DEMONSTRATIO MATHEMATICA
Vol. XXV No 1-2 1992

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APPLICATION OF THE METHOD OF CONTOUR INTEGRATION TO THE INITIAL-BOUNDARY VALUE PROBLEM FOR A PARABOLIC SYSTEM OF EVEN ORDER

1. Statement of the problem

Using the method of contour integration, we will investigate existence and uniqueness of the solution of the mixed problem for a parabolic (in the Petrovsky sense) system of order 2s. For s=1 this problem was investigated in [2]. This paper is a continuation of [1].

Let D be a bounded domain in the Euclidean space E^n, $n\geq 3$; the boundary S of D is a closed Lapunov's surface with exponent $\kappa \in (0,1\rangle$.

Let us consider the following initial-boundary value problem. Find a vector-function $v \in C_x^{2s}(D)\cap C_t^{\infty}(0,+\infty)$ such that

$$(1)\qquad \frac{\partial v(x,t)}{\partial t} = A(x,\frac{\partial}{\partial x})\ v(x,t) \qquad \text{for} \quad (x,t) \in D\times(0,+\infty)\ ,$$

$$(2)\qquad \lim_{t\to 0^+} v(x,t) = \phi(x) \qquad \text{for} \quad x \in D,$$

$$(3)\qquad \lim_{D\ni x\to z\in S} B(z,\frac{\partial}{\partial t},\frac{\partial}{\partial x})\ v(x,t) = \Psi(z,t) \qquad \text{for } (z,t) \in S\times(0,+\infty)\ ,$$

where

$$(4)\qquad A(x,\frac{\partial}{\partial x}) = A^0(x,\frac{\partial}{\partial x}) + A^1(x,\frac{\partial}{\partial x})\ ,$$

$$(5)\qquad A^0(x,\frac{\partial}{\partial x}) =$$

$$= (-1)^s a^0(x) \sum_{i_1 \dots i_{2s}=1}^{n} a_{i_1 i_2}(x) \dots a_{i_{2s-1} i_{2s}}(x) \frac{\partial^{2s}}{\partial x_{i_1} \dots \partial x_{i_{2s}}} ,$$

(6) $$A^1(x, \frac{\partial}{\partial x}) = \sum_{0 \le k_1 + \dots + k_n \le 2s-1} a^1_{k_1 \dots k_n}(x) \frac{\partial^{k_1 + \dots + k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} ,$$

(7) $$B(z, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}) = (b^0(z) + b^1(z) \frac{\partial}{\partial t}) D^{2s-1}_{T_z} + B_1(z, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}) ,$$

(8) $$B_1(z, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}) =$$

$$= \sum_{0 \le k_1 + \dots + k_n \le 2s-2} \sum_{l=0}^{1} b^l_{k_1 \dots k_n}(z) \frac{\partial^{k_1 + \dots + k_n + l}}{\partial x_1^{k_1} \dots \partial x_n^{k_n} \partial t^l} ,$$

(9) $$D^{2s-1}_{T_z} =$$

$$= \sum_{i_1 \dots i_{2s}=1}^{n} a_{i_1 i_2}(x) \dots a_{i_{2s-1} i_{2s}}(x) \cos(n_z, x_{i_{2s}}) \frac{\partial^{2s-1}}{\partial x_{i_1} \dots \partial x_{i_{2s-1}}}$$

with data : $a^0(x)$, $a^1_{k_1 \dots k_n}(x)$, $b^0(z)$, $b^1(z)$, $b^l_{k_1 \dots k_n}(z)$ - the square N×N matrices of functions ($N \ge 1$) ; $\phi(x)$, $\Psi(z,t)$ - vector-functions of order N and $a_{ij}(x)$, $i,j = 1,\dots,n$, - real functions. In limit (3), the point $x \in D$ tends to $z \in S$ along a curve satisfying the following condition (see[4],p.115)

(10) $$\lim_{z \to z_x} \|z - z_x\|^{\kappa} \log \|x - z_x\| = 0,$$

where $\|x - z_x\| = \inf_{\xi \in S} \|x - \xi\|$.

