DEMONSTRATIO MATHEMATICA
Vol. XXV No 1-2 1992

Kazimierz Sienkiewicz

DETERMINANT SYSTEMS IN A CERTAIN CLASS OF ALGEBRAS

0. Introduction

Let $L(X)$ denote the algebra of all endomorphisms on a linear space X (over the real or complex field of scalars). An operator $B \in L(X)$ is called a reciprocal generalized inverse (or almost inverse, or quasi-inverse) to an operator $A \in L(X)$ if $ABA=A$ and $BAB=B$. This notion plays an especially important part in the theory of linear equations. If B is a reciprocal generalized inverse to A, then the general form of solution of the equation

(0.1) $$Ax = x_0$$

where x_0 belongs to the range of A, is

$$x = Bx_0 + x_1$$

where x_1 is an element of Kernel of the operator A.

In the fifties and early sixties R. Sikorski introduced the notion of determinant system $\{D_n\}$ for an operator A. The determinant system for A gives full information of solving the equation (0.1). If we know the determinant system for A, then we can obtain a generalized inverse B of A and therefore we can completely solve the equation (0.1). The Sikorski's and Buraczewski's formulae for solution are generalized version of the well known Cramer formulae from Algebra.

Thus, the main problems which arise in the determinant theory are the following: under what conditions A has a determinant system $\{D_n\}$ and what is the relationship between A

and $\{D_n\}$. The answer is given by the following theorems: A has a determinant system $\{D_n\}$ if and only if A is Fredholm, and then $\{D_n\}$ is determined by A uniquely up to a constant factor different from zero.

The main aim of the theory of determinants in Banach spaces is to give some analytic formulae for determinant systems of operators. We know the analytic formula for a determinant system of an operator of the form S+T where A is Fredholm and T is a quasinuclear (integral) operator.

So far the determinant theory has been applied only in (Banach) algebras of linear (bounded) operators on a linear (Banach) space. The purpose of this paper is to develope the determinant theory for a certain class of algebras over the field K of real or complex numbers in a similar way as in vector space by R. Sikorski and A. Buraczewski.

Definition 0.1. Let A be an algebra with identity and J be any fixed two-sided ideal of A. The element $a \in A$ is calleed a Fredholm element relative to the ideal J iff the coset a+J is invertible in the quotient algebra A/J.

The notions of g-algebra and g-determinant system are defined in § 1 and their fundamental properties are considered. In § 2 we deal with g-total algebras. We prove that every Fredholm element relative to the special ideal S_g has a g-determinant system. In § 3 it is shown that every element in a g-small algebra having a g-determinant system must be Fredholm. Quasinuclei and quasinuclear elements are defined and their properties are considered in § 4.

The author cordially thanks A. Buraczewski for suggestion of the problem and useful discussions.

1. Definition of the g-determinant system

Definition 1.1. An algebra A over the field K of real or complex scalars is called a g-algebra if there exists an element $g \in A$ such that:

1° $gg = g \neq 0$;

2° $gxg = \delta_x g$, where $\delta_x \in K$ for each $x \in A$.

The linear functional $F: A \longrightarrow K$ defined by

$$F(x) = \delta_x, \qquad x \in A$$

is called a g-functional.

Corollary 1.2. If A is a g-algebra and F is a g-functional, then

$$F(x)F(y) = F(xgy)$$

for all $x,y \in A$.

Example 1.3. Let us consider the algebra $A = \left\{ \begin{bmatrix} a b \\ 0 c \end{bmatrix}; \; a,b,c \in K \right\}$ with the usual operations of addition and multiplication, and let $g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to see that A is a g_1-algebra and a function $F_1: A \longrightarrow K$ defined by

$$F_1 \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = a, \qquad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in A$$

is the g_1-functional.

Example 1.4. Let now A be the algebra of all bounded functions defined on the closed interval [0,1] with the usual operations. For any fixed $t_0 \in [0,1]$, let us define the function $g_{t_0} \in A$ by the formula:

$$g_{t_0}(t) = \begin{cases} 1 \text{ if } & t=t_0 \\ 0 \text{ if } & t \neq t_0 \end{cases} \qquad t \in [0,1].$$

The algebra A is a g_{t_0}-algebra for every fixed $t_0 \in [0,1]$, and the functional $F_{t_0}: A \longrightarrow K$

$$F_{t_0}(x) = x(t_0), \qquad x \in A$$

is the g_{t_0}-functional.

Example 1.5. Let Ξ and X be a pair of conjugate linear spaces[1)] over the real or complex field K, i.e. there exists a bilinear functional on $\Xi \times X$ whose the value at the point (ξ, x) is denoted by ξx and which satisfies two additional conditions:

(a) if $\xi x=0$ for every $\xi \in \Xi$, then $x=0$;

(b) if $\xi x=0$ for every $x \in X$, then $\xi=0$.

Let us consider the algebra $U=U(\Xi, X)$ of all endomorphisms E on X such that:

(c) for every fixed $\xi \in \Xi$ there exists an $\eta \in \Xi$ such that $\xi(Ex) = \eta x$ for every $x \in X$.

For fixed $\xi_0 \in \Xi$ and $x_0 \in X$, let $x_0 \cdot \xi_0$ denote the one-dimensional endomorphism on X defined by the formula:

$$(x_0 \cdot \xi_0)(x) = (\xi_0 x) x_0 \qquad x \in X.$$

If moreover $\xi_0 x_0 = 1$, then the algebra U is the $x_0 \cdot \xi_0$-algebra and the $x_0 \cdot \xi_0$-functional $F: U \longrightarrow K$ is defined by

$$F(E) = \xi_0(Ex_0) \qquad E \in U.$$

1.6. In a g-algebra A the following notations will be used:

$$L = Ag; \qquad P = gA;$$

$$S_g = \left\{ \sum_{i=1}^{n} x_i g y_i; \quad x_i, y_i \in A, \quad n \in N \right\};$$

Φ_g - the set of all Fredholm elements of the algebra A with identity relative to the ideal S_g.

Definition 1.7. Let A be a g-algebra. Every infinite sequence $D_0, D_1, \ldots$ is called a g-determinant system (with nonnegative index d) for an element $a \in A$ if the following

1) - see Sikorski [9] and Buraczewski [1].

conditions are satisfied:

(d_0) D_0 is a number if d=0, for otherwise D_0 is a d-linear functional on P^d such that for every fixed $p_1,\dots,p_{i-1}$, $p_{i+1},\dots,p_d \in P$ (i=1,...,d) there exists an element $c \in A$ satisfying the following identity

$$D_0(p_1,\dots,p_d) = F(p_i c);$$

(d_1) D_n is a (2n+d)-linear functional on $P^{n+d} \times L^n$ for $n \geq 1$, the value of D_n at the point $(p_1,\dots,p_{n+d},l_1,\dots,l_n)$ we denote by

$$D_n \begin{pmatrix} p_1,\dots,p_{n+d} \\ l_1,\dots,l_n \end{pmatrix};$$

(d_2) D_0 is skew symmetric on P^d (if d>0), D_n ($n \geq 1$) is skew symmetric on P^{n+d} and skew symmetric on L^n;

(d_3) if D_n ($n \geq 1$) is interpreted as a function of p_i and l_j only ($1 \leq i \leq n+d, 1 \leq j \leq n$), then there exists an element $c \in A$ such that

$$D_n \begin{pmatrix} p_1,\dots,p_{n+d} \\ l_1,\dots,l_n \end{pmatrix} = F(p_i c l_j);$$

(d_4) there exists an integer $r \geq 0$ such that D_r does not vanish identically;

(d_5) the following identities hold for n=0,1,... :

$$D_{n+1} \begin{pmatrix} p_0 a, p_1,\dots,p_{n+d} \\ l_0, l_1,\dots,l_n \end{pmatrix} =$$

$$= \sum_{i=0}^{n} (-1)^i F(p_0 l_i) D_n \begin{pmatrix} p_1,\dots\dots\dots\dots\dots,p_{n+d} \\ l_0,\dots,l_{i-1},l_{i+1},\dots,l_n \end{pmatrix};$$

$$D_{n+1}\begin{pmatrix} p_0,p_1,\dots,p_{n+d} \\ al_0,l_1,\dots,l_n \end{pmatrix} =$$

$$= \sum_{i=0}^{n+d} (-1)^i F(p_i l_0)\, D_n \begin{pmatrix} p_0,\dots,p_{i-1},p_{i+1},\dots,p_{n+d} \\ l_1,\dots\dots\dots,l_n \end{pmatrix}.$$

Analogously we define the g-determinant system with a negative index. Then the number of l_i is larger then that of p_i. The least integer r, such that D_r does not vanish identically, will be called the order of the g-determinant system, noted by $r(D_n)$. The difference d between the numbers of p_i and l_i in D_n $(n\geq 0)$ is called the index of the g-determinant system, denoted by $d(D_n)$.

