\gamma} + \left(h^{\alpha}_{\beta k} + h^{\beta}_{\alpha k}\right)\omega^{k}.$$

Now we substitute $da = a_{\gamma}\omega^{\gamma} + a_{k}\omega^{k}$ and looking at both sides of this equation we have:

$$(9)\qquad 2\delta^{\alpha}_{\beta}a_{\gamma} = h^{\alpha}_{\beta\gamma} + h^{\beta}_{\alpha\gamma},$$

$$(10)\qquad 2\delta^{\alpha}_{\beta}a_{k} = h^{\alpha}_{\beta k} + h^{\beta}_{\beta k} + h^{\beta}_{\alpha k},$$

where a denotes a function on M^n and a_{γ}, a_{k} its Pffafian derivatives.

Similarly we treat the equation (8) with respect to i and k obtaining

$$\delta^k_l db = (h^k_{l\gamma} + h^l_{k\gamma})\omega^\gamma + (h^k_{li} + h^l_{ki})\omega^i,$$

where $db = b_\gamma\omega^\gamma + b_i\omega^i$. Hence

$$2\delta^k_l b_\gamma = h^k_{l\gamma} + h^l_{k\gamma}, \tag{11}$$

$$2\delta^k_l b_i = h^k_{li} + h^l_{ki}. \tag{12}$$

It follows from (9) and (12)

$$h^\alpha_{\beta\gamma} = \delta^\alpha_\beta a_\gamma + \delta^\alpha_\gamma a_\beta - \delta^\gamma_\beta a_\alpha,$$

$$h^k_{li} = \delta^k_l b_i + \delta^k_i b_l - \delta^i_l b_k.$$

We will determine remaining tensor components as follows. We differentiate externally equations (1) of the decomposition and making use of Cartan's lemma we get

$$\begin{cases} \omega^k_\alpha = A^k_{\alpha\beta}\omega^\beta + A^k_{\alpha l}\omega^l, & A^k_{\alpha\beta} = -A^\alpha_{k\beta}, \\ \omega^\alpha_k = A^\alpha_{k\beta}\omega^\beta + A^\alpha_{kl}\omega^l, & A^k_{\alpha l} = -A^\alpha_{kl}, \end{cases} \tag{13}$$

where A is a non-holonomy tensor of the mapping Φ.

Considering remaining equations (6) and (7) we obtain formulae

$$(a^2-b^2)\omega^\alpha_k = (ah^\alpha_{k\beta} + bh^k_{\alpha\beta})\omega^\beta + (ah^\alpha_{kl} + bh^k_{\alpha l})\omega^l, \tag{14}$$

$$(a^2-b^2)A^\alpha_{k\beta} = ah^\alpha_{kl} + bh^k_{\alpha\beta}, \tag{15}$$

$$(a^2-b^2)A^\alpha_{kl} = ah^\alpha_{kl} + bh^k_{\alpha l}. \tag{16}$$

Now alternation and symmetrization of the system (15), (16) with respect to α and β gives rise to equations

$$a(h^\alpha_{k\beta} - h^\beta_{k\alpha}) = (a^2-b^2)(A^\alpha_{k\beta} - A^\beta_{k\alpha}),$$

$$a(h^\alpha_{k\beta} + h^\beta_{k\alpha}) + 2bh^k_{\alpha\beta} = (a^2-b^2)(A^\alpha_{k\beta} + A^\beta_{k\alpha}).$$

Hence

$$h^k_{\alpha\beta} = -\frac{a}{b}\,\delta^\alpha_\beta a_k + \frac{1}{2b}(a^2-b^2)(A^\alpha_{k\beta} + A^\beta_{k\alpha}),$$

$$h^\alpha_{k\beta} = \delta^\alpha_\beta a_k + \frac{1}{2a}(a^2-b^2)(A^\alpha_{k\beta} - A^\beta_{k\alpha}).$$

Other components of the second symmetric tensor we obtain by changing indices: $\alpha \longrightarrow k$, $k \longrightarrow \alpha$:

$$h^\alpha_{ki} = -\frac{b}{a}\,\delta^k_i b_\alpha + \frac{1}{2a}(b^2-a^2)(A^k_{\alpha i} + A^i_{\alpha k})$$

$$h^k_{\alpha i} = \delta^k_i b_\alpha + \frac{1}{2b}(b^2-a^2)(A^k_{\alpha i} - A^i_{\alpha k}).$$

Thus we have proved:

Theorem 1. Components of the second symmetric tensor of biconformal mapping $\Phi:M^n \longrightarrow \bar{M}^n$ are of the form:

$$h^\alpha_{\beta\gamma} = \delta^\alpha_\beta a_\gamma - \delta^\alpha_\gamma a_\beta - \delta^\gamma_\beta a_\alpha,$$

$$h^\alpha_{k\beta} = \delta^\alpha_\beta a_k + \frac{1}{2a}(a^2-b^2)(A^\alpha_{k\beta} - A^\beta_{k\alpha}),$$

$$h^\alpha_{ki} = -\frac{b}{a}\delta^k_i b_\alpha + \frac{1}{2a}(b^2-a^2)(A^k_{\alpha i} + A^i_{\alpha k}),$$

$$h^k_{li} = \delta^k_i b_i + \delta^k_i b_l - \delta^i_l b_k,$$

$$h^k_{\alpha i} = \delta^k_i b_\alpha + \frac{1}{2b}(b^2-a^2)(A^k_{\alpha i} - A^i_{\alpha k}),$$

$$h^k_{\alpha\beta} = -\frac{a}{b}\delta^\alpha_\beta a_k + \frac{1}{2b}(a^2-b^2)(A^\alpha_{k\beta} + A^\beta_{k\alpha}),$$

where a,b are smooth functions on M^n, a_α, a_k, b_α, b_k their Pffafian derivatives and A a non-holonomy tensor.