We make the following assumptions (see[1]):

(I) The matrix $a(x) = [a_{ij}(x)]_{N \times n}$ of functions $a_{ij}(x)$, $i,j=1,\dots,n$, is symmetric and positively defined at any point $x \in \overline{D} = D \cup S$.

(II) The coefficients of the operator given by (4)-(6) are determined in the domain $D_0 \supset D$ and a^0, $a \in C^{2s}(D_0)$, $a^1_{k_1 \ldots k_n} \in C^{k_1+\ldots+k_n}(D_0)$, $0 \le k_1+\ldots+k_n \le 2s-1$.

(III) The matrices b^0, $b^1, b^l_{k_1 \ldots k_n}$, $l=1,2$, are determined and continuous on the boundary S.

(IV) The vector-function $\phi(x)$ is determined on D and satisfies the conditions : $\phi \in C^1(D)$ and there exists a domain D_1 such that $\bar{D}_1 \subset D$ and $\phi(x)=0$ for $x \in D-D_1$.

(V) The components of the vector-function $\Psi(z,t)$ are the originals of the second kind in the Zeynalov sense (see [3], p.1692).

(VI) The matrices :

$$[b^0(z) + \lambda^{2s} b^1(z)]^{-1},\ [b^0(z) + \lambda^{2s} b^1(z)]^{-1} \sum_{l=0}^{1} b^l_{k_1 \ldots k_n}(z)\ \lambda^{2sl}$$

$k_1+\ldots+k_n = 0,1,\ldots,2s-2$, are bounded for $z \in S$ and $\lambda \in R_\delta$, where

(11) $R_\delta := \{\lambda : |\arg\lambda| \le \frac{\pi+2\delta}{4s},\ |\lambda| \ge R > 0,\ \delta \in (0, \frac{\pi}{4})\}$.

(VII) The system (1) is parabolic in the Petrovsky sense.

2. Existence and uniqueness of the solution of the mixed problem

Applying formally the Laplace transformation composed with 2s-th powers of complex parameter λ to (1)-(3), we obtain the spectral problem given by (2)-(3) in [1] for $\lambda \in R_\delta$, where $\psi(z,\lambda)$ is an analytic continuation of the integral $\int_0^{+\infty} \exp(-\lambda^{2s}t)\Psi(z,t)dt$ on the region R_δ.

Now we shall give Lemmas on the existence of a solution of the mixed problem (1)-(3).

Lemma 1. Let D be a bounded domain in E^n with boundary S which is a closed Lapunov's surface with exponent $\kappa\in(0,1\rangle$, and Γ be an infinite curve lying in R_δ (see(11)), coinciding with half-lines $|\arg\lambda| = \frac{\pi+\delta}{4s}$ for sufficiently large $|\lambda|$. If the assumptions (I)-(III).(V)-(VIII) are satisfied, then the problem (1)-(3) with $\phi(x)=0$ has in the space $C_x^{2s}(D)\cap C_t^{\infty}(0,+\infty)$ the solution which may be represented in the following form

$$(12)\qquad v_1(x,t) = \frac{s}{\pi i} \int_\Gamma \lambda^{2s-1}\exp(\lambda^{2s}t)\ u(x,\lambda)\ d\lambda ,$$

where $u(x,\lambda)$ is the solution of the spectral problem given by (47) in [1] for $\lambda \in R_\delta$.

Proof. We shall first prove that the function defined by (12) is of the class $C_x^{2s}(D)\cap C_t^{\infty}(0,+\infty)$. In order to do that we must check locally uniform convergence at any point $(x_0,t_0) \in D\times(0,+\infty)$ of the following integrals

$$(13)\qquad \int_\Gamma \lambda^{2(s+1)-1}\exp(\lambda^{2st})\frac{\partial^k u(x,\lambda)}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}\ d\lambda$$

for $k=0,\dots,2s$, $k=k_1+\dots+k_n$, $1=0,1,\dots$. Let us observe that for $t>0$ and for sufficiently large $|\lambda|$, if $\lambda\in\Gamma$, then

$$|\exp(\lambda^{2s}t)|=\exp(\mathrm{Re}\lambda^{2s}t)=\exp(|\lambda|^{2s}t\ \cos\tfrac{\pi+\delta}{2}) = \exp(-\delta_1|\lambda|^{2s}t),$$