Remark 1.8. If the sequence (D_n) is a g-determinant system for an element $a\in A$ and $\alpha\neq 0$, then the sequence (αD_n) is also a g-determinant system for a, and the sequence $(\alpha^{-n}D_n)$ is a g-determinant system for the element $\alpha a\in A$.

Remark 1.9. If the sequence (D_n) is a g-determinant system for an element a of g-algebra A with identity and $b\in A$ has the inverse $b^{-1}\in A$, then the sequence (D'_n) defined by

$$D'_n\begin{pmatrix} p_1,\dots,p_{n+d} \\ l_1,\dots,l_n \end{pmatrix} = D_n\begin{pmatrix} p_1 b^{-1},\dots,p_{n+d}b^{-1} \\ l_1,\dots\dots,l_n \end{pmatrix} \qquad n=0,1,2,\dots$$

is a g-determinant system for ab. Similarly, the sequence (D''_n) defined by

$$D''_n\begin{pmatrix} p_1,\dots,p_{n+d} \\ l_1,\dots,l_n \end{pmatrix} = D_n\begin{pmatrix} p_1,\dots\dots,p_{n+d} \\ b^{-1}l_1,\dots,b^{-1}l_n \end{pmatrix} \qquad n=0,1,2,\dots$$

is a g-determinant system for ba.

Example 1.10. Let A be the g_{t_0}-algebra defined in Example 1.4. The sequence (D_n)

$$(1.10.1) \qquad D_n\begin{pmatrix} p_1,\dots,p_n \\ l_1,\dots,l_n \end{pmatrix} = \begin{cases} 0 & \text{for } n=0,2,3,\dots \\ (p_1 l_1)(t_0) & \text{for } n=1 \end{cases}$$

is a g_{t_0}-determinant system of the element $h \in A$ defined by

$$h(t) = t - t_0 \qquad t \in [0,1].$$

The reader can easily see that the conditions (d_0)-(d_5) are satisfied.

Example 1.11. Let U be an $x_0 \cdot \xi_0$-algebra defined in Example 1.5. Then the ideals P and L are of the form

$$P = \left\{x_0 \cdot \xi;\ \xi \in \Xi\right\}, \qquad L = \left\{x \cdot \xi_0;\ x \in X\right\}$$

and for $E \in U$

$$F((x_0 \cdot \xi) \circ E \circ (x \cdot \xi_0)) = F((\xi E x)(x_0 \cdot \xi_0)) = \xi E x.$$

If the sequence (W_n) is a determinant system for an operator $E \in U$ in the sense of Buraczewski's definition (see [1]), then the sequence (D_n) defined by the formula

$$D_n \begin{pmatrix} x_0 \cdot \xi_1,\dots,x_o \cdot \xi_{n+d} \\ x_1 \cdot \xi_o,\dots,x_n \cdot \xi_0 \end{pmatrix} = W_n \begin{pmatrix} \xi_1,\dots,\xi_{n+d} \\ x_1,\dots,x_n \end{pmatrix} \qquad n=0,1,2,\dots$$

is a $x_0 \cdot \xi_0$-determinant system for the element E of the $x_0 \cdot \xi_0$-algebra U in the sense of Definition 1.7.

Example 1.12. Now, let A be any g-algebra with identity e. The sequence (θ_n) defined by

$$\theta_n \begin{pmatrix} p_1,\dots,p_n \\ l_1,\dots,l_n \end{pmatrix} = \begin{cases} 1 & \text{for } n=0 \\ \det(F(p_s l_t)) & \text{for } n>0 \end{cases} \qquad (1 \le s,t \le n),$$

is a g-determinant system for the element e. Indeed, the axioms (d_0)-(d_2), (d_4) and (d_5) of Definition 1.7 are satisfied. The condition (d_3) follows from Lemma 1.13. below (putting d=r=0 and $b_k=e$ for k=1,...,n).

If an element $a\in A$ has the inverse $a^{-1}\in A$, then the sequence (θ'_n)

$$\theta'_n\begin{pmatrix} p_1,\dots,p_n \\ l_1,\dots,l_n\end{pmatrix} = \begin{cases} 1 & \text{for } n=0 \\ \det(F(p_s a^{-1} l_t)) & \text{for } n>0\end{cases} \qquad (1\le s,t\le n),$$

is a g-determinant system for a.

Lemma 1.13. Let A be a g-algebra, let π be any permutation of the integers 1,2,...,n+d+r (n,d,r≥0 and n+d>0) and let $l_{n+1},\dots,l_{n+d+r}\in L$, $p_{n+d+1},\dots,p_{n+d+r}\in P$, $b_1,\dots,b_{n+d+r}\in A$ be fixed. Then the formula

$$\Psi(p_1,\dots,p_{n+d},l_1,\dots,l_n) = \prod_{k=1}^{n+d+r} F(p_{\pi(k)}b_{\pi(k)}l_k)$$

defines:

a) a d-linear functional Ψ: $P^d \longrightarrow K$ satisfying condition (d_0) in Definition 1.7 if n=0;

b) a (2n+d)-linear functional Ψ: $P^{n+d}\times L^n \longrightarrow K$ satisfying condition (d_3) in Defintion 1.7 if n>0.

Proof. It is evident that Ψ just defined is a (2n+d)-linear functional. Let n=0 and let $p_1,\dots,p_{i-1},p_{i+1},\dots,p_d$ $(1\le i\le d)$ be fixed. Then by putting

$$c = \left(\prod_{\substack{k=1\\ k\ne\pi^{-1}(i)}}^{d+r} F(p_{\pi(k)}b_{\pi(k)}l_k)\right) b_i l_{\pi^{-1}(i)}$$

the relation (d_0) holds true.

Now, let us assume that n>0 and let $p_1,\dots,p_{i-1},p_{i+1},\dots,$ $p_{n+d}\in P$ $(1\ge i\ge n+d)$ and $l_1,\dots,l_{j-1},l_{j+1},\dots,l_n\in L$ $(1\ge j\ge n)$ be fixed. If $i=\pi(j)$, then by putting

$$c = \left(\prod_{\substack{k=1\\ k\ne j}}^{n+d+r} F(p_{\pi(k)}b_{\pi(k)}l_k)\right) b_i$$

we obtain

$$\Psi(p_1,\dots,p_{n+d},l_1,\dots,l_n) = F(p_i c l_j)$$

for every $p_i \in P$ and $l_j \in L$. If $i \neq \pi(j)$, then by putting

$$c = \left(\prod_{\substack{k=1 \\ k\neq\pi^{-1}(i) \\ k\neq j}}^{n+d+r} F(p_{\pi(k)} b_{\pi(k)} l_k) \right) b_i l_{\pi^{-1}(i)} p_{\pi(j)} b_{\pi(j)}$$

and by Corollary 1.2 the relation (d_3) holds true.

2. g-determinant systems in g-total algebras

It follows from Remark 1.8 that a g-determinant system for the element a of a g-algebra A, if it exists, is not uniquely determined by a. Also it follows from Example 1.10 that many different elements may have the same g-determinant system. (Let us observe that the formula (1.10.1) defines a g_{t_0}-determinant system for each element $x \in A$ such that $x(t_0)=0$).

