DEMONSTRATIO MATHEMATICA
Vol. XXV No 1-2 1992

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SATURATION FOR FOURIER SERIES BY MATRIX OPERATOR

Let f(x) be periodic function with period 2Π integrable in the Lebesgue sense over $(-\pi,\pi)$ and let $\tilde{S}_n(\tilde{f},x)$, $n=1,2,3,\ldots,$ denote the n-th partial sum of conjugate Fourier series associated with f(x), i.e.

$$\tilde{S}_n(\tilde{f},x)=\frac{1}{\pi}\int_0^{\pi}[\tilde{f}(x+t)-\tilde{f}(x-t)]\,\frac{\cos\frac{t}{2}-\cos\frac{n+1}{2}t}{2\sin\frac{t}{2}}\,dt.$$

We call

$$\tilde{f}(x)=\frac{1}{2\pi}\int_0^{\pi}[f(x+t)-f(x-t)]\cot\frac{t}{2}\,dt \tag{1}$$

the conjugate function of f, if the integral in (1) exists and converges absolutely for all $(f)\in K_0$, where K_0 is a class of all continuous functions f such that $\tilde{f}\in \mathrm{Lip}\,1$.

Let $(\lambda_{n,k})$ $(n=0,1,2,\ldots;\ k=0,1,2,\ldots,n;\ \lambda_{n,0}=1)$ be a triangular matrix of real or complex numbers. We define the matrix operator

$$L_n(x)=L_n(f,x)=\sum_{k=0}^{k}\lambda_{n,k}A_k(x) \tag{2}$$

and the norm of f as

$$\|f\| = \sup_{0\le x\le 2\pi}|f(x)|.$$

Given a positive non-increasing function $\phi(n)$ and a class K of functions, we say that a matrix summability method $L_n(x)$ is

AMS (MOS) Subject classification: 41A25

saturated with the order $\phi(n)$ relative to K, if

$$\|f(x)-L_n(x)\|=o(\phi(n)) \Rightarrow f(x)=\text{constant},$$

$$\|f(x)-L_n(x)\|=O(\phi(n)) \Leftrightarrow f\in K.$$

The problem of determining the order of saturation by Cesèaro means operator of the Fourier series of f(x) has been considered in [3]. In [1], [2] was simultaneously discussed the saturation problem for Nörlund and generalized Nörlund means operators, and obtained the same order of saturation, i.e. $(^{p}n/P_n)$ and $(^{r}n/R_n)$, respectively. The main purpose of this paper is to determine the order of saturation of the sequence $L_n(x)$ relative to the class K_0 of all continuous functions f for which $\tilde{f} \in \text{Lip } 1$. Our result generalizes that of Zamansky, Khan and Sahney.

Theorem. If $\left\{\Delta\lambda_{n,k}\right\}_{k=0}^{n}$ is the non-negative and non-decreasing sequence with respect to k, then the order of saturation of sequence $\{L_n(f,x)\}$ relative to the class K_0 is given by $\Delta\lambda_{n,0}$, where $\Delta\lambda_{n,0}$ is a positive non-increasing function of n.

Proof: By (2), we have (cf.[1])

$$\frac{1}{\pi}\int_{-\pi}^{\pi} L_n(x)\cos qx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=0}^{n}\lambda_{n,k}A_k(x)\cos qx\,dx =$$

$$= \frac{1}{\pi}\sum_{k=0}^{n}\lambda_{n,k}\int_{-\pi}^{\pi} a_q\cos kx\cos qx\,dx = \lambda_{n,q}\cdot a_q.$$

Therefore,

$$|a_q-\lambda_{n,q}\cdot a_q| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)-L_n(x)]\cos qx\,dx\right| \le$$

$$\le \frac{1}{\pi}\|f(x)-L_n(x)\| = o(\Delta\lambda_{n,0})$$

or

$$a_q|1-\lambda_{n,q}| = a_q\left[\sum_{p=0}^{q-1}\Delta\lambda_{n,p}\right] = o(\Delta\lambda_{n,0}),$$

$$a_q\left(\frac{\sum_{p=0}^{q-1}\Delta\lambda_{n,p}}{\Delta\lambda_{n,0}}\right) = o(1).$$

Taking the limit as $n\to\infty$, we see that $a_q=0$ for each $q-1\geq 0$. Similarly (cf.[1]) $b_q=0$ for each $q-1\geq 0$. Hence, $f(x)=\frac{1}{2}a_0$ is constant.

Let $Q_n(x)=f(x)-L_n(x)$, and $\|Q_n(x)\|=O(\Delta\lambda_{n,0})$. Now, taking the N-th arithmetic mean $\sigma_n[x;Q_n]$ of the series

$$Q_n(x) \sim \sum_{k=1}^{\infty}(1-\lambda_{n,k})A_k(x),$$

we get

$$\sigma_N[x;Q_n[=\sum_{k=1}^{N}\left(1-\lambda_{n,k}\right)\left(1-\frac{k}{N+1}\right)A_k(x). \tag{3}$$

We have $\|Q_n\|>\|\sigma_N(x;Q_n)\|$ and so

$$\left\|\sum_{k=1}^{N}\left(\frac{1-\lambda_{n,k}}{\Delta\lambda_{n,0}}\right)\left(1-\frac{k}{N+1}\right)A_k(x)\right\|=O(1) \text{ for } N\leq n.$$