where $\delta_1= -\cos\frac{\pi+\delta}{2} = \sin\frac{\delta}{2} > 0$. Let Q_m, Q_p denote points of the curve Γ such that $|OQ_m|=r_m$ and $|OQ_p|=r_p$, respectively. We assume that $p>m$ and $\lim\limits_{m\to+\infty} r_m=+\infty$. We shall prove that

$$(14)\qquad \int_{\widetilde{Q_mQ_p}} \lambda^{2(s+1)-1}\ \exp(\lambda^{2s}t)\frac{\partial^k u(x,\lambda)}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}\ d\lambda \quad \underset{m\to+\infty}{\rightrightarrows} 0$$

for all $t\in (\frac{1}{2}t_0,\ \frac{3}{2}t_0)\subset (0,+\infty)$, $x\in K(x_0,\frac{1}{2}d(x_0))\subset D$,

where $\widetilde{Q_mQ_p}$ is an arc of the curve Γ, $K(x_0,\frac{1}{2}d(x_0))$ is the ball with center x_0 and radius $\frac{1}{2}d(x_0)$ and $d(x_0)$ is the distance of the point x_0 from the boundary S.

From the inequality (62) in [1] we obtain

$$\left| \int\limits_{\underbrace{Q_m Q_p}} \lambda^{2(s+1)-1} \exp(\lambda^{2s}t) \frac{\partial^k u(x,\lambda)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \, d\lambda \right| \le$$

$$\le C \int\limits_{r_m}^{r_p} r^{2(s+1)-1} \exp(-\delta_1 r^{2s} t) \, \mu(r) \frac{\exp(-\varepsilon \, r \, d(x))}{d(x)^{n-2s+k}} \, dr ,$$

where, on the basis of the assumption (V), we have

$$\mu(r) := \sup_{|\lambda|=r} \sup_{\xi\in S} |\Psi(\xi,\lambda)| \underset{r\to+\infty}{\longrightarrow} 0 .$$

Thus

$$\left| \int\limits_{\underbrace{Q_m Q_p}} \lambda^{2(s+1)-1} \exp(\lambda^{2s}t) \frac{\partial^k u(x,\lambda)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \, d\lambda \right| \le$$

$$\le \frac{(r_p^{2(s+1)} - r_m^{2(s+1)})}{2(s+1)} \; \frac{C \, \mu(r_m) \exp(-\delta_1 r_m^{2s} t) \exp(-\varepsilon r_m d(x))}{d(x_0)^{n-2s+k}} \le$$

$$\le 2^{n-2s+k} \frac{r_p^{2(s+1)}}{2(s+1)} \; \frac{C \, \mu(r_m) \exp(-\delta_1 r_m^{2s} t_0/2) \exp(-\varepsilon r_m d(x_0)/2)}{d(x_0)^{n-2s+k}}$$

for all $t\in (\frac{1}{2}t_0, \frac{3}{2}t_0)$, $x\in K(x_0, \frac{1}{2}d(x_0))$.

It means that the condition (14), i.e. locally uniform convergence of the integrals (13), is satisfied in points $(x_0,t_0) \in D\times(0,+\infty)$.

Applying the results (63),(65) in [1], by means of similar considerations as above, one can show the locally uniform convergence of integrals

$$\int\limits_{\Gamma} \lambda^{2(s+1)-1} \exp(\lambda^{2st}) \frac{\partial^k u(z,\lambda)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \, d\lambda$$

for $k=0,\dots,2s-2$, $k=k_1+\dots+k_n$, $l=0,1,\dots$ and

$$\int_{\Gamma} \lambda^{2(s+1)-1} \exp(\lambda^{2st})\, D_{T_z}^{2s-1}\, u(z,\lambda)\, d\lambda \quad \text{for } l=0,1,\ldots$$

in each point $(z,t) \in S\times(0,+\infty)$.