Therefore we have to restrict oureselves to a smaller class of algebras.

Definition 2.1. A g-algebra A is called a g-total algebra if for each $x_0 \in A$ the following two conditions are satisfied:

(a) if $gx_0xg = 0$ for every $x \in A$, then $gx_0 = 0$;

(b) if $gxx_0g = 0$ for every $x \in A$, then $x_0g = 0$.

We will show that every element $a \in \Phi_g$ of any g-total algebra A with identity has a g-determinant system but the inverse theorem does not hold.

Remark 2.2. If a g-algebra A is commutative, then A is a g-total algebra.

Consequently, the algebra from Example 1.4 is g_{t_0}-total.

Remark 2.3. A g-algebra A is a g-total algebra iff the linear spaces P and L are conjugate relative to the bilinear functional $P\times L\ni(p,l) \longrightarrow F(pl) \in K$, (see Example 1.5.)

Remark 2.4. The conditions (a) and (b) in Definition 2.1 are independent. The g_1-algebra in Example 1.3 does not satisfy condition (a). Indeed, let $x_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then for every $x\in A$ we obtain $g_1x_0xg_1 = 0$ with $g_1x_0 \neq 0$. Observe that the condition (b) in 2.1 is satisfied.

It is easy to verify that the algebra U in Example 1.5 is $x_0\cdot\xi_0$-total.

Now, let A be a g-algebra. For any $a\in A$, we will denote by L_a the following operator L_a: $L\longrightarrow L$ defined by $L_a(l) = al$. It is evident that $L_a\circ L_b = L_{ab}$ and

$\lambda_1L_a + \lambda_2L_b = L_{\lambda_1a+\lambda_2b}$ for every $a,b\in A$ and $\lambda_1,\lambda_2\in K$.

If $f = \sum_{i=1}^{n} x_igy_i\in S_g$, then for $xg\in L$ we obtain

$$L_f(xg) = fxg = \sum_{i=1}^{n} x_igy_ixg = \sum_{i=1}^{n} F(y_ix)x_ig.$$

It means that L_f is a finitely dimensional endomorphism in L for every $f\in S_g$. If, moreover, A is a g-total algebra with identity, then for any $a\in A$ the operator L_a is in the algebra $U = U(P,L)$ (see Remark 2.3 and Example 1.5). If $a\in\Phi_g$, then there exists elements $b_1,b_2 \in A$ and $f_1,f_2 \in S_g$ such that $ab_1 = e-f_1$ and $b_2a = e-f_2$. Hence, the following identities hold

$$L_a\circ L_{b_1} = I - L_{f_1}, \qquad L_{b_2}\circ L_a = I - L_{f_2}.$$

So, it follows from this immediately that L_a is a Fredholm operator. By the theory of Fredholm operators (see [1]) we obtain the following

Corollary 2.5. If A is a g-total algebra with identity and $a\in\Phi_g$, then

1° $\dim\{l\in L;\ al=0\} < \infty$, $\dim\{p\in P;\ pa=0\} < \infty$;

2° the codimension of the subspace aL in the space L is finite , the codimension of the subspace Pa in the P is finite.

For any $a \in A$ let us introduce the following notation

$$Z_L(a) = \{l \in L;\ al=0\}, \qquad Z_P(a) = \{p \in P;\ pa=0\}.$$

Definition 2.6. Let A be a g-total algebra with identity. The functions

$$r\colon \Phi_g \longrightarrow N\cup\{0\}, \quad r(a) = \min(\dim Z_L(a), \dim Z_P(a));$$

$$d\colon \Phi_g \longrightarrow Z, \quad d(a) = \dim Z_L(a) - \dim Z_P(a)),$$

are called the order and the index, respectively. The value of order and the index at the point $a \in \Phi_g$ will be called the order and the index of a, respectively.

The index of the Fredholm element $a \in \Phi_g$ is equal to the index of the Fredholm operator $L_a \in U(P,L)$. Hence the following

Corollary 2.7. If A is a g-total algebra with identity and $a,b \in \Phi_g$, $f \in S_g$, then

1° $d(ab) = d(a)+a(b)$;

2° $d(a+f) = d(a)$.

Definition 2.8. Let A be an algebra. An element $b \in A$ is called a generalized inverse of $a \in A$ if $aba=a$. If in addition b satisfies the condition $bab=b$, then b is called a reciprocal generalized inverse of a.

Remark 2.9. If $b' \in A$ is a generalized inverse of $a \in A$, then $b = b'ab'$ is a reciprocal generalized inverse of a.

Theorem 2.10. Let A be a g-total algebra with identity e. If $a \in \Phi_g$, then a has a reciprocal generalized inverse $b \in \Phi_g$.

Proof. Let $a \in \Phi_g$. There exist elements $b_1 \in A$ and $f_1 \in S_g$ such that $ab_1 = e - f_1$, where $f_1 = \sum_{i=1}^{n} x_i g y_i$. By Corollary 2.5, there exists finitely dimensional subspace L′ of L such that $L = aL \oplus L'$. Let $x_1'g, \dots, x_k'g$ form a basis of L′. For every

$i=1,\dots,n$ there exists an element $x_i''\in A$ and suitable scalars $\alpha_{i1},\dots,\alpha_{ik}$ such that

$$x_i g = a x_i'' g + \sum_{j=1}^{k} \alpha_{ij} x_j' g.$$

Putting $b' = b_1 + \sum_{i=1}^{n} x_i'' g y_i$ and $y_j' = \sum_{i=1}^{n} \alpha_{ij} y_i$ for $j=1,\dots,k$ we obtain

$$ab' = ab_1 + \sum_{i=1}^{n} a x_i'' g y_i = e - \sum_{i=1}^{n} x_i g y_i + \sum_{i=1}^{n} a x_i'' g y_i =$$

$$= e - \sum_{i=1}^{n} (a x_i'' g + \sum_{j=1}^{k} \alpha_{ij} x_j' g) y_i + \sum_{i=1}^{n} a x_i'' g y_i =$$

$$= e - \sum_{j=1}^{k} x_j' g (\sum_{i=1}^{n} \alpha_{ij} y_i) = e - \sum_{j=1}^{k} x_j' g y_j'.$$

Then $ab' axg = axg - \sum_{j=1}^{k} x_j' g y_j' axg$ for every $x\in A$. Hence $\sum_{j=1}^{k} F(y_j' axg) x_j' g = a(x-b' ax)g \in aL$ for every $x\in A$. Since $x_j' g\in L'$ for $j=1,\dots,k$ we conclude that $F(y_j' axg) = 0$ for every $x\in A$ and $j=1,\dots,k$ and so $g y_j' a = 0$ $(j=1,\dots,k)$. It follows from this that $ab' a=a$ and by Remark 2.9 a has a reciprocal generalized inverse $b\in A$. Evidently, $b\in\Phi_g$.

Lemma 2.11. Let A be a g-total algebra and $a,x_1,\dots,x_n, y_1,\dots,y_n\in A$;

1° if $x_1 g,\dots,x_n g$ are linearly independent elements and

$$(\sum_{i=1}^{n} x_i g y_i) a = 0,$$

then $g y_i a = 0$ for every $i=1,\dots,n$;

2° if $g y_1,\dots,g y_n$ are linearly independent elements and

$$a\left(\sum_{i=1}^{n} x_i g y_i\right) = 0,$$

then $ax_ig=0$ for every $i=1,\dots,n$.

Proof. Let $\left(\sum_{i=1}^{n} x_i g y_i\right)a = 0$. Then for every $x\in A$ we obtain

$$0 = \sum_{i=1}^{n} x_i g y_i axg = \sum_{i=1}^{n} F(y_i ax)x_i g.$$

Since $x_1g,\dots,x_ng$ are linearly independent it follows that $F(y_iax) = 0$ for every $x\in A$ and $i=1,\dots,n$. Hence $gy_ia=0$ for $i=1,\dots,n$. The proof of 2° is analogous.