Upon taking limit as $n\to\infty$, we get

$$\left\|\sum_{k=1}^{N} kA_k(x)\left(1-\frac{k}{N+1}\right)\right\| = O(1)$$

which is the (C,1) means of the Fourier series $\sum_{k=1}^{\infty}[-kA_k(x)]$. Since $-kA_k(x)=B_k'(x)$, we have $\|\tilde{\sigma}_N'(f)\|< M$ ($\tilde{\sigma}_N'(f)$ being the (C,1) means of the conjugate series). Applying Lagrange's mean value theorem, we get

$$|\tilde{\sigma}_N(f,x+t)-\tilde{\sigma}_N(f,x)|=O(|t|) \text{ as } t\to 0.$$

Finally,

$$|\tilde{f}(x+t)-\tilde{f}(x)|\leq|\tilde{f}(x+t)-\tilde{\sigma}_N(f,x+t)|+|\tilde{\sigma}_N(f,x+t)-\tilde{\sigma}_N(f,x)|+$$
$$+|\tilde{\sigma}_N(f,x)-\tilde{f}(x)|=o(1)+O(|t|)+o(t)=O(|t|).$$

Therefore $\tilde{f}\in \text{Lip } 1$.

To prove the converse part, we consider

$$\tilde{S}_n(\tilde{f},x)=\frac{1}{2\pi}\int_0^{\pi}[\tilde{f}(x+t)-\tilde{f}(x-t)]\,\frac{\cos\frac{t}{2}-\cos\frac{n+1}{2}t}{2\sin\frac{t}{2}}\,dt$$

and

$$L_n(\tilde{S}_n(\tilde{f},x))=\sum_{k=0}^{n}\Delta\lambda_{n,k}\tilde{S}_k(\tilde{f},x)=$$

$$= \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{1}{2\pi} \int_0^{\pi} [f(x+t)-f(x-t)] \cot\frac{t}{2}\,dt-$$

$$- \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{1}{2\pi} \int_0^{\pi} [\tilde{f}(x+t)-\tilde{f}(x-t)] \frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt.$$

Since $\tilde{f} \in \mathrm{Lip}\ 1$ and $-f+\frac{1}{2}a_0$ is identical to $\tilde{\tilde{f}}$, we see that

$$|f(x)-L_n(x)| = \left|\frac{1}{2\pi}\int_0^{\pi} [\tilde{f}(x+t)-\tilde{f}(x-t)] \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt\right| \le$$

$$\le \frac{1}{2\pi}\int_0^{\pi/n} |\tilde{f}(x+t)-\tilde{f}(x-t)| \left|\sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right| dt+$$

$$+ \frac{1}{2\pi}\int_{\pi/n}^{\pi} |\tilde{f}(x+t)-\tilde{f}(x-t)| \frac{1}{\sin\frac{t}{2}} \sum_{k=0}^{n} \Delta\lambda_{n,k}\cdot|\cos(k+\tfrac{1}{2})t|\,dt = I_1+I_2.$$

Since $|\tilde{f}(x-t)-\tilde{f}(x-t)| \le M|t|$, M being a constant, we have

$$I_1 \le \sum_{k=0}^{n} \Delta\lambda_{n,k} \int_0^{\pi/n} \frac{2M|t|}{|t|}\,dt = O(\Delta\lambda_{n,0}).$$

If we write

$$F_n(t) = \int_t^{\pi} \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\cos(k+\frac{1}{2})u}{\sin\frac{u}{2}}\,du,$$

then

$$I_2 \le \frac{1}{2\pi}\left[f\left(\frac{x+\pi}{n}\right) - f\left(\frac{x-\pi}{n}\right)\right] F_n\left(\frac{\pi}{n}\right)+$$

$$+ \frac{1}{2\pi}\int_{\pi/n}^{\pi} \frac{d}{dt}[\tilde{f}(x+t)-\tilde{f}(x-t)] F_n(t)\,dt = I_{21}+I_{22}.$$

Since

$$F_n(t) = \int_t^{\pi} \frac{1}{2\sin^2\frac{u}{2}} \sum_{k=0}^{n} \Delta\lambda_{n,k} \sin(k+1)u\,du + o(1),$$

the second mean value theorem gives for $t \le \xi \le \pi$,

$$F_n(t) = \frac{1}{2\sin^2\frac{t}{2}} \int_t^{\zeta} \sum_{k=0}^{n} \Delta^2\lambda_{n,k} \sin(k+1)u\,du + o(1)$$

and, consequently,

$$F_n(t) \le \frac{1}{2\sin^2 \frac{t}{2}} \sum_{k=0}^{n} \frac{|\Delta^2\lambda_{n,k}|}{K+1} + o(1) \le M \frac{1}{t^2} \frac{\Delta\lambda_{n,0}}{n} + o(1),$$

M being a constant. Hence, we have the estimates

$$I_{21} = 0\left(\frac{1}{\pi} \cdot \frac{\pi}{n} \cdot \frac{\Delta\lambda_{n,0}}{n} \cdot \frac{n^2}{\pi^2}\right) = 0(\Delta\lambda_{n,0}),$$

$$I_{22} \le M \int_{\pi/n}^{\pi} |F_n(t)|dt = M \int_{\pi/n}^{\pi} \frac{\Delta\lambda_{n,0}}{t^2 n} dt = 0(\Delta\lambda_{n,0})$$

implying $\|f(x)-L_n(x)\| = 0(\Delta\lambda_{n,0})$, which completes the proof of the theorem.

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Received May 22, 1989.