Thus, the function $v_1(x,t)$ can be substituted in the equation (1) and in the boundary condition (3). Since $u(x,\lambda)$ is the solution of the spectral problem (2)-(3) from [1], therefore we obtain the following equation

$$(15) \qquad A(x,\tfrac{\partial}{\partial x})v_1(x,t) - \frac{\partial\, v_1(x,t)}{\partial\, t} =$$

$$= \frac{s}{\pi i} \int_{\Gamma} \lambda^{2s-1} [A(x,\tfrac{\partial}{\partial x}) - \lambda^{2s} I]\, d\lambda = 0.$$

It means that $v_1(x,t)$ is the solution of the system (1). Similarly, using the Zeynalov theorem (cf.[3], p.1692), we obtain

$$(16) \qquad \lim_{D\ni x\to z\in S} B(z,\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial x})\; v_1(x,t) =$$

$$= \frac{s}{\pi i} \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s}t) \lim_{D\ni x\to z\in S} B(z,\lambda^{2s},\tfrac{\partial}{\partial x})\; u(x,\lambda)\; d\lambda =$$

$$= \frac{s}{\pi i} \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s}t)\; \psi(z,\lambda)\; d\lambda = \Psi(z,t)\ .$$

Thus, v_1 satisfies the boundary condition (3) . To check the initial condition we can show that

$$(17) \qquad \int_{\Gamma} \lambda^{2s-1}\, u(x,\lambda)\; d\lambda = 0 \quad \text{for } x\in D.$$

From the inequality (62) in [1] it results that

$$(18) \qquad |\, \lambda^{2s-1}\, u(x,\lambda)| \le$$

$$\le C_1 |\lambda|^{2s-1} \sup_{\xi\in S} |\psi(z,\lambda)|\; \frac{\exp(-\varepsilon|\lambda| d(x))}{d(x)^{n-2s}} \le \frac{C(x)}{|\lambda|^{1+b}}\ ,$$

where $b>0$.

Let us denote by Γ_m the portion of the curve Γ lying inside the circle with radius r_m (we assume that $\lim_{m\to\infty} r_m = +\infty$), and by θ_m the arc of this circle lying in R_δ. Since $\lambda^{2s-1}u(x,\lambda)$ is the analytic function in R_δ, therefore, on the basis of the Cauchy theorem, we obtain

$$\int_\Gamma \lambda^{2s-1} u(x,\lambda)\, d\lambda = \lim_{m\to+\infty} \int_{\Gamma_m} \lambda^{2s-1} u(x,\lambda) d\lambda = \lim_{m\to+\infty} \int_{\theta_m} \lambda^{2s-1} u(x,\lambda) d\lambda.$$

From the inequality (18), for sufficiently large m, we have

$$\left|\int_{\theta_m} \lambda^{2s-1}u(x,\lambda)d\lambda\right| \le C(x) \int_{-\frac{\pi+\delta}{4s}}^{\frac{\pi+\delta}{4s}} \frac{r_m}{r_m^{b+1}}\, d\theta = \frac{C(x)\,(\pi+\delta)}{2s\ r_m^b} .$$

Thus, we obtain the equality (17) which ends the proof of Lemma 1.

Lemma 2. Let the assumptions of Lemma 1 on D,S,Γ and (I)-(VII) be satisfied. If $G(x,\xi,\lambda)$ is the Green matrix for the problem (2)-(3) from [1] (cf.(66)), then the problem (1)-(3) with $\Psi(z,t)=0$ has, in the class $C_x^{2s}(D)\cap C_t^\infty(0,+\infty)$ the solution which may be represented in the form

(19) $$v_2(x,t) = \frac{s}{\pi i} \int_\Gamma \lambda^{2s-1}\exp(\lambda^{2s}t) \int_D G(x,\xi,\lambda)\phi(\xi)d\xi\, d\lambda .$$

Proof. Let us rewrite the function $v_2(x,t)$ in the form

(20) $$v_2(x,t) = v_{21}(x,t) + v_{22}(x,t),$$

where

(21) $$v_{21}(x,t) = \frac{s}{\pi i} \int_\Gamma \lambda^{2s-1}\exp(\lambda^{2s}t) \int_D P(x,\xi,\lambda)\phi(\xi)d\xi\, d\lambda ,$$

(22) $$v_{22}(x,t) = -\frac{s}{\pi i} \int_\Gamma \lambda^{2s-1}\exp(\lambda^{2s}t) \int_D Q(x,\xi,\lambda)\phi(\xi)d\xi\, d\lambda ,$$

and $P(x,\xi,\lambda)$ is the fundamental solution of the equation (2) in [1] (cf.(16),(16')).