Definition 2.12. Let A be a g-algebra. A representation $f = \sum_{i=1}^{n} x_i g y_i$, $x_i,y_i\in A$, of an element f in A is said to be bilinearly independent if both $x_1g,\dots,x_ng$ and $gy_1,\dots,gy_n$ are linearly independent.

Remark 2.13. Every element $f\in S_g-\{0\}$ of the g-algebra A has a bilinearly independent representation.

In what follows, the notation $f = \sum_{i=1}^{n} x_i g y_i$ means that the elements $x_1g,\dots,x_ng$ are linearly independent and so are $gy_i,\dots,gy_n$.

Theorem 2.14. Let A be a g-total algebra with identity e and let $b\in A$ be a reciprocal generalized inverse of the element $a\in\Phi_g$. Then

1° there exist elements f_1, $f_2\in S_g$ such that $ab=e-f_1$ and $ba=e-f_2$;

2° if $f_1 = \sum_{i=1}^{n} x_i g y_i$ and $f_2 = \sum_{i=1}^{m} z_i g t_i$, then

the elements $gy_1,\dots,gy_n$ form a basis of $Z_P(a)$;

the elements $z_1g,\dots,z_mg$ form a basis of $Z_L(a)$;

the elements $gt_1,\dots,gt_m$ form a basis of $Z_P(b)$;

the elements $x_1g,\dots,x_ng$ form a basis of $Z_L(b)$;

$F(y_ix_j) = \delta_{ij}$ $i,j=1,\dots,n$;

$F(t_iz_j) = \delta_{ij}$ $i,j=1,\dots,m$, where δ_{ij} means the Kronecker symbol;

3° if $gu_i,\dots,gu_n$ is a basis of $Z_P(a)$ and $v_1g,\dots,v_mg$ is a basis of $Z_L(a)$, then there exist elements $w_1g,\dots,w_ng,gs_1,\dots,gs_m \in A$ such that

$$ab = e - \sum_{i=1}^{n} w_igu_i, \qquad ba = e - \sum_{i=1}^{m} v_igs_i.$$

Proof. If $a\in\Phi_g$, then there exist elements $f'_1,f'_2\in S_g$ and $b_1\in A$ such that $ab_1=e-f'_1$ and $b_1a=e-f'_2$. Multiplying the equality $aba=a$ by b_1 we obtain

$$b_1aba = b_1a, \qquad abab_1 = ab_1,$$

$$(e-f'_2)ba = e-f'_2, \qquad ab(e-f'_1) = e-f'_1,$$

$$ba = e-f'_2+f'_2ba = e-f'_2(e-ba) \qquad ab = e-f'_1+abf'_1 = e-(e-ab)f'_1.$$

Putting $f_2=f'_2(e-ba)\in S_g$ and $f_1 = (e-ab)f'_1\in S_g$ we prove 1°.

Now, let $f_1 = \sum_{i=1}^{n} x_igy_i$ and $f_2 = \sum_{i=1}^{m} z_igt_i$. Since

$a = aba = (e-f_1)a = a-f_1a$ so $f_1a = 0$ and hence

$$\left(\sum_{i=1}^{n} x_igy_i\right)a = 0.$$

By Lemma 2.11 the elements $gy_i,\dots,gy_n$ are in $Z_P(a)$. If an element $p\in Z_P(a)$, then

$$0 = pab = p - \sum_{i=1}^{n} gpx_igy_i = p - \sum_{i=1}^{n} F(px_i)gy_i,$$

and $p = \sum_{i=1}^{n} F(px_i)gy_i$. Thus $gy_1,\dots,gy_n$ form a basis of the space $Z_P(a)$. To prove that $F(y_ix_j) = \delta_{ij}$ for $i,j=1,\dots,n$ let us observe, by Remark 2.3, that for every $i=1,\dots,n$ there exists $p'_i \in P$ such that $F(p'_i x_j) = \delta_{ij}$ for $j=1,\dots,n$. Moreover

$$gp'_i f_1 = gp'_i \left(\sum_{j=1}^{n} x_j gy_j\right) = \sum_{j=1}^{n} F(p'_i x_j) gy_j = gy_i \quad \text{for } i=1,\dots,n.$$

On the other hand $(e-f_1)^2 = abab = ab = e-f_1$, hence $f_1^2 = f_1$. Thus for every $i=1,\dots,n$

$$gy_i = gp'_i f_1 = gp'_i f_1^2 = (gp'_i f_1) f_1 = (gy_i) f_1 =$$

$$= (gy_i) \sum_{j=1}^{n} x_j gy_j = \sum_{j=1}^{n} F(y_i x_j) gy_j.$$

Since $gy_1,\dots,gy_n$ are linearly independent, it follows that $F(y_i x_j) = \delta_{ij}$ for $i,j=1,\dots,n$.

Now, let $ab = e - \sum_{i=1}^{n} x_i gy_i$ and let elements $gu_1,\dots,gu_n$ form a basis of the space $Z_P(a)$. Then for every $i=1,\dots,n$ we have

$$gy_i = \sum_{j=1}^{n} \alpha_{ij} gu_j$$

for suitable scalars α_{ij}. Both $gy_1,\dots,gy_n$ and $gu_1,\dots,gu_n$ are linearly independent, hence $\det(\alpha_{ij}) \neq 0$ $(i,j=1,\dots,n)$. Moreover

$$ab = e - \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} \alpha_{ij} gu_j\right) = e - \sum_{i,j=1}^{n} \alpha_{ij} x_i gu_j =$$

$$= e - \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \alpha_{ij} x_i\right) gu_j.$$

Let us put $w_j = \sum_{i=1}^{n} \alpha_{ij} x_i$ for $j=1,\dots,n$. Since $\det(\alpha_{ij}) \neq 0$, then elements $w_1 g,\dots,w_n g$ are linearly independent. Thus, we obtain

$$ab = e - \sum_{j=1}^{n} w_j g u_j .$$

The proof of the second part of 3° is analogous.

Theorem 2.15. Let A be a g-total algebra with identity e. If $a \in \Phi_g$, then

1° a has a g-determinant system;

2° if (D_n) and (D'_n) are g-determinant systems of a, then

a) there exists a scalar $\alpha \neq 0$ such that $(D'_n) = (\alpha D_n)$;

b) $d(D_n) = d(a)$ and $r(D_n) = r(a)$.

Proof. Let us assume that $d(a) = d \geq 0$. The proof in the other case is analogous. Let $r(a) = r$ and let $b \in A$ be a reciprocal generalized inverse of a. Then $\dim Z_L(a) = r+d$, $\dim Z_P(a) = r$ and there exist $x_1 g, \ldots, x_r g, z_1 g, \ldots, z_{r+d} \in L$, $gy_1, \ldots, gy_r, gt_1, \ldots, gt_{r+d} \in P$ such that

$$ab = e - \sum_{i=1}^{r} x_i g y_i ,$$

$$ba = e - \sum_{i=1}^{r+d} z_i g t_i .$$

Put $\theta_n = 1$ if $n=d=r=0$ and

(2.15.1)[2]
$$\theta_n \begin{pmatrix} p_1, \ldots, p_{n+d} \\ l_1, \ldots, l_n \end{pmatrix} =$$

$$= \begin{vmatrix} F(p_1 b l_1) \ldots\ldots F(p_1 b l_n) & F(p_1 z_1) \ldots\ldots F(p_1 z_{r+d}) \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ F(p_{n+d} b l_1) \ldots F(p_{n+d} b l_n) & F(p_{n+d} z_1) \ldots F(p_{n+d} z_{r+d}) \\ F(y_1 l_1) \ldots\ldots F(y_1 l_n) & 0 \ \ldots\ldots \ 0 \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ F(y_r l_1) \ldots\ldots F(y_r l_n) & 0 \ \ldots\ldots \ 0 \end{vmatrix}$$

---

[2] The formula (2.15.1) in the case of linear operators is given by Buraczewski [5]

otherwise. We will show that the sequence (θ_n) is a g-determinant system for a. At first, to prove the condition (d_0) of Definition 1.7, let us consider the following cases:

a) d=0 and r=0, then $\theta_0=1$;

b) d=0 and r>0, then $\theta_0=0$;

c) d>0 and r>0, then $\theta_0\equiv 0$ and the condition (d_0) is satisfied for c=0;

d) d>0 and r=0, then

$$\theta_0(p_1,\dots,p_d) = \begin{vmatrix} F(p_1z_1) & \dots & F(p_1z_d) \\ \vdots & & \vdots \\ F(p_dz_1) & \dots & F(p_dz_d) \end{vmatrix}$$

and the condition (d_0) follows from Lemma 1.13a). It is sufficient to put $l_s=z_s$ (s=1,...,d) and $b_1=b_2=\dots=b_d=e$.