Now we will determine components of the tensor h_2 for inverse mapping Φ^{-1}. Let us consider the map $\Phi^{-1} = \bar{\Phi}:\bar{M}^n \longrightarrow M^n$.

Theorem 2. Let $\Phi:M^n \longrightarrow \bar{M}^n$ be a biconformal mapping and $\Phi^{-1}=\bar{\Phi}:\bar{M}^n \longrightarrow M^n$ its inverse. Then the components of the second symmetric tensor of the mapping $\Phi^{-1}=\bar{\Phi}$ are related to components of the second symmetric tensor of Φ as follows

$$\bar{h}^\alpha_{\beta\gamma} = -\frac{1}{a^3}h^\alpha_{\beta\gamma}, \quad \bar{h}^k_{li} = -\frac{1}{b^3}h^k_{li},$$

$$\bar{h}^\alpha_{\beta k} = -\frac{1}{a^2 b}h^\alpha_{\beta k}, \quad \bar{h}^k_{l\alpha} = -\frac{1}{b^2 a}h^k_{l\alpha},$$

$$\bar{h}^\alpha_{kl} = -\frac{1}{b^2 a}h^\alpha_{kl}, \quad \bar{h}^k_{\alpha\beta} = -\frac{1}{a^2 b}h^k_{\alpha\beta}.$$

Now we look at several examples in order to explain the geometric sense of the second symmetric tensor of Φ. To do this we have consider equations (2) along with condition (13). Consequently we get equations:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + A^\alpha_{k\beta}\omega^\beta \wedge \omega^k + A^\alpha_{kl}\omega^k \wedge \omega^l,$$

$$d\omega^k = \omega^k_l \wedge \omega^l + A^k_{l\alpha}\omega^\alpha \wedge \omega^l + A^k_{\alpha\beta}\omega^\beta \wedge \omega^\alpha,$$

where $A^\alpha_{k\beta} = -A^k_{\alpha\beta}$ and $A^\alpha_{kl} = -A^k_{\alpha l}$.

We describe the following five cases:

1° Coordinates $h^\alpha_{k\beta}=h^\alpha_{ki}=h^k_{\alpha i}=h^k_{\alpha\beta}$ vanish and coordinates $h^\alpha_{\beta\gamma}$, h^k_{li} are non-zero. This case is a decomposable one i.e. $M^n=M^p\times M^q$, $\bar{M}^n=\bar{M}^p\times\bar{M}^q$, $\Phi=(\Phi_1,\Phi_2)$, where $\Phi_1:M^p\longrightarrow\bar{M}^p$, $\Phi_2:M^q\longrightarrow\bar{M}^q$ are conformal mappings.

2° Coordinates $h^\alpha_{\beta\gamma}=h^\alpha_{k\beta}=h^\alpha_{ki}=h^k_{li}=h^k_{\alpha i}$ vanish and $h^k_{\alpha\beta}\neq 0$. Then the systems $\omega^\alpha=0$ and $\omega^i=0$ are completely integrable and the integral submanifold determined by orthogonal decomposition is totally geodesic.

Moreover the metric tensor under the mapping Φ is multiplited by constants a (b) on the fibre Λ^p (Λ^q respectively).

3° The following coordinates vanish $h^\alpha_{\beta\gamma}=h^\alpha_{ki}=h^k_{li}=h^k_{\alpha i}=0$ and $h^\alpha_{k\beta}$, $h^k_{\alpha\beta}$ are non-zero. In this situation the vertical system $\omega^i=0$ is completely integrable while the horizontal one $\omega^\alpha=0$ is not, The metric tensor of the mapping Φ of orthogonal decompositions is multiplied by a constant a. Differentiation of the equation $da=a_k\cdot\omega^k$ implies $a_k\cdot A^k_{\alpha\beta}=0$, hence tensors $A^k_{\alpha\beta}$ are globally constant.

4° Coordinates $h^\alpha_{k\beta}=h^k_{\alpha i}=h^k_{li}=h^k_{\alpha\beta}$ vanish while $h^\alpha_{\beta\gamma}\neq 0$, $h^\alpha_{ki}\neq 0$ are non-zero. Then the horizontal system $\omega^\alpha=0$ is completely integrable, but the horizontal decomposition is not geodesic. The system $\omega^i=0$ is not completely integrable. The metric tensor of Φ for horizontal decomposition is multiplied by a constant a. Similarly as above we get that tensors A^α_{kl} are globally constant.

5° All coordinates except of $h^\alpha_{\beta\gamma}$ and $h^k_{\alpha\beta}$ are zero. The both decompositions are completely integrable. Horizontal fibres are totally geodesic and their metrics transform by means of a non-constant coefficient a which does not change among fibres.

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