Let us check the differentiability of the function (21), i.e. the locally uniform convergence of integrals

$$(23)\qquad \int_{\Gamma} \lambda^{2(s+1)-1}\exp(\lambda^{2s}t)\int_{D} \frac{\partial^{k}P(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\,\phi(\xi)d\xi\, d\lambda$$

for $(x,t)\in D\times(0,+\infty)$.

Applying the inequality (19) from [1] and using polar coordinates, we have

$$(24)\qquad \left|\int_{D} \frac{\partial^{k}P(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\,\phi(\xi)d\xi\right| \le C_1 \int_{D} \frac{\exp(-\varepsilon|\lambda|\,\|x-\xi\|)}{\|x-\xi\|^{n-2s+k}}\,d\xi \le$$

$$\le C_2 \int_{0}^{+\infty} r^{2s-k-1}\exp(-\varepsilon|\lambda|r)dr \le \frac{C}{|\lambda|^{2s-k}} \le \frac{C}{|\lambda|}$$

for $k=k_1+\ldots+k_n$, $k=0,..,2s-1$, $x\in D$. Similarly as in the proof of Lemma 1, we can conclude that from (24) it follows the locally convergence of integrals (23) for $k=0,\ldots,2s-1$ in each point $(x,t)\in D\times(0,+\infty)$. Let us estimate the integral

$$\frac{\partial^{2s}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\int_{D} P(x,\xi,\lambda)\ \phi(\xi)d\xi\ ,\qquad k_1+\ldots k_n=2s,\ x\in D\ ,$$

where

$$(25)\qquad P(x,\xi,\lambda) = P_0(x,\xi,\lambda) + P_1(x,\xi,\lambda)\ .$$

Since $P_1(x,\xi,\lambda)$ satisfies the condition (18) from [1], we obtain the following inequalities

$$(26)\qquad \left|\frac{\partial^{2s}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\int_{D} P_1(x,\xi,\lambda)\ \phi(\xi)d\xi\right| \le$$

$$\le C_1\int_{D} \frac{\exp(-\varepsilon|\lambda|\,\|x-\xi\|)}{\|x-\xi\|^{n-1}}d\xi \le C_2 \int_{0}^{+\infty} \exp(-\varepsilon|\lambda|r)dr \le \frac{C}{|\lambda|}\quad \text{for } x\in D.$$

By the formula (13) from [1] determining P_0 ,we have

$$(27)\qquad \frac{\partial P_0(x,\xi,\lambda)}{\partial x_j} + \frac{\partial P_0(x,\xi,\lambda)}{\partial \xi_j} =$$

$$= \frac{i}{(2\pi)^n} \int_{E^n} e^{i(x-\xi,\alpha)} \frac{\partial}{\partial \xi_j} [(a(\xi)\alpha,\alpha)^s a^0(\xi)-\lambda^{2s}I]^{-1} d\alpha .$$

Let us observe that, by assumption (II), the matrix $\frac{\partial}{\partial \xi_j}[(a(\xi)\alpha,\alpha)^s a^0(\xi)-\lambda^{2s}I]^{-1}$ exists and has singularity at $\|\alpha\| \to +\infty$ of the same order as the matrix $[(a(\xi)\alpha,\alpha)^s a^0(\xi)-\lambda^{2s}I]^{-1}$. Thus, by the inequality (15) in [1], we get the estimate

$$(28) \quad \left| \frac{\partial^{2s-1}}{\partial x_1^{k_1}\ldots\partial x_j^{k_j-1}\ldots\partial x_n^{k_n}} \left\{ \frac{\partial P_0(x,\xi,\lambda)}{\partial x_j} + \frac{\partial P_0(x,\xi,\lambda)}{\partial \xi_j} \right\} \right| \le$$

$$\le C \frac{\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-1}} .$$

Thus

$$(29) \quad \frac{\partial^{2s}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} \int_D P_0(x,\xi,\lambda) \; \phi(\xi)d\xi =$$