It is evident that the conditions (d_1) and (d_2) of Definition 1.7 are satisfied. The condition (d_3) follows from Lemma 1.13b). To show this, it is sufficient to put $p_{n+d+q} = y_q$ (q=1,...,r), $l_{n+s} = z_s$ (s=1,...,r+d) and

$$b_{\pi(k)} = \begin{cases} b \text{ if } \pi(k)\le n+d \text{ and } k\le n \\ 0 \text{ if } \pi(k)>n+d \text{ and } k>n \\ e \text{ otherwise} \end{cases}$$

for any permutation π of the set $\{1,\dots,n+d+r\}$. By Theorem 2.14

$$\theta_r\begin{pmatrix} gt_1,\dots,gt_{r+d} \\ x_1g,\dots,x_rg \end{pmatrix} =$$

$$= \begin{vmatrix} F(t_1bx_1) \ldots\ldots F(t_1bx_r) & F(t_1z_1) \ldots\ldots F(t_1z_{r+d}) \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ F(t_{r+d}bx_1) \ldots F(t_{r+d}bx_r) & F(t_{r+d}z_1) \ldots F(t_{r+d}z_{r+d}) \\ F(y_1x_1) \ldots\ldots F(y_1x_r) & 0 \ \ldots\ldots\ 0 \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ F(y_rx_1) \ldots\ldots F(y_rx_r) & 0 \ \ldots\ldots\ 0 \end{vmatrix} =$$

$$= \begin{vmatrix} \begin{matrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{matrix} & \begin{matrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{matrix} \\ \underbrace{\qquad\qquad}_{r} & \underbrace{\qquad\qquad}_{r+d} \end{vmatrix} \neq 0.$$

Hence condition (d_4) is satisfied. Applying Theorem 2.14 and basic properties of determinants, we obtain for every $n=0,1,\ldots$

$$\theta_{n+1} \begin{pmatrix} p_0a, p_1, \ldots, p_{n+d} \\ l_0, l_1, \ldots, l_n \end{pmatrix} =$$

$$= \begin{vmatrix} F(p_0abl_0) & F(p_0abl_1) \ldots F(p_0abl_n) & F(p_0az_1) \ldots F(p_0az_{r+d}) \\ F(p_1bl_0) & F(p_1bl_1) \ldots F(p_1bl_n) & F(p_1z_1) \ldots F(p_1z_{r+d}) \\ \vdots & \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ F(p_{n+d}bl_0) & F(p_{n+d}bl_1) \ldots F(p_{n+d}bl_n) & F(p_{n+d}z_1) \ldots F(p_{n+d}z_{r+d}) \\ F(y_1l_0) & F(y_1l_1) \ldots F(y_1l_n) & 0 \ \ldots\ 0 \\ \vdots & \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ F(y_rl_0) & F(y_rl_1) \ldots F(y_rl_n) & 0 \ \ldots\ 0 \end{vmatrix} =$$

$$
= \left| \begin{matrix}
F(p_0l_0)-\sum\limits_{i=1}^{r}F(p_0x_i)F(y_il_0) & F(p_0l_1)-\sum\limits_{i=1}^{r}F(p_0x_i)F(y_il_1) & \cdots\cdots \\
F(p_1bl_0) & F(p_1bl_1) & \cdots\cdots \\
\vdots & \vdots & \\
F(p_{n+d}bl_o) & F(p_{n+d}bl_1) & \cdots\cdots \\
F(y_1l_0) & F(y_1l_1) & \cdots\cdots \\
\vdots & \vdots & \\
F(y_rl_0) & F(y_rl_1) & \cdots\cdots
\end{matrix} \right.
$$

$$
\left. \begin{matrix}
\cdots & F(p_0l_n)-\sum\limits_{i=1}^{r}F(p_0x_i)F(y_il_n) & 0 & \cdots\cdots & 0 \\
\cdots\cdots & F(p_1bl_n) & F(p_1z_1) & \cdots\cdots & F(p_1z_{r+d}) \\
 & \vdots & \vdots & & \vdots \\
\cdots\cdots & F(p_{n+d}bl_n) & F(p_{n+d}z_1) & \cdots & F(p_{n+d}z_{r+d}) \\
\cdots\cdots & F(y_1l_n) & 0 & \cdots\cdots & 0 \\
 & \vdots & \vdots & & \vdots \\
\cdots\cdots & F(y_rl_n) & 0 & \cdots\cdots & 0
\end{matrix} \right| =
$$

$$
= \begin{vmatrix}
F(p_0l_0) & F(p_0l_1) & \cdots & F(p_0l_n) & 0 & \cdots & 0 \\
F(p_1bl_0) & F(p_1bl_1) & \cdots & F(p_1bl_n) & F(p_1z_1) & \cdots & F(p_1z_{r+d}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
F(p_{n+d}bl_o) & F(p_{n+d}bl_1) & \cdots & F(p_{n+d}bl_n) & F(p_{n+d}z_1) & \cdots & F(p_{n+d}z_{r+d}) \\
F(y_1l_0) & F(y_1l_1) & \cdots & F(y_1l_n) & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
F(y_rl_0) & F(y_rl_1) & \cdots & F(y_rl_n) & 0 & \cdots & 0
\end{vmatrix} .
$$

Expanding the above determinant along the terms of its first row, we obtain

$$\theta_{n+1}\begin{pmatrix} p_0a,p_1,\dots,p_{n+d} \\ l_0,l_1,\dots,l_n \end{pmatrix} =$$

$$= \sum_{i=0}^{n} (-1)^i F(p_0 l_i)\theta_n \begin{pmatrix} p_1,\dots\dots\dots\dots\dots\dots,p_{n+d} \\ l_0,\dots,l_{i-1},l_{i+1},\dots,l_n \end{pmatrix}.$$

The proof of the second condition in (d_5) is analogous.

If (D_n) and (D'_n) are g-determinant systems for $a\in A$, then they are determinant systems for the Fredholm operator $L_a\in U(P,L)$ in the sense of Buraczewski's definition ([1]). Hence, there exists a scalar $\alpha\neq 0$ such that $(D'_n)=(\alpha D_n)$.

One can easily see that $d(\theta_n)=d$ and $r(\theta_n)=r$. If (D_n) is a g-determinant system for a, then $(D_n)=(\alpha\theta_n)$ for suitable $\alpha\neq 0$ and hence $d(D_n)=d=d(a)$, $r(D_n)=r=r(a)$.

Remark 2.16. The inverse theorem is not true. The element h of the g_{t_0}-total algebra A (see Example 1.10) has a g_{t_0}-determinant system definined by (1.10.1) but $h\notin\Phi_{g_{t_0}}$.

The inverse theorem is true in a smaller class of algebras.

3. g-determinant systems in g-small algebras

Definition 3.1. A g-algebra A is called a g-small algebra if the following condition is satisfied:

$$gxayg = 0 \quad \text{for every } x,y\in A \quad \text{if and only if } a=0.$$

The algebra $U=U(\Xi,X)$ (see Example 1.5) is an $x_0\cdot\xi_0$-small algebra.

The g_{t_0}-total algebra A in Example 1.4 is not a g_{t_0}-small algebra.

We now show that an element a of a g-small algebra A has a g-determinant system if and only if $a\in\Phi_g$.

Theorem 3.2. If A is a g-small algebra, then:

1° A is a g-total algebra;

2° A is commutative if and only if dimA=1.

Proof. If $gx_0yg=0$ for every $y\in A$, then $gxgx_0yg=0$ for every $x,y\in A$. Hence $gx_0=0$. The proof of condition (b) of Definition 2.1 is analogous.

If a g-small algebra A is commutative, then L is a onedimensional space and the algebra End(L) of all linear endomorphisms in L is a onedimensional algebra. It is easy to show that the map $A\ni a\longrightarrow\varphi(a)\in End(L)$ defined by

$$\varphi(a)(l) = al, \qquad l\in L$$

is an algebra isomorphism. Hence dimA=dimEnd(L)=1.

Theorem 3.3. If A is a g-small algebra with identity e, then an element $a\in A$ has a g-determinant system if and only if $a\in\Phi_g$.

Proof. If $a\in\Phi_g$, then a has a g-determinant system by Theorem 2.15. Now, let (D_n) be a g-determinant system for $a\in A$ such that $r(D_n)=r$ and $d(D_n)=d\geq 0$. (If $d<0$, the proof is analogous). We will consider the following three cases.

a) $r>0$. Then by condition (d_4) of Definition 1.7, there exist elements $x_1,\dots,x_r\in L$ and $t_1,\dots,t_{r+d}\in P$ such that

$$\alpha = D_r\begin{pmatrix} t_1,\dots,t_{r+d} \\ x_1,\dots,x_r \end{pmatrix} \neq 0. \tag{3.3.1}$$

By condition (d_3) of Definition 1.7, there exist elements $y_1,\dots,y_r\in P$ and $z_1,\dots,z_{r+d}\in L$ such that

$$F(y_i l) = \frac{1}{\alpha}D_r\begin{pmatrix} t_1,\dots\dots\dots\dots\dots\dots,t_{r+d} \\ x_1,\dots,x_{i-1},l,x_{i+1},\dots,x_r \end{pmatrix} \tag{3.3.2}$$

for every $l\in L$, and

$$(3.3.3) \qquad F(pz_i) = \frac{1}{\alpha} D_r \begin{pmatrix} t_1,\dots,t_{i-1},p,t_{i+1},\dots,t_{r+d} \\ x_1,\dots\dots\dots\dots,x_r \end{pmatrix}$$

for every $p \in P$. It follows from (d_2) that

$$F(y_i x_j) = \delta_{ij} \qquad \text{for } i,j=1,\dots,r,$$

$$F(t_i z_j) = \delta_{ij} \qquad \text{for } i,j=1,\dots,r+d.$$

Thus both $y_1,\dots,y_r$ and $z_1,\dots,z_{r+d}$ are linearly independent. Now, let $b \in A$ be such that

$$(3.3.4) \qquad F(pbl) = \frac{1}{\alpha} D_{r+1} \begin{pmatrix} p,t_1,\dots,t_{r+d} \\ l,x_1,\dots,x_r \end{pmatrix}$$

for every $p \in P$ and $l \in L$. (see condition (d_3) of Definition 1.7). By (d_5) we obtain

$$F(pabl) = \frac{1}{\alpha} D_{r+1} \begin{pmatrix} pa,t_1,\dots,t_{r+d} \\ l,x_1,\dots x_r \end{pmatrix} = \frac{1}{\alpha} \left(F(pl) D_r \begin{pmatrix} t_1,\dots,t_{r+d} \\ x_1,\dots,x_r \end{pmatrix} + \right.$$

$$\left. + \sum_{i=1}^{r} (-1)^i F(px_i) D_r \begin{pmatrix} t_1,\dots\dots\dots\dots,t_{r+d} \\ l,x_1,\dots,x_{i-1},x_{i+1},\dots,x_r \end{pmatrix} \right) =$$

$$= F(pl) + \sum_{i=1}^{r} (-1)^i F(px_i) \frac{1}{\alpha} (-1)^{i-1} D_r \begin{pmatrix} t_1,\dots\dots\dots\dots,t_{r+d} \\ x_1,\dots,x_{i-1},l,x_{i+1},\dots,x_r \end{pmatrix} =$$

$$= F(pl) - \sum_{i=1}^{r} F(px_i) F(y_i l) = F(pl) - \sum_{i=1}^{r} F(px_i g y_i l) =$$

$$= F(p(e - \sum_{i=1}^{r} x_i g y_i) l)$$

for every $p \in P$, $l \in L$. It follows from the definition of g-small algebra that

$$(3.3.5) \qquad ab = e - \sum_{i=1}^{r} x_i g y_i .$$

It can be proved in a similar manner that

$$(3.3.6) \qquad ba = e - \sum_{i=1}^{r+d} z_i g t_i .$$

Hence, we conclude that $a \in \Phi_g$.

b) $r=0$ and $d>0$. Let $t_1,\ldots,t_d \in P$ are fixed elements such that

$$\alpha = D_0(t_1,\ldots,t_d) \neq 0.$$

It follows from (d_0) in Definition 1.7 that there exist elements $z_1,\ldots,z_d \in L$ such that

$$F(pz_i) = D_0(t_1,\ldots,t_{i-1},p,t_{i+1},\ldots,t_d)$$

for every $p \in P$. The reader can easily see that for the element $b \in A$ such that

$$F(pbl) = \alpha^{-1} D_1 \begin{pmatrix} p,t_1,\ldots,t_d \\ l \end{pmatrix} \qquad p \in P,\ l \in L$$

the following identities hold:

$$ab = e;$$

$$ba = e - \sum_{i=1}^{d} z_i g t_i.$$

It means that $a \in \Phi_g$.

c) $r=0$ and $d=0$. Then D_0 is u number different from zero. By (d_3) there exists an element $b \in A$ such that

$$F(pbl) = D_0^{-1} D_1 \begin{pmatrix} p \\ l \end{pmatrix}$$

for every $p \in P$, $l \in L$. By (d_5) we obtain

$$F(pabl) = D_0^{-1} D_1 \begin{pmatrix} pa \\ l \end{pmatrix} = D_0^{-1} F(pl) D_0 = F(pel)$$

for every $p \in P$, $l \in L$. Hence $ab=e$. It can be proved in a similar manner that $ba=e$. Thus, $a \in \Phi_g$.

Remark 3.4. If A is a g-small algebra and (D_n) is a g-determinant system for $a \in A$, then the element $b \in A$ defined in the proof of Theorem 3.3 is a reciprocal generalized inverse of a.

Indeed, it is evident for $r=r(D_n)=0$. Let $r=r(D_n)>0$ and let $x_1,\ldots,x_r,\ z_1,\ldots,z_{r+d} \in L,\ t_1,\ldots,t_{r+d}, y_1,\ldots,y_r \in P$ be fixed elements such that (3.3.1), (3.3.2) and (3.3.3) hold. By (d_2) and (d_3) we have

$$F(y_i al) = \frac{1}{\alpha} D_r \begin{pmatrix} t_1, \ldots\ldots\ldots\ldots\ldots\ldots, t_{r+d} \\ x_1,\ldots,x_{i-1},al,x_{i+1},\ldots,x_r \end{pmatrix} = 0$$

for every $l \in L$ and $i=1,\ldots,r$. It means that

$$(3.4.1) \qquad y_i a = 0$$

for $i=1,\ldots,r$. By (3.3.4) and (d_2) we obtain

$$(3.4.2) \qquad bx_i = 0$$

for $i=1,\ldots,r$. It follows from (3.3.5), (3.4.1) and (3.4.2) that b is a reciprocal generalized inverse of a.