$$= \int_D \frac{\partial^{2s-1}}{\partial x_1^{k_1}\ldots\partial x_j^{k_j-1}\ldots\partial x_n^{k_n}} \left\{ \frac{\partial P_0(x,\xi,\lambda)}{\partial x_j} + \frac{\partial P_0(x,\xi,\lambda)}{\partial \xi_j} \right\} \phi(\xi)d\xi -$$

$$- \int_D \frac{\partial^{2s} P_0(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_j^{k_j-1}\ldots\partial x_n^{k_n} \partial \xi_j} \phi(\xi) \; d\xi = U_1(x,\xi,\lambda)+U_2(x,\xi,\lambda),$$

where, by assumption (IV), we have

$$U_2(x,\xi,\lambda) = -\int_D \frac{\partial^{2s} P_0(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_j^{k_j-1}\ldots\partial x_n^{k_n} \partial \xi_j} \phi(\xi) \; d\xi =$$

$$= \int_D \frac{\partial^{2s-1} P_0(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_j^{k_j-1}\ldots\partial x_n^{k_n}} \frac{\partial \phi(\xi)}{\partial \xi_j} d\xi .$$

From (28),(29) and the inequality (15) from [1] we obtain

$$(30)\qquad \left|\frac{\partial^{2s}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}\int_D P_0(x,\xi,\lambda)\phi(\xi)d\xi\right| \le C_1\int_D \frac{\exp(-\varepsilon|\lambda|\,\|x-\xi\|)}{\|x-\xi\|^{n-1}}\,d\xi \le$$

$$\le C_2\int_0^{+\infty}\exp(-\varepsilon|\lambda|r)dr \le \frac{C}{|\lambda|}\qquad \text{for } x\in D.$$

The inequalities (26),(30) imply the locally uniform convergence of integrals (23) for $k=2s$ in any point $(x,t)\in D\times(0,+\infty)$, so $v_{21}\in C_x^{2s}(D)\cap C_t^{\infty}(0,+\infty)$.

In a similar way, applying the inequality (77) from [1],we prove the locally uniform convergence of integrals

$$\int_\Gamma \lambda^{2(s+1)-1}\exp(\lambda^{2st})\int_D \frac{\partial^k Q(x,\xi,\lambda)}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}\phi(\xi)d\xi\ d\lambda$$

for $k=0,1,\dots,2s$, $l=0,1,\dots$ in the point $(x,t)\in D\times(0,+\infty)$, which finally means, by (19), (20), that $v_2\in C_x^{2s}(D)\cap C_t^{\infty}(0,+\infty)$. Using the inequalities (19),(78), (80) from [1] we can, in a way as above, show the locally uniform convergence of integrals

$$\int_\Gamma \lambda^{2(s+1)-1}\exp(\lambda^{2st})\int_D \frac{\partial^k G(z,\xi,\lambda)}{\partial z_1^{k_1}\dots\partial z_n^{k_n}}\phi(\xi)d\xi\ d\lambda$$

for $k=0,\dots,2s-2$, $k=k_1+\dots+k_n$, $l=0,1,\dots$ and

$$\int_\Gamma \lambda^{2(s+1)-1}\exp(\lambda^{2st})\int_D D_{T_z}^{2s-1} G(z,\xi,\lambda)\phi(\xi)d\xi\ d\lambda$$

for $l=0,1,\dots$in each point $(z,t)\in S\times(0,+\infty)$. Hence, we can introduce the boundary operator $B(z,\frac{\partial}{\partial t},\frac{\partial}{\partial x})$ for $t>0$ under the integral sign in (19).

Since $Q(x,\xi,\lambda)$ is the regular part of the Green function satisfying the homogeneous equation (2) from [1], we obtain

$$(31)\qquad A(x,\tfrac{\partial}{\partial x})\ v_{22}(x,t) - \tfrac{\partial}{\partial t}v_{22}(x,t) =$$

$$= \frac{s}{\pi i} \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s} t) \int_{D} (A(x,\frac{\partial}{\partial x}) - \lambda^{2s} I) Q(x,\xi,\lambda)\ \phi(\xi) d\xi d\lambda = 0.$$