Remark 3.5. The thesis of Remark 3.4 is not true in a g-total algebra. To this end, let us consider the algebra A in Example 1.4 and the element $a=e-g_{t_0} \in A$. It is evident that a is a reciprocal generalized inverse of a and $a \in \Phi_{g_{t_0}}$. By Theorem 2.15 and (2.15.1) we obtain

$$\theta_n\begin{pmatrix} p_1,\ldots,p_n \\ l_1,\ldots,l_n \end{pmatrix} = \begin{vmatrix} (p_1al_1)(t_0)\ldots\ldots(p_1al_1)(t_0) & (p_1g_{t_0})(t_0) \\ \vdots \qquad\qquad \vdots & \vdots \\ (p_nal_1)(t_0)\ldots\ldots(p_nal_n)(t_0) & (p_ng_{t_0})(t_0) \\ (g_{t_0}l_1)(t_0)\ldots\ldots(g_{t_0}l_n)(t_0) & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & \ldots & 0 & p_1(t_0) \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & p_n(t_0) \\ l_1(t_0) & \ldots & l_n(t_0) & 0 \end{vmatrix} = \begin{cases} 0 & \text{for } n=0,2,3,\ldots \\ -(l_1p_1)(t_0) & \text{for } n=1 \end{cases}$$

so that the sequence (θ_n) is a g_{t_0}-determinant system for a. The element $b \in A$ defined by

$$b(t) = \begin{cases} 0 & \text{for } t=t_0 \\ t+3 & \text{for } t \neq t_0 \end{cases} \qquad t \in [0,1]$$

satisfies the following condition:

$$F(pbl) = 0 = \frac{\theta_2\begin{pmatrix} p, g_{t_0} \\ l, g_{t_0} \end{pmatrix}}{\theta_1\begin{pmatrix} g_{t_0} \\ g_{t_0} \end{pmatrix}} \qquad p\in P,\ l\in L$$

but it is not a generalized inverse of a.

Let A be a g-algebra. The set

$$A_0 = \left\{ a\in A;\ \bigwedge_{x,y\in A} gxayg=0 \right\}$$

is an ideal of A.

Definition 3.6. The ideal A_0 of a g-algebra A is called the ideal of zeros of A.

Theorem 3.7. If A is a g-algebra, then the quotient algebra A/A_0 (where A_0 is the ideal of zeros) is a [g]-small algebra and the function $\bar{F}:A/A_0 \longrightarrow K$ defined by

$$\bar{F}([x]) = F(x) \qquad x\in A$$

is a [g]-functional.

Proof. First, we will show the function $\bar{F}$ is well defined. Let $x_0\in[x]$, then there exists $b_0\in A_0$ such that $x_0=x+b_0$. Hence

$$\bar{F}(x_0]) = F(x_0) = F(x+b_0) = F(x) = F([x]).$$

Let us observe that $[g]\cdot[g] = [g]\neq A_0$ and

$$[g][x][g] =[gxg] = [F(x)g] = F(x)[g] = \bar{F}([x])[g]$$

for every coset $[x]\in A/A_0$. Hence A/A_0 is a [g]-algebra and $\bar{F}$ is a [g]-functional.

Finally, suppose that an element $[a]\in A/A_0$ satisfies the equality

$$[g][x][a][y][g] = A_0$$

for every $[x],[y]\in A/A_0$. Then $F(xay)[g] = A_0$ for every $x,y\in A$. Thus $gxayg = F(xay)g = 0$ for $x,y\in A$. Hence $a\in A_0$ and [a] is the zero of algebra A/A_0.

Theorem 3.8. If a g-algebra (g-total, g-small algebra) A has no identity, then the algebra $\tilde{A} = A\oplus\{\lambda e\}$ is a g-algebra (g-total, g-small algebra respectively) and the function $\tilde{F}:\tilde{A}\longrightarrow K$

$$\tilde{F}(x+\lambda e) = F(x)+\lambda \qquad x\in A, \quad \lambda\in K$$

is a g-functional on the algebra $\tilde{A}$.

Proof. Let $x+\lambda e\in\tilde{A}$, then

$$g(x+\lambda e)g = gxg + \lambda g = F(x)g + \lambda g = (F(x)+\lambda)g.$$

If A is a g-total algebra and

$$g(x_0+\lambda_0 e)(x+\lambda e)g = 0$$

for every $x+\lambda e\in\tilde{A}$, then

$$g(x_0+\lambda_0 e)xg = 0$$

for $x\in A$. Hence, $g(x_0+\lambda_0 e) = 0$. The proof of (b) in Definition 2.1 is analogous.

Finally, let us suppose that A is a g-small algebra. If

$$g(x+\lambda_1 e)(a+\lambda e)(y+\lambda_2 e)g = 0$$

for every $x+\lambda_1 e, y+\lambda_2 e \in \tilde{A}$, then $gx(a+\lambda e)yg = 0$ for every $x,y\in A$. Hence

$$gxayg = -\lambda gxyg. \tag{3.8.1}$$

If $\lambda=0$, then $a=0$ and $a+\lambda e=0$. Let $\lambda\neq 0$. We will show that the element $-\frac{1}{\lambda}a$ is the identity of the algebra A. Indeed, let $b\in A$. By (3.8.1) we obtain

$$gx(-\tfrac{1}{\lambda}ab)yg = -\tfrac{1}{\lambda}gxa(by)g = -\tfrac{1}{\lambda}(-\lambda)gx(by)g = gxbyg$$

for every $x,y\in A$. Hence $(-\frac{1}{\lambda}a)b=b$. The proof that $b(\frac{-1}{\lambda}a)=b$ is analogous.

4. The algebra of quasinuclei

Definition 4.1. Let A be a g-algebra. A linear functional $\xi\in A$ is a called a quasinucleus relative to the ideals S_g if there exists an element $q\in A$ such that

(4.1.1) $$\xi(xgy) = F(yqx)$$

for every $x,y\in A$. The element $q\in A$ defined in this way is called a quasinuclear element relative to the ideal S_g.

We will denote by $Q_F(Q_g)$ the set of all quasinuclei (quasinuclear elements respectively) relative to the ideal S_g.

It follows from Corollary 1.2 that

$$F(xgy) = F(x)(Fy) = F(y)F(x) = F(ygx)$$

for every $x,y\in A$. Hence $F\in Q_F$ and $g\in Q_g$.

Example 4.2. The set M of all infinite square matrices $a=(a_{ij})$ satisfying the condition

$$\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$$

with the usual operations is an algebra over the field K. For a fixed natural m, let us denote by g the matrix $(g_{ij})\in M$ such that

$$g_{ij} = \begin{cases} 1 \text{ for } i=j=m \\ 0 \text{ otherwise.} \end{cases}$$

Then M is a g-small algebra and the g-functional F is defined by $F((a_{ij})) = a_{mm}$. Every $q=(q_{ij})\in M$ such that

(4.2.1) $$\sum_{j=1}^{\infty} \sup_i |q_{ij}| < \infty$$

is quasinuclear. Every functional $\xi\in M'$

$$\xi((a_{ij})) = \sum_{i,j=1}^{\infty} q_{ij}a_{ji} \qquad (a_{ij})\in M,$$

where the matrix (q_{ij}) satisfies (4.2.1), is a quasinucleus.

The matrix $a=(a_{ij})\in M$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } i>r\geq 0 \text{ and } j-i=d\geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a Fredholm element relative to the ideal S_g with $d(a)=d$ and $r(a)=r$.

Theorem 4.3. If A is a g-small algebra, then

1° $S_g \subset Q_g$;

2° Q_F and Q_g are subspaces of the linear spaces A′ and A respectively;

3° the function $k:Q_F \longrightarrow Q_g$ defined at $\xi \in Q_F$ by

(4.3.1) $$\xi(xgy) = F(yk(\xi)x) \qquad x,y\in A$$

is well-defined and it is a linear homomorphism.

Proof. Suppose $\sum_{i=1}^{n} x_i g y_i \in S_g$. Then the function $\xi:A\longrightarrow K$ defined by

$$\xi(x) = F\left(\sum_{i=1}^{n} y_i x x_i\right)$$

for $x\in A$ is a linear functional such that

$$\xi(xgy) = F\left(\sum_{i=1}^{n} g_i xgyx_i\right) = \sum_{i=1}^{n} F(y_i xgyx_i) = \sum_{i=1}^{n} F(yx_i g y_i x) =$$
$$= F\left(y\left(\sum_{i=1}^{n} x_i g y_i\right)x\right).$$

for every $x,y\in A$. Hence $\sum_{i=1}^{n} x_i g y_i \in Q_g$.