Similarly, applying the formula (20) from [1], we have

$$(32) \qquad A(x,\frac{\partial}{\partial x})\ v_{21}(x,t) - \frac{\partial}{\partial t}\ v_{21}(x,t) =$$

$$= \frac{s}{\pi i} \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s} t) \int_{D} (A(x,\frac{\partial}{\partial x}) - \lambda^{2s} I) P(x,\xi,\lambda)\ \phi(\xi) d\xi d\lambda =$$

$$= - \frac{s}{\pi i}\ \phi(x) \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s} t) d\lambda \ .$$

We shall prove that

$$(33) \qquad \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s} t)\ d\lambda = 0.$$

Let us denote by Γ_m the arc of Γ lying inside the circle with a radius r_m (we assume that $\lim_{m\to\infty} r_m = +\infty$). So we have

$$\int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s} t)\ d\lambda = \lim_{m\to\infty} \int_{\Gamma_m} \lambda^{2s-1} \exp(\lambda^{2s} t)\ d\lambda \ ,$$

$$\left| \int_{\Gamma_m} \lambda^{2s-1} \exp(\lambda^{2s} t)\ d\lambda \right| = \frac{1}{2st} \left| \exp(\ r_m^{2s} t(\cos\frac{\pi+\delta}{2} + i \sin\frac{\pi+\delta}{2})) - \right.$$

$$\left. - \exp(\ r_m^{2s} t(\cos\frac{\pi+\delta}{2} - i \sin\frac{\pi+\delta}{2})) \right| \le \frac{1}{st} \exp(-r_m^{2s} t\delta_1),$$

where $\delta_1 = \sin \frac{\delta}{2} > 0$. Thus, for $t>0$ we obtain

$\lim_{m\to\infty} \int_{\Gamma_m} \lambda^{2s-1} \exp(\lambda^{2s} t)\ d\lambda = 0$ which implies (33). So (32) becomes $A(x,\frac{\partial}{\partial x})\ v_{21}(x,t) - \frac{\partial}{\partial t}\ v_{21}(x,t) = 0$ which shows, together with (31), that the function $v_2(x,t)$ satisfies the equation (1).

Since the Green matrix $G(x,\xi,\lambda)$ satisfies the homogeneous boundary condition of the spectral problem (cf. (3) in [1]), we have the equality

$$\lim_{D\ni x\to z\in S} B(z,\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial x})\; v_2(x,t) =$$

$$= \frac{s}{\pi i}\int_\Gamma \lambda^{2s-1}\exp(\lambda^{2s}t)\int_D \lim_{D\ni x\to z\in S} B(z,\lambda^{2s},\tfrac{\partial}{\partial x})G(x,\xi,\lambda)\phi(\xi)d\xi d\lambda=0.$$

Now, we will show that

$$(34)\qquad \lim_{t\to 0^+} v_2(x,t) = \phi(x) \qquad \text{for } x\in D.$$

From (22) we have

$$\lim_{t\to 0^+} v_{22}(x,t) = -\frac{s}{\pi i}\int_\Gamma \lambda^{2s-1}\int_D Q(x,\xi,\lambda)\phi(\xi)d\xi d\lambda .$$

The inequality (79) for k=0 from [1] implies

$$\left|\int_D Q(x,\xi,\lambda)\phi(\xi)d\xi\right| \le \frac{C}{|\lambda|^{2s+n-1}} .$$

Thus, applying analyticity of $\int_D Q(x,\xi,\lambda)\phi(\xi)d\xi$ in the region R_δ, we have

$$(35)\qquad \lim_{t\to 0^+} v_{22}(x,t) = 0.$$

Now, for t>0, we can transform $v_{21}(x,t)$ as follows (see (21))

$$(36)\quad v_{21}(x,t)=\frac{s}{\pi i}\int_\Gamma \lambda^{-1}\exp(\lambda^{2s}t)[\lambda^{2s}I-A(x,\tfrac{\partial}{\partial x})]\int_D P(x,\xi,\lambda)\phi(\xi)d\xi d\lambda+$$
$$+\frac{s}{\pi i}\int_\Gamma \lambda^{-1}\exp(\lambda^{2s}t)A(x,\tfrac{\partial}{\partial x})\int_D P(x,\xi,\lambda)\phi(\xi)d\xi d\lambda .$$

By (20) from [1], we can further write

$$(37)\qquad v_{21}(x,t) = \frac{s}{\pi i}\phi(x)\int_\Gamma \lambda^{-1}\exp(\lambda^{2s}t)d\lambda +$$
$$+\frac{s}{\pi i}\int_\Gamma \lambda^{-1}\exp(\lambda^{2s}t)\; A(x,\tfrac{\partial}{\partial x})\int_D P(x,\xi,\lambda)\phi(\xi)d\xi d\lambda .$$

Let us notice that applying the Zeynalov theorem (cf.[3], p.1692) we have

$$\frac{s}{\pi i} \int_{\Gamma} \lambda^{-1} \exp(\lambda^{2s} t) d\lambda = 1 .$$

From (24),(25),(26), (30) we obtain the estimate

$$|A(x,\frac{\partial}{\partial x}) \int_{D} P(x,\xi,\lambda)\phi(\xi)d\xi| \le \frac{C}{|\lambda|} .$$

So, taking the limit at $t\to 0$, we conclude that the integrand function in the second integral of the formula (37) descreases when $O(|\lambda|^{-2})$ at $|\lambda|\to+\infty$ .By means of this, we can easily prove that the limit at $t\to 0^+$ of the second component of the sum (37) equals zero, so finally gives the following condition

$$\lim_{t\to 0^+} v_{21}(x,t) = \phi(x) \qquad \text{for } x \in D. \tag{38}$$

By (20), (35), (38), we conclude that the function $v_2(x,t)$ satisfies the initial condition (2).

Lemmas 1 and 2 imply the following theorem .

Theorem 1. If the assumptions of Lemma 1 and Lemma 2 are satisfied, then the initial-boundary value problem (1)-(3) has the solution in the following form

$$v(x,t) = \frac{s}{\pi i} \int_{\Gamma} \lambda^{2s-1} \exp(\lambda^{2s} t)[u(x,\lambda) + \int_{D} G(x,\xi,\lambda)\phi(\xi)d\xi]d\lambda ,$$

where $u(x,\lambda)$ is the solution of the appropriate spectral problem, and $G(x,\xi,\lambda)$ is the Green function for this problem.

Besides, we have the result as follows.

Theorem 2. If the assumptions of Lemma 1 and Lemma 2 are satisfied, then the problem (1)-(3) has exactly one solution $u \in C_x^{2s}(D)\cap C_t^{\infty}(0,+\infty)$ satisfying conditions of the second-kind orginal in Zeynalov's sense (cf.[3], p.1692).

Proof. Let $v_1(x,t)$ and $v_2(x,t)$ be two different solutions of the problem (1)-(3). Then the function $v(x,t)=v_1(x,t)-v_2(x,t)$ is the solution of the homogeneous problem

$$\frac{\partial v(x,t)}{\partial t} = A(x,\frac{\partial}{\partial x}) v(x,t) \qquad \text{for } (x,t)\in D\times(0,+\infty) ,$$

$$\lim_{t\to 0^+} v(x,t) = 0 \qquad \text{for } x\in D,$$

$$\lim_{D\ni x\to z\in S} B(z,\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial x})\ v(x,t) = 0 \qquad \text{for } (z,t)\in S\times(0,+\infty)\ .$$

Using the Laplace transformation composed with 2s-th powers of the complex parameter λ^{2s}, we obtain the proper spectral problem

$$A(x,\tfrac{\partial}{\partial x})\ u(x,\lambda) - \lambda^{2s}u(x,\lambda) = 0 \qquad \text{for } x\in D,\ \lambda\in R_\delta\ ,$$

$$\lim_{D\ni x\to z\in S} B(z,\lambda^{2s},\tfrac{\partial}{\partial x})\ u(x,\lambda) = 0 \qquad \text{for } z\in S,\ \lambda\in R_\delta .$$

If $u(x,\lambda)$ is its solution, then on the basis of the inequalities (62) and (63) from [1] we have $u(x,\lambda) = 0$ for $x\in\bar{D}$ and $\lambda\in R_\delta$. From the Zeynalov theorem (cf.[3], p.1692) it results that $v(x,t)=0$ for $(x,t)\in\bar{D}\times[0,+\infty)$, which is contradictory to the assumption that $v_1(x,t),v_2(x,t)$ are different.

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Received May 8, 1989.