Suppose now that for $\xi\in Q_F$ there exist $q_1,q_2\in A$ such that

$$F(yq_1x) = \xi(xgy) = F(yq_2x)$$

for every $x,y\in A$. Since A is a g-small algebra so $q_1=q_2$ and therefore k is well-defined.

If $\xi_1,\xi_2\in Q_F$ and $\lambda_1,\lambda_2\in K$, then

$$(\lambda_1\xi_1+\lambda_2\xi_2)(xgy) = \lambda_1\xi_1(xgy) + \lambda_2\xi_2(xgy) = \lambda_1 F(yk(\xi_1)x) +$$
$$+ \lambda_2 F(yk(\xi_2)x) = F(y(\lambda_1 k(\xi_1) + \lambda_2 k(\xi_2))x)$$

for every $x,y\in A$. This means that Q_F is a subspace of A′ and k is a linear homomorphism. Hence $Q_g=k(Q_F)$ is a subspace of A.

Definition 4.4. Let A be a g-small algebra. The linear homomorphism $k:Q_F\longrightarrow Q_g$ defined by (4.3.1) is called a natural homomorphism.

Let $a\in A$ and $\xi\in A'$. We denote by ξa and $a\xi$ the linear functionals defined on A by

$$(\xi a)(x) = \xi(ax),$$

$$(a\xi)(x) = \xi(xa), \qquad x\in A.$$

Theorem 4.5. If A is a g-small algebra, then

1° Q_g is an ideal of the algebra A;

2° each of the following formulae

$$\xi_1 \blacktriangleright \xi_2 = \xi_1 k(\xi_2),$$

$$\xi_1 \blacktriangleleft \xi_2 = k(\xi_1)\xi_2, \qquad \xi_1,\xi_2 \in Q_F,$$

defines multiplication $\blacktriangleright$ and $\blacktriangleleft$ on the set Q_F respectively;

3° the linear space Q_F is an F-algebra with multiplication $\blacktriangleright$ and $\blacktriangleleft$;

4° the natural homomorphism $k:Q_F \longrightarrow Q_g$ is an algebra homomorphism and the ideal of zeros of the F-algebra Q_F is the kernel of k.

Proof. If $a\in A$ and $q=k(\xi)\in Q_g$, then

$$(\xi a)(xgy) = \xi(axgy) = F(yqax),$$

$$(a\xi)(xgy) = \xi(xgya) = F(yaqx)$$

for every $x,y\in A$. Hence Q_g is an ideal of the algebra A.
Let $\xi_1,\xi_2 \in Q_F$, then

$$(\xi_1 \blacktriangleright \xi_2)(xgy) = \xi_1(k(\xi_2)xgy) = F(yk(\xi_1)k(\xi_2)x)$$

for every $x,y\in A$. Thus, the multiplication $\blacktriangleright$ is well-defined and $k(\xi_1 \blacktriangleright \xi_2) = k(\xi_1)k(\xi_2)$. Similarly we show that $\blacktriangleleft$ is also multilication on Q_F and $k(\xi_1 \blacktriangleleft \xi_2) = k(\xi_1)k(\xi_2)$.

Now we shall show that the linear space Q_F is an F-algebra with multipliaction $\blacktriangleright$. Let $\xi_1,\xi_2,\xi_3 \in Q_F$. Then

$$((\xi_1 \blacktriangleright \xi_2) \blacktriangleright \xi_3)(x) = (\xi_1 \blacktriangleright \xi_2)(k(\xi_3)x) = \xi_1(k(\xi_2)(k(\xi_3)x)) =$$
$$= \xi_1((k(\xi_2)k(\xi_3))x) = \xi_1(k(\xi_2 \blacktriangleright \xi_3)x) = (\xi_1 \blacktriangleright (\xi_2 \blacktriangleright \xi_3))(x)$$

for every $x\in A$. It proves the associativity of $\blacktriangleright$. Moreover

$$(\xi_1 \blacktriangleright (\xi_2+\xi_3))(x) = \xi_1(k(\xi_2+\xi_3)x) = \xi_1(k(\xi_2)x+k(\xi_3)x) =$$
$$= \xi_1(k(\xi_2)x) + \xi_1(k(\xi_3)x) = (\xi_1 \blacktriangleright \xi_2)(x) + (\xi_1 \blacktriangleright \xi_3)(x) =$$
$$= (\xi_1 \blacktriangleright \xi_2 + \xi_1 \blacktriangleright \xi_3)(x)$$

for every $x \in A$. In a similar way we can show that

$$(\xi_1+\xi_2) \blacktriangleright \xi_3 = \xi_1 \blacktriangleright \xi_3 + \xi_2 \blacktriangleright \xi_3.$$

If $\lambda \in K$, then

$((\lambda\xi_1) \blacktriangleright \xi_2)(x) = (\lambda\xi_1)(k(\xi_2)x) = \lambda\xi_1(k(\xi_2)x) = \lambda(\xi_1 \blacktriangleright \xi_2)(x) = (\lambda(\xi_1 \blacktriangleright \xi_2))(x)$ for every $x \in A$. Further, it evident that $\xi_1 \blacktriangleright (\lambda\xi_2) = \lambda(\xi_1 \blacktriangleright \xi_2)$. Also $(F \blacktriangleright F)(x) = F(gx) = F(x)$ for $x \in A$, which shows that $F \blacktriangleright F = F$. Moreover, if $\xi \in Q_F$, then by Corollary 1.2

$$(F \blacktriangleright \xi \blacktriangleright F)(x) = F(k(\xi \blacktriangleright F)x) = F(k(\xi)gx) = F(k(\xi))F(x) =$$
$$= F(gk(\xi)g)F(x) = \xi(ggg)F(x) = \xi(g)F(x) =$$
$$= (\xi(g)F)(x)$$

for every $x \in A$. Hence $F \blacktriangleright \xi \blacktriangleright F = \xi(g)F$ i.e. the algebra Q_F is an F-algebra and the map $\xi \longrightarrow \xi(g)$ is an F-functional.

The proof in the case of $\blacktriangleleft$ is analogous.

Finally, we shall proof that the ideal of zeros of the algebra Q_F is the kernel of k. Indeed, if $k(\xi_0)=0$, then

$$(F \blacktriangleright \xi \blacktriangleright \xi_0 \blacktriangleright \eta \blacktriangleright F)(x) = F(k(\xi \blacktriangleright \xi_0 \blacktriangleright \eta \blacktriangleright F)x) =$$
$$= F(k(\xi)k(\xi_0)k(\eta)gx) = F(0) = 0$$

for every $x \in A$ and $\xi, \eta \in Q_F$. Hence ξ_0 belongs to the ideal of zeros of the F-algebra Q_F. Suppose now that $F \blacktriangleright \xi \blacktriangleright \xi_0 \blacktriangleright \eta \blacktriangleright F = 0$ for every $\xi, \eta \in Q_F$. Then $gk(\xi)k(\xi_0)k(\eta)g = 0$ and $gxk(\xi_0)yg = 0$ for every $x, y \in Q_g$. Since $L \subset Q_g$, $P \subset Q_g$ and A is a g-small algebra, so $k(\xi_0) = 0$.

The proof of the case with the multiplication $\blacktriangleright$ is analogous.

The following corollary is a direct consequence of Theorems 4.5 and 3.7.

Corollary 4.6. If A is a g-small algebra, then the quotient algebra $Q_{F/\ker k}$ is an [F]-small algebra.

Theorem 4.7. Let A be a g-algebra without identity. If $\xi \in Q_F$, then the functional $\tilde{\xi}:\tilde{A}=A\oplus\{\lambda e\}\longrightarrow K$ defined by

$$\tilde{\xi}(x+\lambda e) = \xi(x)+\lambda$$

is a quasinucleus relative to the ideal S_g of the g-algebra $\tilde{A}$. If $q\in Q_g\subset A$ then q is a quasinuclear element relative to the ideal $S_g\subset\tilde{A}$. The proof is similar to the proof of Theorem 3.7.

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INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY,
10-561 OLSZTYN, POLAND

Received April 18, 1989.